BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **34***(1) (2011), 127–135*

An Existence Result on Periodic Solutions of an Ordinary p-Laplacian System

[1]Xingyong Zhang and [2]Peixiang Zhou
[1,2]School of Mathematical Science and Computing Technology,
Central South University, Changsha, Hunan 410083, P. R. China
[1]zhangxingyong1@163.com, [2]zhoupeixiang@163.com

Abstract. A solvability condition of periodic solutions is obtained for an ordinary p-Laplacian system by the Linking Theorem in critical point theory.

2010 Mathematics Subject Classification: 37J45, 34C25, 58E50

Keywords and phrases: Periodic solution, p-Laplacian system, critical point, Linking Theorem.

1. Introduction

This paper is concerned with the existence of periodic solutions for the following p-Laplacian system:

$$-(|u'(t)|^{p-2}u'(t))' = \nabla F(t,u(t)), \quad \text{a.e.} \quad t \in \mathbb{R}, \tag{1.1}$$

where $p \geq 2, T > 0$ and $F : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ with $N \geq 2$ is T-periodic in its first variable and satisfies the following assumption:

(A) $F(t,x)$ is measurable in t for all $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$ and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^1(0,T;\mathbb{R}^+)$ such that

$$|F(t,x)| \leq a(|x|)b(t), \quad |\nabla F(t,x)| \leq a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

When $p = 2$, there are many existence results of periodic solutions for system (1.1) (see [5–11] and references therein). However, in these references, all authors studied only the existence of solutions. In [12] and [15], by using the local Linking Theorem, the authors considered the existence of nontrivial solutions. In [13], Tao and Tang considered the existence of non-constant solutions and they obtained the following theorem:

Communicated by Lee See Keong.
Received: September 19, 2008; *Revised:* March 24, 2010.

Theorem 1.1. *Assume that F satisfies* (A) *and the following conditions:*

$$F(t,x) \geq 0, \quad \forall\ (t,x) \in [0,T] \times \mathbb{R}^N,$$
$$\lim_{|x|\to 0} \frac{F(t,x)}{|x|^2} < \frac{\omega^2}{2} \text{ uniformly for a.e. } t \in [0,T],$$
$$\liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^2} > \frac{\omega^2}{2} \text{ uniformly for a.e. } t \in [0,T],$$

where $\omega = 2\pi/T$. There exist constants $r > 2$ and $\mu > r-2$ such that

$$\limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^r} < \infty \text{ uniformly for a.e. } t \in [0,T],$$
$$\liminf_{|x|\to\infty} \frac{\big(\nabla F(t,x), x\big) - 2F(t,x)}{|x|^\mu} > 0 \text{ uniformly for a.e. } t \in [0,T].$$

Then system (1.1) *with $p = 2$ has a non-constant T-periodic solution.*

When $p > 1$, in [14] and [16], the authors considered system (1.1) by using the dual least action principle and the Saddle Point Theorem, respectively, and they also only obtained the existence results of solutions for system (1.1). In our paper, motivated by idea of [13, 14, 16], we shall use the method in [13] to study the existence of non-constant solutions for system (1.1) with $p \geq 2$. The corresponding conditions in Theorem 1.1 are generalized and it is proved that under these conditions, the corresponding energy functional also satisfies (C) condition. Then an existence result for problem (1.1) is obtained by Linking Theorem.

2. Main results

Let

$$W_T^{1,p} = \{u : \mathbb{R} \to \mathbb{R}^N |\ u(t) \text{ is absolutely continuous on } \mathbb{R},\ u(t) = u(t+T) \text{ and } \dot{u} \in L^p(0,T;\mathbb{R}^N)\}.$$

On $W_T^{1,p}$, we define the norm as follows:

$$\|u\| = \left[\int_0^T |u(t)|^p dt + \int_0^T |\dot{u}(t)|^p dt\right]^{1/p}, \qquad u \in W_T^{1,p}.$$

Then, $\Big(W_T^{1,p}, \|\cdot\|\Big)$ is a reflexive and uniformly convex Banach space (see e.g. [1, Theorem 3.3 and Theorem 3.6]). From [4], one can know that a locally uniformly convex Banach space has the Kadec-Klee property, that is for any sequence $\{u_n\}$ satisfying $u_n \rightharpoonup u$ weakly in Banach space $(X, \|\cdot\|)$ and $\|u_n\| \to \|u\|$, one has $u_n \to u$ strongly in X. This property will be used later.

Let $\varphi : W_T^{1,p} \to \mathbb{R}$ be defined by

$$\varphi(u) = \frac{1}{p}\int_0^T |\dot{u}(t)|^p dt - \int_0^T F(t, u(t))dt.$$

It follows from assumption (A) that φ is continuously differentiable on $W_T^{1,p}$ and

$$\langle \varphi'(u), v\rangle = \int_0^T \big(|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t)\big)\, dt - \int_0^T (\nabla F(t, u(t)), v(t))dt$$

for $u, v \in W_T^{1,p}$ (see [6, Theorem 1.4]). Similar to [6, Corollary 1.1], it is easy to know that the solutions of problem (1.1) correspond to the critical points of φ. By [6, Proposition 1.1], there is a constant $C_0 > 0$ such that

$$\|u\|_\infty := \max_{t\in[0,T]} |u(t)| \le C_0\|u\|, \text{ for every } u \in W_T^{1,p}. \tag{2.1}$$

Let

$$\tilde{W}_T^{1,p} = \left\{ u \in W_T^{1,p} \;\middle|\; \int_0^T u(t)dt = 0 \right\}.$$

It is easy to know that $\tilde{W}_T^{1,p}$ is a closed subspace of $W_T^{1,p}$ and $W_T^{1,p} = \mathbb{R}^N \oplus \tilde{W}_T^{1,p}$. For $u \in W_T^{1,p}$, let $\bar{u} = \frac{1}{T}\int_0^T u(t)dt$ and $\tilde{u}(t) = u(t) - \bar{u}$. It follows from the proof of in [6, Proposition 1.1] that

$$\int_0^T |u(t)|^p dt \le T^p \int_0^T |u'(t)|^p dt, \text{ for every } u \in \tilde{W}_T^{1,p}, \tag{2.2}$$

(Wirtinger's inequality). Hence,

$$\|u\|^p \le (T^p + 1)\int_0^T |u'(t)|^p dt, \text{ for every } u \in \tilde{W}_T^{1,p}. \tag{2.3}$$

The main result of this paper is the following theorem:

Theorem 2.1. *Assume that F satisfies* (A) *and the following conditions:*

$$F(t, x) \ge 0, \quad \forall\ (t, x) \in [0, T] \times \mathbb{R}^N, \tag{2.4}$$

$$\lim_{|x|\to 0} \frac{F(t, x)}{|x|^p} < \frac{1}{pT^p} \quad \text{uniformly for a.e. } t \in [0, T], \tag{2.5}$$

$$\liminf_{|x|\to\infty} \frac{F(t, x)}{|x|^p} > \frac{\omega^p}{p} \quad \text{uniformly for a.e. } t \in [0, T], \tag{2.6}$$

where $\omega = 2\pi/T$. There exist constants $r > p$ and $\mu > r - p$ such that

$$\limsup_{|x|\to\infty} \frac{F(t, x)}{|x|^r} < \infty \quad \text{uniformly for a.e. } t \in [0, T], \tag{2.7}$$

$$\liminf_{|x|\to\infty} \frac{\big(\nabla F(t, x), x\big) - pF(t, x)}{|x|^\mu} > 0 \quad \text{uniformly for a.e. } t \in [0, T]. \tag{2.8}$$

Then system (1.1) *has a non-constant T-periodic solution.*

3. Proof of theorem

Lemma 3.1. *Assume that condition* (A), (2.7) *and* (2.8) *hold, then the functional φ satisfies condition* (C), *that is $\{u_n\}$ has a convergent subsequence in $W_T^{1,p}$, whenever $\varphi(u_n)$ is bounded and $\|\varphi'(u_n)\| \times (1 + \|u_n\|) \to 0$ as $n \to \infty$.*

Proof. Let $\{u_n\}$ be a sequence in $W_T^{1,p}$ such that $\varphi(u_n)$ is bounded and $\|\varphi'(u_n)\| \times (1+\|u_n\|) \to 0$ as $n \to \infty$. Then there exists a constant M such that

$$|\varphi(u_n)| \le M, \quad \|\varphi'(u_n)\|(1+\|u_n\|) \le M \tag{3.1}$$

for all $n \in N$. By (2.7), there are constants $C_1 > 0$ and $\delta_1 > 0$ such that

$$F(t,x) \le C_1|x|^r, \text{ for all } |x| > \delta_1 \text{ and a.e. } t \in [0,T]. \tag{3.2}$$

It follows from by assumption (A) that

$$F(t,x) \le \max_{s\in[0,\delta_1]} a(s)b(t), \text{ for all } |x| \le \delta_1 \text{ and a.e. } t \in [0,T].$$

Hence, for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, one has

$$F(t,x) \le \max_{s\in[0,\delta_1]} a(s)b(t) + C_1|x|^r. \tag{3.3}$$

It follows from (3.1), (3.3) and Hölder's inequality that

$$\begin{aligned}
\frac{1}{p}\|u_n\|^p &= \varphi(u_n) + \int_0^T F(t,u_n(t))dt + \frac{1}{p}\int_0^T |u_n(t)|^p dt \\
&\le M + C_2 + C_1\int_0^T |u_n(t)|^r dt + \frac{1}{p}\int_0^T |u_n(t)|^p dt \\
&\le M + C_2 + C_1\int_0^T |u_n(t)|^r dt + \frac{1}{p}T^{\frac{r-p}{r}}\left(\int_0^T |u_n(t)|^r dt\right)^{\frac{p}{r}}, \qquad (3.4)
\end{aligned}$$

where $C_2 = \max_{s\in[0,\delta_1]} a(s)\int_0^T b(t)dt$. By (2.8), there are constants $C_3 > 0$ and $\delta_2 > 0$ such that

$$(\nabla F(t,x),x) - pF(t,x) \ge C_3|x|^\mu > 0, \text{ for all } |x| > \delta_2 \text{ and a.e. } t \in [0,T].$$

By assumption (A), one has

$$\Big|(\nabla F(t,x),x) - pF(t,x)\Big| \le C_4 b(t), \text{ for all } |x| \le \delta_2 \text{ and a.e. } t \in [0,T],$$

where $C_4 = (p+\delta_2)\max_{s\in[0,\delta_2]} a(s)$. Hence one can obtain that

$$(\nabla F(t,x),x) - pF(t,x) \ge C_3|x|^\mu - C_3\delta_2^\mu - C_4 b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. Then one has

$$\begin{aligned}
(p+1)M &\ge p\varphi(u_n) - (\varphi'(u_n),u_n) \\
&= \int_0^T |u_n'(t)|^p dt - p\int_0^T F(t,u_n(t))dt - \int_0^T \big(|u_n'(t)|^{p-2}u_n'(t),u_n'(t)\big)dt \\
&\quad + \int_0^T \big(\nabla F(t,u_n(t)),u_n(t)\big)dt \\
&= \int_0^T \big(\nabla F(t,u_n(t)),u_n(t)\big)dt - p\int_0^T F(t,u_n(t))dt \\
&= \int_0^T \Big[\big(\nabla F(t,u_n(t)),u_n(t)\big) - pF(t,u_n(t))\Big]dt
\end{aligned}$$

$$\geq C_3 \int_0^T |u_n(t)|^\mu dt - TC_3\delta_2^\mu - C_4 \int_0^T b(t)dt,$$

which implies that $\int_0^T |u_n(t)|^\mu dt$ is bounded. If $\mu \geq r$, by (3.4) and Hölder's inequality

$$\int_0^T |u_n(t)|^r dt \leq T^{\frac{\mu-r}{\mu}} \Big(\int_0^T |u_n(t)|^\mu dt\Big)^{\frac{r}{\mu}},$$

one has $\|u_n\|$ is bounded. If $\mu \leq r$, by (2.1), one has

$$\begin{aligned}\int_0^T |u_n(t)|^r dt &= \int_0^T |u_n(t)|^{r-\mu}|u_n(t)|^\mu dt \\ &\leq \|u_n\|_\infty^{r-\mu} \int_0^T |u_n(t)|^\mu dt \\ &\leq C_0^{r-\mu}\|u_n\|^{r-\mu} \int_0^T |u_n(t)|^\mu dt.\end{aligned}$$

Thus, by (3.4) and $r-\mu < p$, one can know that $\|u_n\|$ is bounded, too. Hence $\|u_n\|$ is bounded in $W_T^{1,p}$. Since for the set in a reflexive Banach space, boundedness and weak compactness are equivalent, then there is a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u \text{ weakly in } W_T^{1,p}. \tag{3.5}$$

Furthermore, by [6, Proposition 1.2], one has

$$u_n \to u \text{ strongly in } C([0,T],\mathbb{R}^N).$$

Then we can use the same argument as in [16] to obtain that $\|u_n\| \to \|u\|$. Because of the uniform convexity of $W_T^{1,p}$ and (3.5), it follows that $u_n \to u$ strongly in $W_T^{1,p}$ from the *Kadec-Klee* property. The proof is completed. ∎

The Linking Theorem introduced in [9] (also see [3, Theorem 2.1 and Example 3 in Chapter 3]) by Rabinowitz will be used to obtain the critical point of φ.

Theorem 3.1 (Linking Theorem). *Let $E = E_1 \oplus E_2$ be a Banach space, where E_1 is a finite dimensional subspace of E and $E_2 = E_1^\perp$. Suppose that $\varphi(\cdot) \in C^1(E,\mathbb{R})$ satisfies the Palais-Smale condition and the following conditions:*

(i) *There are constants $\rho > 0$ and α such that $\varphi|_{\partial B_\rho \cap E_2} \geq \alpha$, where $B_\rho = \{u \in E : \|u\|_E < \rho\}$,*

(ii) *there is a constant $d < \alpha$ and $e \in E_2$, $\|e\|_E = 1$, $s_1 > 0, s_2 > \rho$ such that $\varphi|_{\partial Q} \leq d$ where $Q = \{u \in E | u = z + \lambda e, z \in E_1, \|z\| \leq s_1, \lambda \in [0, s_2]\}$.*

Then φ possesses a critical value.

Proof of Theorem 2.1. As shown in [2], a deformation lemma can be proved with the weaker condition (C) replacing the usual Palais-Smale condition, and it turns out that the Linking Theorem holds under the condition (C). Let $E_1 = \mathbb{R}^N$, $E_2 = \tilde{W}_T^{1,p} = \{u \in W_T^{1,p} | \int_0^T u(t)dt = 0\}$. Then, by Lemma 3.1, one only needs to prove (i) and (ii) in Linking Theorem hold.

By (2.5), there is $\varepsilon_0 > 0$ such that

$$\lim_{|x|\to 0} \frac{F(t,x)}{|x|^p} \le \frac{1}{pT^p} - 2\varepsilon_0.$$

Thus, there is a constant $\delta_0 \in (0, \delta_1)$ such that

$$F(t,x) \le \left(\frac{1}{pT^p} - \varepsilon_0\right)|x|^p, \text{ for all } |x| \le \delta_0 \text{ and a.e. } t \in [0,T]. \tag{3.6}$$

It follows from assumption (A) that

$$|F(t,x)| \le \max_{s\in[\delta_0,\delta_1]} a(s)b(t), \quad \text{for all } \delta_0 \le |x| \le \delta_1 \text{ and a.e. } t \in [0,T]. \tag{3.7}$$

Then, by (3.2), (3.6) and (3.7), for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, one has

$$F(t,x) \le \left(\frac{1}{pT^p} - \varepsilon_0\right)|x|^p + \Big(\max_{s\in[\delta_0,\delta_1]} a(s)b(t)\delta_0^{-r} + C_1\Big)|x|^r. \tag{3.8}$$

Hence, by (2.2), (2.3), (3.8) and (2.1), for every $u \in \tilde{W}_T^{1,p}$, one has

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\int_0^T |u'(t)|^p dt - \int_0^T F(t,u(t))dt \\
&\ge \frac{1}{p}\int_0^T |u'(t)|^p dt - \left(\frac{1}{pT^p} - \varepsilon_0\right)\int_0^T |u(t)|^p dt \\
&\quad - \max_{s\in[\delta_0,\delta_1]} a(s)\int_0^T b(t)dt\ \delta_0^{-r}\|u\|_\infty^r - C_1\|u\|_\infty^{r-p}\int_0^T |u(t)|^p dt \\
&\ge \frac{1}{p}\int_0^T |u'(t)|^p dt - \left(\frac{1}{pT^p} - \varepsilon_0\right)T^p\int_0^T |u'(t)|^p dt \\
&\quad - \max_{s\in[\delta_0,\delta_1]} a(s)\int_0^T b(t)dt\ \delta_0^{-r}\|u\|_\infty^r - C_1\|u\|_\infty^{r-p}\int_0^T |u(t)|^p dt \\
&= \varepsilon_0 T^p\int_0^T |u'(t)|^p dt - \max_{s\in[\delta_0,\delta_1]} a(s)\int_0^T b(t)dt\ \delta_0^{-r}\|u\|_\infty^r \\
&\quad - C_1\|u\|_\infty^{r-p}\int_0^T |u(t)|^p dt \\
&\ge \varepsilon_0 T^p(T^p+1)^{-1}\|u\|^p - (C_1C_0^{r-p} + C_2\delta_0^{-r}C_0^r)\|u\|^r.
\end{aligned}$$

Thus it is easy to know that there exist constants $\alpha > 0$ and $\rho \in (0,1)$ such that

$$\varphi(u) \ge \alpha, \text{ for every } u \in \tilde{W}_T^{1,p} \text{ and } \|u\| = \rho.$$

which shows that (i) holds. Next it will be shown that (ii) also holds. By (2.6), for

$$\varepsilon_1 = \operatorname*{essinf}_{t\in[0,T]} \liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^p} - \frac{\omega^p}{p} > 0,$$

there exists $\delta_3 > \rho T^{-\frac{1}{p}}$ such that

$$F(t,x) \ge \left(\frac{\omega^p}{p} + \varepsilon_1\right)|x|^p, \text{ for all } |x| \ge \delta_3 \text{ and a.e. } t \in [0,T].$$

Hence, by (2.4), for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, one has

$$F(t,x) \geq \left(\frac{\omega^p}{p} + \varepsilon_1\right) |x|^p - \left(\frac{\omega^p}{p} + \varepsilon_1\right) \delta_3^p. \tag{3.9}$$

Let

$$e = \frac{\sin \omega t}{(T + \omega^p T)^{\frac{1}{p}}} e_1 + \frac{\cos \omega t}{(T + \omega^p T)^{\frac{1}{p}}} e_2,$$
$$e_1 = (1,0,0,\ldots,0) \in \mathbb{R}^N,$$
$$e_2 = (0,1,0,\ldots,0) \in \mathbb{R}^N.$$

Then it is easy to know that

$$|e| = \frac{1}{(T + \omega^p T)^{\frac{1}{p}}},$$
$$|\dot{e}| = \frac{\omega}{(T + \omega^p T)^{\frac{1}{p}}},$$

$\|e\| = 1$ and $e \in \tilde{W}_T^{1,p}$. Let

$$Q = \left\{ x + se | x \in \mathbb{R}^N,\ |x| \leq \left(\frac{\omega^p}{p\varepsilon_1} + 1\right)^{\frac{1}{p}} \delta_3,\ 0 \leq s \leq \left(\frac{\omega^p}{p\varepsilon_1} + 1\right)^{\frac{1}{p}} \delta_3 (T + T\omega^p)^{\frac{1}{p}} \right\}$$

and $C_6 = \left(\frac{\omega^p}{p} + \varepsilon_1\right) T\delta_3^p$. It follows from Hölder's inequality that

$$\int_0^T |x + se|^2 dt \leq \left[\int_0^T \left(|x + se|^2\right)^{\frac{p}{2}} dt\right]^{\frac{2}{p}} \cdot T^{1-\frac{2}{p}}.$$

Thus one has

$$\begin{aligned}
\omega^p \int_0^T |x + se|^p dt &\geq \omega^p T^{1-\frac{p}{2}} \left(\int_0^T |x + se|^2 dt\right)^{\frac{p}{2}} \\
&= \omega^p T^{1-\frac{p}{2}} \left(\int_0^T |x|^2 dt + \int_0^T |se|^2 dt\right)^{\frac{p}{2}} \\
&\geq \omega^p T^{1-\frac{p}{2}} \left(\int_0^T |se|^2 dt\right)^{\frac{p}{2}} \\
&= \frac{T s^p \omega^p}{T + \omega^p T} \\
&= \int_0^T |s\dot{e}|^p dt. \qquad (3.10)
\end{aligned}$$

Then for every $x + se \in Q$, by (3.9) and (3.10), one has

$$\begin{aligned}
\varphi(x + se) &= \frac{1}{p} \int_0^T |s\dot{e}|^p dt - \int_0^T F(t, x + se) dt \\
&\leq \frac{\omega^p}{p} \int_0^T |x + se|^p dt - \left(\frac{\omega^p}{p} + \varepsilon_1\right) \int_0^T |x + se|^p dt + C_6
\end{aligned}$$

$$= -\varepsilon_1 \int_0^T |x+se|^p dt + C_6$$

$$\leq -\varepsilon_1 T^{1-\frac{p}{2}} \left(\int_0^T |x|^2 dt + \int_0^T \frac{s^2}{(T+\omega^p T)^{\frac{2}{p}}} dt \right)^{\frac{p}{2}} + C_6. \tag{3.11}$$

For every $x + se \in Q$, where $|x| = \left(\frac{\omega^p}{p\varepsilon_1} + 1\right)^{\frac{1}{p}} \delta_3$, one has

$$\varphi(x+se) \leq -\varepsilon_1 T^{1-\frac{p}{2}} \left(\int_0^T |x|^2 dt \right)^{\frac{p}{2}} + C_6 = 0. \tag{3.12}$$

For every $x + se \in Q$, where $s = \left(\frac{\omega^p}{p\varepsilon_1} + 1\right)^{\frac{1}{p}} \delta_3 (T + T\omega^p)^{\frac{1}{p}}$, one has

$$\varphi(x+se) \leq -\varepsilon_1 T^{1-\frac{p}{2}} \left(\int_0^T \frac{s^2}{(T+\omega^p T)^{\frac{2}{p}}} dt \right)^{\frac{p}{2}} + C_6 = 0. \tag{3.13}$$

If $s = 0$, for all $x \in \mathbb{R}^N$, by (2.4), one has

$$\varphi(x) = -\int_0^T F(t,x) dt \leq 0. \tag{3.14}$$

Therefore, by (3.12)–(3.14), one has $\varphi|_{\partial Q} \leq 0$. Let $d = 0, s_1 = T^{\frac{1}{p}} \left(\frac{\omega^p}{p\varepsilon_1} + 1\right)^{\frac{1}{p}} \delta_3$ and $s_2 = \left(\frac{\omega^p}{p\varepsilon_1} + 1\right)^{\frac{1}{p}} \delta_3 (T + T\omega^p)^{\frac{1}{p}} > \rho$. Thus (ii) is proved. Hence, by Linking Theorem, system (1.1) has a non-constant T-periodic solution. ∎

References

[1] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, (Second Edition), Pure and Applied Mathematics (Amsterdam), 140, Elsevier/Academic Press, Amsterdam, 2003.

[2] P. Bartolo, V. Benci and D. Fortunato, Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity, *Nonlinear Anal.* **7** (1983), no. 9, 981–1012.

[3] K. C. Chang, *Critical Point Theory and Its Applications* (Chinese), Xiandai Shuxue Congshu. Shanghai Kexue Jishu Chubanshe, Shanghai, 1986.

[4] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer, New York, 1975.

[5] Y. M. Long, Nonlinear oscillations for classical Hamiltonian systems with bi-even subquadratic potentials, *Nonlinear Anal.* **24** (1995), no. 12, 1665–1671.

[6] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Applied Mathematical Sciences, 74, Springer, New York, 1989.

[7] J. Mawhin and M. Willem, Critical points of convex perturbations of some indefinite quadratic forms and semilinear boundary value problems at resonance, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **3** (1986), no. 6, 431–453.

[8] P. H. Rabinowitz, On subharmonic solutions of Hamiltonian systems, *Comm. Pure Appl. Math.* **33** (1980), no. 5, 609–633.

[9] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conference Series in Mathematics, 65, Published for the Conference Board of the Mathematical Sciences (CBMS), Washington, DC, 1986.

[10] C. Tang, Periodic solutions of non-autonomous second order systems with γ-quasisubadditive potential, *J. Math. Anal. Appl.* **189** (1995), no. 3, 671–675.

[11] C.-L. Tang and X.-P. Wu, Notes on periodic solutions of subquadratic second order systems, *J. Math. Anal. Appl.* **285** (2003), no. 1, 8–16.

[12] Z.-L. Tao, S. Yan and S.-L. Wu, Periodic solutions for a class of superquadratic Hamiltonian systems, *J. Math. Anal. Appl.* **331** (2007), no. 1, 152–158.

[13] Z.-L. Tao and C.-L. Tang, Periodic and subharmonic solutions of second-order Hamiltonian systems, *J. Math. Anal. Appl.* **293** (2004), no. 2, 435–445.

[14] Y. Tian and W. Ge, Periodic solutions of non-autonomous second-order systems with a p-Laplacian, *Nonlinear Anal.* **66** (2007), no. 1, 192–203.

[15] J. F. Wu and X. P. Wu, Existence of nontrivial periodic solutions for a class of superquadratic seconder-order Hamiltonian systems, *Journal of Southwest University* (*Natural Science Edition*) (*A Journal of China*) **30** (2008), no. 4, 26–31.

[16] B. Xu and C.-L. Tang, Some existence results on periodic solutions of ordinary p-Laplacian systems, *J. Math. Anal. Appl.* **333** (2007), no. 2, 1228–1236.