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Characterization of a Signed Graph Whose Signed Line Graph is S-Consistent

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Abstract. A *signed graph* is a graph in which every edge is designated to be either positive or negative; it is *balanced* if every cycle contains an even number of negative edges. A *marked signed graph* is a signed graph each vertex of which is designated to be positive or negative and it is *consistent* if every cycle in the signed graph possesses an even number of negative vertices. *Signed line graph* $L(S)$ of a given signed graph $S = (G, \sigma)$, as given by Behzad and Chartrand [7], is the signed graph with the standard *line graph* $L(G)$ of G as its underlying graph and whose edges are assigned the signs according to the rule: for any $e_ie_j \in E(L(S))$, $e_ie_j \in E^-\ (L(S)) \iff$ the edges e_i and e_j of S are both negative in S. Further, $L(S)$ is *S-consistent* if to each vertex e of $L(S)$, which is an edge of S, one assigns the sign $\sigma(e)$ then the resulting marked signed graph $(L(S))_\mu$ is consistent. In this paper, we give a characterization of signed graphs S whose signed line graphs $L(S)$ are S-consistent.

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1. Consistency in marked signed graphs

Unless mentioned or defined otherwise, for all terminology and notation in graph theory the reader is referred to [23]. We consider only finite, simple graphs free from self-loops.

Cartwright and Harary [10] considered graphs in which vertices represent persons and the edges represent symmetric dyadic relations amongst persons each of which is designated as being *positive* or *negative* according to whether the nature of the relationship is positive (friendly, like, etc.) or negative (hostile, dislike, etc.). Such a *network* (*i.e., weighted* graph) S is called a *signed graph* (Chartrand [11]; Harary *et al.* [15]). We regard graph as signed graph in which all the edges are positive, or the

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so-called *all-positive signed graph* (*all-negative signed graph* is defined similarly). A signed graph is said to be *homogeneous* if it is either all-positive or all-negative and *heterogeneous* otherwise.

Signed graphs are much studied in literature because of their extensive use in modeling a variety of socio-psychological processes (e.g., see Acharya [4, 5], Katai and Iwai [17], Roberts [19], Roberts and Xu [20]) and also because of their interesting connections with many classical mathematical systems (Zaslavsky [24]).

A cycle in a signed graph S is said to be *positive* if the product of the signs of its edges is positive. A cycle which is not positive is said to be *negative*. A signed graph is then said to be *balanced* if every cycle in it is positive (Harary [12], Cartwright and Harary [10], Harary *et al.* [15], Acharya and Acharya [5]). A spectral characterization of balanced signed graphs was given by Acharya [1]. Harary and Kabell [13, 14] developed a simple algorithm to detect balance in signed graphs as also enumerated them.

A *marked signed graph* is an ordered pair $S_\mu = (S, \mu)$ where $S = (G, \sigma)$ is a signed graph on its *underlying* graph $G = (V, E)$, and $\sigma : E(G) \to \{+1, -1\}$ is a function from the edge set $E(G)$ of G, called a *signing* of G, into the set $\{+1, -1\}$ whose elements are called *signs* and $\mu : V(G) \to \{+1, -1\}$ is a function from the vertex set $V(G)$ of G into the set $\{+1, -1\}$ whose elements are called *marks*. The *mark* (*sign*) $\mu(G')$ of a nonempty subgraph S' of S_μ is then defined as the product of the marks (signs) of the vertices (edges) in S'. A cycle Z in S_μ is said to be *consistent* if $\mu(Z) = +1$; otherwise, it is said to be *inconsistent*. Further, S is said to be *consistent* if every cycle in it is consistent. Beineke and Harary [8, 9] were the first to pose the problem of characterizing consistent marked graphs, which was eventually settled independently by Acharya [2, 3], Hoede [16] and Rao [18]. Recently, new characterizations of consistent marked graphs have been obtained by Roberts and Xu [20]. We will need the following result, known for graphs.

Theorem 1.1. [16] *A marked graph G_μ is consistent if and only if for any spanning tree T of G all fundamental cycles are consistent and all maximal common paths of pairs of fundamental cycles have end vertices with the same marking.*

The following definition of the *signed line graph* $L(S)$ (which they called 'line sigraph') of a given signed graph $S = (G, \sigma)$ was given by Behzad and Chartrand [7]: $L(S)$ is the signed graph, with the standard *line graph* $L(G)$ of G as its underlying graph and whose edges are assigned the signs according to the rule: for any $e_ie_j \in E(L(S))$, $e_ie_j \in E^-(L(S)) \Longleftrightarrow$ the edges e_i and e_j of S are both negative in S. Given any signed graph H, let $\pounds_H = \{S : S \text{ is a signed graph for which } L(S) \cong H\}$. Call H a *signed line graph* if $\pounds_H \neq \phi$ and then any $S \in \pounds_H$ a *signed line root* of H (cf. Acharya and Sinha [6]). A signed graph S and its signed line graph $L(S)$ are shown in Figure 1.

A signed graph $\Gamma = (H, \xi)$ is $(S, \mathcal{R})$-*marked* if there exists a signed graph $S = (G, \sigma)$, a bijection $\varphi : E(S) \to V(H)$, a binary relation $\mathcal{R}$ on $E(S)$ and marking $\mu : V(H) \to \{+1, -1\}$ of H satisfying the following *compatibility conditions*:

(CC1): $uv \in E(H) \Longleftrightarrow \{\varphi^{-1}(u), \varphi^{-1}(v)\} \in \mathcal{R}$.

(CC2): $\{\mu(u), \mu(v)\} = \{\sigma(\varphi^{-1}(u)), \sigma(\varphi^{-1}(v))\}\ \forall uv \in E(H)$.

Further, Γ is $(S, \mathcal{R})$-*consistent* if the following additional condition is satisfied:

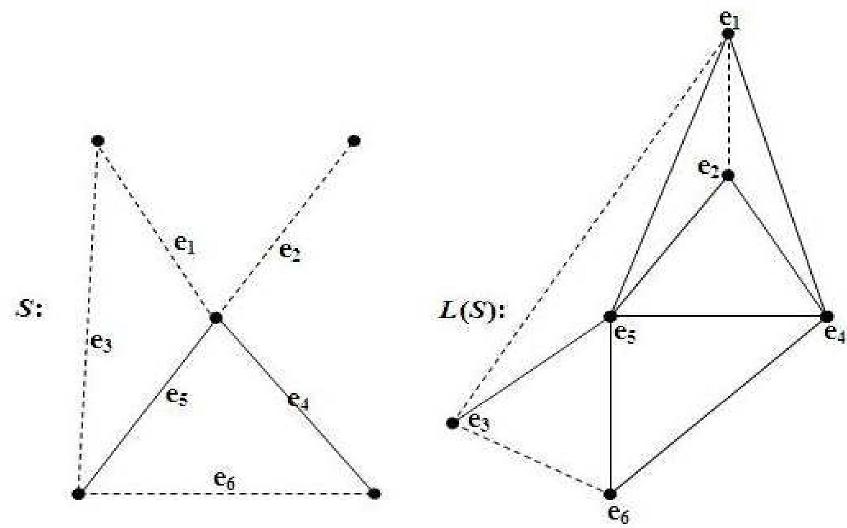

Figure 1. A signed graph S and its signed line graph $L(S)$

(CC3): $\prod_{v\in V(Z)} \mu(v) = 1, \ \forall Z \in \mathcal{C}_\Gamma$, where $\mathcal{C}_\Gamma$ denotes the set of all cycles in Γ.

The case when $\mathcal{R}$ is defined by the condition that $\varphi^{-1}(u)\cap\varphi^{-1}(v) \neq \phi$ is treated in Sinha [22] in respect of *signed graph equations* involving signed line graphs; in this particular case, the term '$(S, \mathcal{R})$-marked' and '$(S, \mathcal{R})$-consistent' will be reduced to 'S-marked' and 'S-consistent' respectively and similarly reduced terminology and notation will be adopted without specific mention in other notions using '$(S, \mathcal{R})$-format' in the above sense.

Clearly, if Γ_μ is S-consistent then S must be balanced. In this paper, we settle the problem of determining a signed graph S whose signed line graph $L(S)$ is S-consistent. This is equivalent to asking for an answer to the following question: Precisely, which signed graphs S are the signed line roots of H so that H is S-consistent?

2. Signed line-roots of S-consistent signed line graphs

We begin with the following formal definitions: Given any signed graph $S = (G, \sigma)$, $L(S)$ is *S-consistent* if $L(S)$ is consistent with respect to the marking $\mu_\sigma : V(L(S)) \rightarrow \{+1, -1\}$ which assigns to each vertex e in $L(S)$ the sign $\sigma(e)$ of the edge e in S, i.e., $\mu_\sigma(e) = \sigma(e)$ for every $e \in V(L(S)) = E(S)$. The number of positive (negative) edges incident at a vertex v, denoted by $d^+(v)$ (respectively, $d^-(v)$), is called the *positive* (*negative*) *degree* of v in S. The *total degree* $d(v)$ of the vertex v in S is then given by $d(v) = d^+(v) + d^-(v)$. The following is the main result of this paper.

Theorem 2.1. *For any isolate-free signed graph $S = (G, \sigma)$ of order p, $L(S)$ is S-consistent if and only if the following conditions hold in S:*

(1) *S is balanced; and*

(2) *for every vertex* v_i, $1 \leq i \leq p$, *in* S, $d(v_i) \geq 3$,
  (a) *if* $d(v_i) > 3$, *then* $d^-(v_i) = 0$; *or*
  (b) *if* $d(v_i) = 3$, *then either* $d^-(v_i) = 0$ *or* $d^-(v_i) = 2$; *and*
  (c) *if* $d^-(v_i) = 2$ *and* v_i *lies on a cycle of* S, *then the negative degree of* v_i *is due to the negative edges of the cycle.*

Proof. Necessity: Suppose $L(S)$ is S-consistent and suppose it is due to the S-marking $\mu : V(L(S)) \rightarrow \{+1, -1\}$ with respect to the bijection $\varphi : E(S) \rightarrow V(L(S)) := E(S)$ in accordance with the *compatibility conditions*:

(CC1): $e_ie_j \in E(L(S)) \Longleftrightarrow \varphi^{-1}(e_i) \cap \varphi^{-1}(e_j) \neq \phi$
(CC2): $\{\mu(e_i), \mu(e_j)\} = \{\sigma(\varphi^{-1}(e_i)), \sigma(\varphi^{-1}(e_j))\}\ \forall e_ie_j \in E(L(S))$
(CC3): $\prod_{e \in E(Z)} \sigma_L(e) = 1\ \forall Z \in \mathcal{C}_{L(S)}$

where σ_L is the signing of $L(S)$.

Then every cycle Z' in $(L(S))_\mu$ must have an even number of negative vertices. Since every vertex of $L(S)$ is marked by the signs of the edges in S and since the edges of every cycle in S create a cycle in $L(S)$ it follows that every cycle in S must be positive. Thus, by the definition of balanced signed graphs, (1) follows. Let v_i be a vertex of S having degree at least three. If in S, $d^-(v_i) \geq 3$, then any of the three negative edges incident to v_i will form an inconsistent triangle in $(L(S))_\mu$, a contradiction to the assumption that $(L(S))_\mu$ is S-consistent. Therefore, $d^-(v_i) < 3$. If $d(v_i) > 3$, then, $d^-(v_i)$ being equal to one or two would again contradict the assumption that $(L(S))_\mu$ is S-consistent. Thus, (2)(a) follows. Therefore, $d(v_i) = 3$. Since $(L(S))_\mu$ is consistent, it is clear that either $d^-(v_i) = 0$ or $d^-(v_i) = 2$. Thus, (2)(b) follows. Now suppose v_i lies on a cycle with $d(v_i) = 3$ and $d^-(v_i) = 2$. Then one of the two negative edges incident at v_i lies on a cycle Z. Further, in $(L(S))_\mu$ three cycles are created, namely the consistent cycle $\mathbf{C}_1$ consisting of the edges of the balanced cycle Z in S by condition (1), the consistent cycle $\mathbf{C}_2$ containing an edge incident to v_i by condition (2)(a) and (b) and also another cycle $\mathbf{C} = \mathbf{C}_1 \oplus \mathbf{C}_2$ is created, where $\mathbf{C}_1 \oplus \mathbf{C}_2$ represents the *symmetric difference* of the edge sets of $\mathbf{C}_1$ and $\mathbf{C}_2$. Now, $\mathbf{C}_1$ and $\mathbf{C}_2$ have two common vertices say u_1 and u_2 which form a positive chord in $\mathbf{C}$. Then, since $(L(S))_\mu$ is S-consistent, it follows from Theorem 1.1 that both u_1 and u_2 should receive the same marks and hence (2)(c) follows.

Sufficiency: For sufficiency, suppose conditions (1) and (2) hold for a given signed graph S. We shall show that $L(S)$ is S-consistent. If S is all positive, $L(S)$ is trivially S-consistent. So, let S be heterogeneous and suppose for any S-marking $\mu : V(L(S)) \rightarrow \{+1, -1\}$ of $L(S)$, $(L(S))_\mu$ is not S-consistent. Then there exists an inconsistent cycle in $(L(S))_\mu$. Let Z' be an inconsistent cycle of least length in $(L(S))_\mu$. Now if all the vertices of Z' correspond to the edges of a cycle in S, it would imply by our assumption and the definition of $L(S)$ that such a cycle in S is unbalanced, a contradiction to condition (1). Therefore, Z' must contain a vertex e'_i which corresponds to an edge e_i not lying on any cycle but incident to a vertex v with $d(v) \geq 3$ in S. Let e'_{i-1} and e'_{i+1} be the vertices adjacent to e'_i in Z'. Suppose Z' is of length three and $Z' = (e'_{i-1}, e'_i, e'_{i+1}, e'_{i-1})$. Since e_i does not belong to a cycle in S, the edges e_{i-1} and e_{i+1} in S corresponding to vertices e'_{i-1} and e'_{i+1} in Z' respectively must both be incident at the vertex v in S. If $d(v_i) = 3$, then by condition (2)(b), $d^-(v) = 0$ or $d^-(v) = 2$; in either case Z'

would be consistent, contrary to our assumption. Therefore, $d(v) > 3$, whence according to (2)(a) $d^-(v) = 0$, a contradiction to the inconsistency of Z'. Thus the length of Z' must be at least four. Let $Z' = (e'_1, e'_2, \ldots, e'_{i-1}, e'_i, e'_{i+1}, \ldots, e'_k, e'_1)$, $k \geq 4$, where e_i does not belong to any cycle in S. Suppose none of the vertices $e'_1, e'_2, \ldots, e'_{i-1}, e'_{i+1}, \ldots, e'_k$ corresponds to an edge of a cycle in S. Claim that all the edges $e_1, e_2, \ldots, e_{i-1}, e_i, e_{i+1}, \ldots, e_k$ in S are incident at the vertex v. If not, then, since length of Z' is at least four e'_{i-1} and e'_{i+1} are not adjacent in $L(S)$ and therefore the corresponding edges e_{i-1} and e_{i+1} in S are not incident. However, e_i is incident to both e_{i-1} and e_{i+1} so that e_{i-1} and e_i are incident at a vertex v_i, and e_i and e_{i+1} are incident at a vertex v_{i+1} whence $e_i = v_i v_{i+1}$. Since e_i does not belong to any cycle in S, the edges e_{i-1} and e_{i+1} would belong to different blocks B_{i-1} and B_{i+1} respectively in S. Hence, by a theorem of Sampathkumar [21] applied to $L(S)$, it follows that e_1 belongs to a block B^- in the maximal chain of blocks whose initial block is B_{i-1} in S and similarly e_k belongs to a block B^+ in the maximal chain of blocks whose initial block is B_{i+1} in S. However, since e'_1 and e'_k are adjacent in Z', by the definition of $L(S)$, we see that the corresponding edges e_1 and e_k must have a common vertex, a contradiction to our assumption that e_i is a bridge in S. Therefore, all the edges $e_1, e_2, \ldots, e_k$ must be incident at a single vertex say v in S. Since $k \geq 4$, so $d(v) \geq 4$ whence by condition (2)(a), $d^-(v_i) = 0$ which would mean that Z' is consistent, a contradiction to our assumption. Therefore, some of the vertices $e'_1, e'_2, \ldots, e'_{i-1}, e'_{i+1}, \ldots, e'_k$ must correspond to the edges $f_1, f_2, \ldots, f_t$ that belong to some cycles $Z_1, Z_2, \ldots, Z_t$ in S. Let $f'_1, f'_2, \ldots, f'_t$ be the corresponding vertices of $L(S)$ in the set $E' = \{e'_1, e'_2, \ldots, e'_{i-1}, e'_{i+1}, \ldots, e'_k\}$ that belong to the corresponding cycles $Z'_1, Z'_2, \ldots, Z'_t$ in $(L(S))_\mu$. Without loss of generality, we assume that all these cycles are fundamental. Note that more than one of the f_is may belong to a single fundamental cycle as also more than one fundamental cycle might share a single f'_i. Correspondingly, more than two edges among f_is might belong to the same fundamental cycle Z_i as also more than one of these fundamental cycles might share a single vertex in S. Since Z' is a minimal inconsistent cycle in $(L(S))_\mu$, all the cycles making up Z' due to its chords as the *direct sum* with respect to the symmetric difference of their edge sets must be consistent. Hence, by Theorem 1.1 we see that the ends of such chords if any must both be of opposite parity. Since e_i does not belong to any cycle in S, it follows that $e'_{i-1}e'_{i+1}$ is indeed a chord of Z' and hence e'_{i-1} and e'_{i+1} must be of opposite parity. Since e_{i-1} and e_{i+1} are incident at v in S, whose degree is at least three, we see that $d^-(v) \neq 0$. Therefore, by the contraposition of condition (2)(a), we get $d(v) \leq 3$. Hence, if $d(v) = 3$ then $d^-(v) = 2$ as $d^-(v) \neq 0$. Further, since v lies on a cycle of S, by condition (2)(c) the negative degree of v is due to the negative edges of the cycle, *viz.*, e_{i-1} and e_{i+1}, a contradiction to the above derivation that they are of opposite parity. Thus, by contraposition, the proof is complete. ∎

Corollary 2.1. *If S is an all-negative signed graph, then $L(S)$ is S-consistent if and if every component of S is either a path or a cycle of even length.*

3. Conclusion and scope

In the case of graphs, a graph G is uniquely determined by $L(G)$ unless $G \cong K_{1,3}$ or K_3. But for signed graphs, given $L(S)$, S need not be unique (Acharya and Sinha [6]). Clearly, given a signed graph H, the set $\pounds_H = \{S : S$ is a signed graph such that $H \cong L(S)\}$ is either empty or contains only a finite number of balanced signed graphs (called *signed line roots* of H). Hence, there are only a finite number of signed graphs S such that $L(S)$ is S-consistent; exactly how many may be a problem of further interest.

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