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The Existence and Uniqueness of Positive Solutions for Integral Boundary Value Problems

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Abstract. This paper investigates the existence and uniqueness of $C[0,1]$ positive solutions for a second order integral boundary value problem. We mainly use the method of lower and upper solutions and the maximal principle. Our nonlinearity $f(t,u)$ may be singular at $u=0,\ t=0,\ 1$.

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1. Introduction and the main result

The theory of boundary value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo elasticity and plasma physics can be reduced to the nonlocal problems with integral boundary conditions. For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to the papers by Gallardo [1], Karakostas and Tsamatos [4], Lomtatidze and Malaguti [5] and the references therein.

In this paper, we shall consider the existence and uniqueness of positive solutions to the following second order singular boundary value problems with integral boundary conditions:

$$-u''(t) = f(t,u(t)), \quad t \in (0,1), \tag{1.1}$$

$$u(0) = \int_0^1 u(t)d\phi(t),\ u(1) = 0, \tag{1.2}$$

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where $f:(0,1)\times(0,+\infty)\to[0,+\infty)$, $f(t,u)$ is decreasing with respect to u and $\int_0^1 u(t)d\phi(t)$ denotes the Riemann-Stiejies integral.

The existence of positive solutions for nonlocal, including three-point, m-point, and integral boundary value problems with nondecreasing nonlinearities has been widely studied in recent years; the author refers the reader to [6, 7, 9] and references therein. However, when $f(t,u)$ is decreasing on u, there are only few published papers that deal with the study on it.

Inspired by the above papers, the aim of the present paper is to establish a sufficient condition for the existence of $C[0,1]$ positive solutions for the second order integral boundary value problem. Obviously, what we discuss is different from those in [1, 4–7, 9]. The main new features presented in this paper are as follows: Firstly, $f(t,u)$ is allowed to be not only singular at $t=0$ and 1, but also singular at $u=0$, which brings about many difficulties. Secondly, we require that $f(t,u)$ is decreasing on u, which is seldom researched. Thirdly, we not only obtain the existence of $C[0,1]$ positive solutions, but also obtain the uniqueness. Finally, the techniques used in this paper are the method of lower and upper solutions and the maximal principle.

In this paper, we first introduce some preliminaries in Section 2, then we state our main results in Section 3. Finally in Section 4 further discussions and remarks are given. Now we are ready to state the main result in this paper.

(H_1) $f(t,u)\in C((0,1)\times(0,+\infty),[0,+\infty))$ and $f(t,u)$ is decreasing with respect to u. ϕ is an increasing nonconstant function defined on [0,1], $\phi(0)=0$ and $\int_0^1(1-s)d\phi(s)\in[0,1)$.

(H_2) $f(t,\lambda)\not\equiv 0$ for all $t\in(0,1)$ and $\lambda>0$ and $\int_0^1 t(1-t)f(t,\lambda t(1-t))dt<+\infty$ for all $\lambda>0$.

Theorem 1.1. *Suppose that* (H_1), (H_2) *hold. Then the second order singular boundary value problems with integral boundary conditions* (1.1), (1.2) *has a unique* $C[0,1]$ *positive solution* ω *for which there exists* $m>0$ *such that*

$$mt(1-t)\le\omega(t). \tag{1.3}$$

When referring to singularity we mean that the function f in (1.1) is allowed to be unbounded at the points $u=0,\ t=0,\ 1$. A function $u\in C[0,1]\bigcap C^2(0,1)$ is called a $C[0,1]$ (positive) solution to (1.1) and (1.2) if it satisfies (1.1) and (1.2) ($u(t)>0$ for $t\in(0,1)$). A $C[0,1]$ (positive) solution of (1.1) and (1.2) is called a $C^1[0,1]$ (positive) solution if both $u'(0+)$ and $u'(1-)$ exist ($u(t)>0$, for $t\in(0,1)$).

A function α is called a lower solution to the problem (1.1), (1.2), if $\alpha\in C[0,1]\bigcap C^2$ $(0,1)$ and satisfies

$$\begin{cases}\alpha''(t)+f(t,\alpha(t))\ge 0, & t\in(0,1),\\ \alpha(0)-\int_0^1\alpha(t)d\phi(t)\le 0, & \alpha(1)\le 0.\end{cases}$$

Upper solution is defined by reversing the above inequality signs. If there exist a lower solution α and an upper solution β to problem (1.1), (1.2) such that $\alpha(t)\le\beta(t)$, then $(\alpha(t),\beta(t))$ is called a couple of upper and lower solution to problem (1.1), (1.2).

2. Preliminaries

In our main results, we will make use of the following lemmas.

Lemma 2.1. *Assume that* (H_1), (H_2) *hold. Then for* $y \in C((0,1),[0,+\infty))$, *boundary value problem*

$$\left\{ \begin{array}{l} -u''(t) = y(t), \quad t \in (0,1), \\ u(0) = \int_0^1 u(t)d\phi(t), \ u(1) = 0, \end{array} \right. \tag{2.1}$$

has a unique solution $u(t)$ *and* $u(t)$ *can be expressed in the form*

$$u(t) = \int_0^1 H(t,s)y(s)ds, \tag{2.2}$$

where

$$H(t,s) = G(t,s) + \frac{1-t}{1-\sigma}\int_0^1 G(s,\tau)d\phi(\tau), \quad \sigma = \int_0^1 (1-s)d\phi(s), \tag{2.3}$$

$$G(t,s) = \left\{ \begin{array}{ll} t(1-s), & 0 \le t \le s \le 1; \\ s(1-t), & 0 \le s \le t \le 1; \end{array} \right. \tag{2.4}$$

and we define $e(t) = G(t,t) = t(1-t), \ t \in [0,1]$.

Proof. First suppose that u is a solution of problem (2.1). It is easy to see by integration of (2.1) that

$$u'(t) = u'(0) - \int_0^t y(s)ds.$$

Integrate again, we can get

$$u(t) = u(0) + u'(0)t - \int_0^t (t-s)y(s)ds. \tag{2.5}$$

Letting $t = 1$ in (2.5), we find

$$u(0) + u'(0) = \int_0^1 (1-s)y(s)ds. \tag{2.6}$$

Substituting $u(0) = \int_0^1 u(s)d\phi(s)$ and (2.6) into (2.5), we obtain

$$u(t) = \int_0^1 G(t,s)y(s)ds + (1-t)\int_0^1 u(s)d\phi(s) \tag{2.7}$$

where

$$\begin{aligned} \int_0^1 u(s)d\phi(s) &= \int_0^1 \left[\int_0^1 G(s,\tau)y(\tau)d\tau + (1-s)\int_0^1 u(\tau)d\phi(\tau)\right] d\phi(s) \\ &= \int_0^1 \left[\int_0^1 G(s,\tau)y(\tau)d\tau + (1-s)\int_0^1 u(\tau)d\phi(\tau)\right] d\phi(s) \\ &= \int_0^1 \left[\int_0^1 G(s,\tau)y(\tau)d\tau\right] d\phi(s) + \int_0^1 (1-s)d\phi(s)\int_0^1 u(s)d\phi(s), \end{aligned}$$

and so,

$$\int_0^1 u(s)d\phi(s) = \frac{1}{1-\int_0^1(1-s)d\phi(s)}\int_0^1\left[\int_0^1 G(s,\tau)y(\tau)d\tau\right]d\phi(s). \tag{2.8}$$

Substituting (2.8) into (2.7), we have

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)y(s)ds + \frac{1-t}{1-\int_0^1(1-s)d\phi(s)}\int_0^1\left[\int_0^1 G(s,\tau)y(\tau)d\tau\right]d\phi(s) \\ &= \int_0^1 H(t,s)y(s)ds, \qquad (2.9) \end{aligned}$$

where $H(t,s)$ is defined by (2.3).

Conversely, suppose that $u(t) = \int_0^1 H(t,s)y(s)ds$. Then

$$\begin{aligned} u(t) = &\int_0^t s(1-t)y(s)ds + \int_t^1 t(1-s)y(s)ds \\ &+ \frac{1-t}{1-\int_0^1(1-s)d\phi(s)}\int_0^1\left[\int_0^1 G(s,\tau)y(\tau)d\tau\right]d\phi(s). \qquad (2.10) \end{aligned}$$

Direct differentiation of (2.10) implies

$$\begin{aligned} u'(t) = &-\int_0^t sy(s)ds + t(1-t)y(t) + \int_t^1 (1-s)y(s)ds - t(1-t)y(t) \\ &- \frac{1}{1-\int_0^1(1-s)d\phi(s)}\int_0^1\left[\int_0^1 G(s,\tau)y(\tau)d\tau\right]d\phi(s) \\ = &\int_t^1 (1-s)y(s)ds - \int_0^t sy(s)ds \\ &- \frac{1}{1-\int_0^1(1-s)d\phi(s)}\int_0^1\left[\int_0^1 G(s,\tau)y(\tau)d\tau\right]d\phi(s), \end{aligned}$$

and

$$u''(t) = -ty(t) - (1-t)y(t) = -y(t).$$

It is easy to verify that $u(0) = \int_0^1 u(t)d\phi(t)$, $u(1) = 0$ and, so, our lemma is proved. ∎

From (2.3) and (2.4), we can prove that $H(t,s), G(t,s)$ have the following properties.

Proposition 2.1. *Assume that* (H_1), (H_2) *hold. Then for* $t,s\in[0,1]$, *we have*

$$H(t,s)\ge 0, G(t,s)\ge 0. \tag{2.11}$$

Proposition 2.2. *For* $t,s\in[0,1]$, *we have*

$$e(t)e(s)\le G(t,s)\le G(t,t) = t(1-t) = e(t)\le \max_{t\in[0,1]} e(t) = \frac{1}{4}. \tag{2.12}$$

Proposition 2.3. *Assume that* (H_1), (H_2) *hold. Then for* $t,s\in[0,1]$, *we have*

$$\rho e(t)e(s)\le H(t,s)\le \gamma t(1-t) = \gamma e(t), \tag{2.13}$$

where

$$\gamma = \frac{1+\int_0^1 s d\phi(s)}{1-\sigma}, \ \ \rho = \frac{\int_0^1 e(\tau)d\phi(\tau)}{1-\sigma}. \tag{2.14}$$

Proof. By (2.3) and (2.12), we have

$$\begin{aligned} H(t,s) &= G(t,s) + \frac{1-t}{1-\sigma}\int_0^1 G(s,\tau)d\phi(\tau) \\ &\geq \frac{1-t}{1-\sigma}\int_0^1 G(s,\tau)d\phi(\tau) \\ &\geq \frac{\int_0^1 G(s,\tau)d\phi(\tau)}{1-\sigma} t(1-t) \\ &\geq \frac{\int_0^1 e(\tau)d\phi(\tau)}{1-\sigma} t(1-t)s(1-s) \\ &= \rho e(t)e(s), \quad t\in[0,1]. \end{aligned} \tag{2.15}$$

On the other hand, since $G(t,s) \leq s(1-s)$, we obtain

$$\begin{aligned} H(t,s) &= G(t,s) + \frac{1-t}{1-\sigma}\int_0^1 G(s,\tau)d\phi(\tau) \\ &\leq s(1-s) + \frac{1-t}{1-\sigma}\int_0^1 s(1-s)d\phi(\tau) \\ &\leq s(1-s)[1+\frac{1}{1-\sigma}\int_0^1 d\phi(\tau)] \\ &= s(1-s)\frac{1+\int_0^1 s d\phi(\tau)}{1-\sigma} \\ &= \gamma e(s), \quad t\in[0,1]. \end{aligned}$$

∎

Lemma 2.2. (*Maximal principle*) *Suppose that*

$$F_n = \{u(t) \in C[0,b_n]\bigcap C^2(0,b_n), u(0) - \int_0^1 u(t)d\phi(t) \geq 0, \ u(b_n) \geq 0\}.$$

If $u \in F_n$ *such that* $-u''(t) \geq 0, t\in(0,1)$, *then* $u(t) \geq 0, t \in [0,b_n]$.

Proof. Let

$$-u''(t) = \delta(t), \quad t\in(0,b_n), \tag{2.16}$$

$$u(0) - \int_0^1 u(t)d\phi(t) = r_1, \ u(b_n) = r_2. \tag{2.17}$$

Then $r_1 \geq 0, r_2 \geq 0$ and $\delta(t) \geq 0, t\in(0,b_n)$.

By integrating (2.17) twice and noticing (2.18), we have

$$u(t) = (1-\frac{t}{b_n})r_1 + r_2 + (1-\frac{t}{b_n})\int_0^1 u(t)d\phi(t) + \int_0^{b_n} G_n(t,s)\delta(s)ds \tag{2.18}$$

where

$$G_n(t,s) = \frac{1}{b_n}\begin{cases} t(b_n-s), & 0\leq t\leq s\leq b_n; \\ s(b_n-t), & 0\leq s\leq t\leq b_n. \end{cases}$$

In view of (2.19) and the definition of $G_n(t,s)$, we can obtain $u(t) \geq 0, t \in [0, b_n]$. This completes the proof of Lemma 2.2. ∎

Lemma 2.3. [2] *Suppose that E is a real Banach space and $D \subset E$ is convex and bounded. $A : D \to D$ is continuous and $A(D)$ is pre-compact. Then A has at least one fixed point in D.*

3. The proof of the main result

3.1. The existence of lower and upper solutions

Let E be the Banach space $C[0,1]$. Define the set P and the operator T as follows:

(3.1)
$$P = \{u \in E \mid \text{there exists a positive number } k_u \text{ such that } u(t) \geq k_u e(t),\ t \in [0,1]\},$$

$$Tu(t) = \int_0^1 H(t,s) f(s,u(s))ds. \tag{3.2}$$

Evidently $e \in P$. Therefore, P is not empty.

For all $u \in P$, by the definition of P, there exists a positive number k_u such that $u(t) \geq k_u e(t),\ t \in [0,1]$. It follows from (H_2) that

$$\int_0^1 e(s) f(s,u(s))ds \leq \int_0^1 e(s) f(s, k_u e(s))ds < +\infty.$$

By the definition of $H(t,s)$ and (3.2) we have

$$Tu(t) \leq \gamma e(t) \int_0^1 f(s,u(s))ds.$$

Let $B = \max_{t\in[0,1]} u(t)$. By the condition (H_2) and the continuity of f, we have that $\int_0^1 e(s) f(s,\ B)ds > 0$. Thus,

$$\int_0^1 e(s) f(s,u(s))ds \geq \int_0^1 e(s) f(s,B)ds > 0.$$

On the other hand, by the definition of $H(t,s)$ we see that

$$Tu(t) \geq \rho e(t) \int_0^1 e(s) f(s,u(s))ds = k_{Tu} e(t), \quad t \in [0,1],$$

where $k_{Tu} = \rho \int_0^1 e(s) f(s,u(s))ds$. So Tu is well defined on P and

$$Tu \in P, \quad \forall u \in P. \tag{3.3}$$

Let

$$b_1(t) = \min\{e(t), (Te)(t)\}, \quad b_2(t) = \max\{e(t), (Te)(t)\}. \tag{3.4}$$

Obviously $b_1(t), b_2(t)$ make sense and $b_1(t) \leq b_2(t)$. Since $Te \in P$ it follows that there exists a positive number k_{Te} such that $(Te)(t) \geq k_{Te} e(t)$. Therefore, $b_1(t) \geq \min\{1, k_{Te}\} e(t) = k_1 e(t)$. This implies $b_1 \in P$ and $b_2 \in P$. Furthermore, $Tb_1(t), Tb_2(t)$ make sense and

$$Tb_2(t) \leq Tb_1(t) \leq T(k_1 e)(t),\ t \in [0,1]. \tag{3.5}$$

With the aid of (3.4) and the decreasing property of the operator T it follows that

$$Tb_2(t) \leq (Te)(t) \leq b_2(t), \quad Tb_1(t) \geq (Te)(t) \geq b_1(t), \; t \in [0,1]. \tag{3.6}$$

Therefore

$$(Tb_2)''(t) + f(t, Tb_2(t)) \geq (Tb_2)''(t) + f(t, b_2(t)) = 0, \tag{3.7}$$

$$(Tb_1)''(t) + f(t, Tb_1(t)) \leq (Tb_1)''(t) + f(t, b_1(t)) = 0. \tag{3.8}$$

From (2.3) and (3.2), it is easy to verify that

$$Tb_i(0) = \int_0^1 Tb_i(t) d\phi(t), \; Tb_i(1) = 0, \; \imath = 1, 2. \tag{3.9}$$

From (3.7) (3.8) and (3.9), it follows that

$$(H(t), Q(t)) = (Tb_2(t), Tb_1(t)), \; t \in [0,1] \tag{3.10}$$

is a couple of upper and lower solution to (1.1) and (1.2). Furthermore, we have $H, \; Q$ are in P and $H, \; Q$ are in $C[0,1] \bigcap C^2(0,1)$.

3.2. The existence of positive solution to (1.1) and (1.2)

First of all, we define a partial ordering in $C[0,1] \bigcap C^2(0,1)$ by $u \leq v$, if and only if

$$u(t) \leq v(t), \forall t \in [0,1].$$

Then for every function $u(t) \in C[0,1] \bigcap C^2(0,1)$ we define

$$(gu)(t) = \begin{cases} f(t, H(t)), & \text{if } u \ngeq H, \\ f(t, u(t)), & \text{if } H \leq u \leq Q, \\ f(t, Q(t)), & \text{if } u \nleq Q. \end{cases} \tag{3.11}$$

By the assumptions of Theorem 1.1, we have that the function $g : (0,1) \times (-\infty, +\infty) \to [0, +\infty)$ is continuous.

Let b_n be a sequence satisfying $b_1 < \ldots < b_n < b_{n+1} < \ldots < 1$, and $b_n \to 1$ as $n \to +\infty$, and let r_n be a sequence satisfying

$$H(b_n) \leq r_n \leq Q(b_n), \quad n = 1, 2, \ldots$$

For each n, let us consider the following nonsingular problem

$$\begin{cases} -u''(t) = (gu)(t), & t \in [0, b_n], \\ u(0) = \int_0^1 u(t) d\phi(t), \; u(b_n) = r_n. \end{cases} \tag{3.12}$$

Obviously, it follows from the proof of Lemma 2.2 that the problem (3.12) is equivalent to the integral equation

$$\begin{aligned} u(t) &= A_n u(t) \\ &= r_n + \left(1 - \frac{t}{b_n}\right) \int_0^1 u(t) d\phi(t) \\ &\quad + \int_0^{b_n} G_n(t,s)(gu)(s) ds, \quad t \in [0, b_n], \end{aligned} \tag{3.13}$$

where $G_n(t,s)$ is defined in Lemma 2.2. It is easy to verify that $A_n : X_n \to X_n = C[0,b_n]$ and

$$e(t) \le e_n(t) \le \frac{1}{b_n} e(t). \tag{3.14}$$

For any $u \in X_n$, from (H_2), (2.12), (3.11) and (3.14), we have

$$\begin{aligned} A_n u(t) &= r_n + (1 - \frac{t}{b_n})u(0) + \int_0^{b_n} G_n(t,s)(gu)(s)ds \\ &\le r_n + u(0) + \int_0^{b_n} e_n(s) f(s, H(s))ds \\ &\le r_n + u(0) + \int_0^{b_n} e_n(t) f(t, \lambda e_n(t))dt \\ &\le r_n + u(0) + \frac{1}{b_n} \int_0^{b_n} e(t) f(t, \lambda e(t))dt < +\infty, \end{aligned}$$

where $e_n(t) = G_n(t,t)$. Thus, $A_n(X_n)$ is bounded. For any $u \in X_n,\ t_1,\ t_2 \in [0,1]$, we have

$$|A_n u(t_2) - A_n u(t_1)| \le \int_0^{b_n} |G_n(t_2,s) - G_n(t_1,s)| f(s, H(s))ds.$$

This, together with the continuity of $G_n(t,s)$, implies that $\{A_n u \mid u \in X_n\}$ is equicontinuous. So $A_n(X_n)$ is pre-compact.

Furthermore, it is easy to verify that A_n is continuous. From Lemma 2.3, we assert that A_n has at least one fixed point $u_n \in C[0,b_n] \cap C^2(0,b_n)$.

We claim that

$$H \le u_n \le Q,$$

that is

$$H(t) \le u_n(t) \le Q(t), \quad t \in [0,b_n]. \tag{3.15}$$

From this it follows that

$$-u_n''(t) = f(t, u_n(t)), \quad t \in [0,b_n]. \tag{3.16}$$

Suppose by contradiction that $u_n \not\le Q$. Because of the definition of g we have

$$(gu_n)(t) = f(t, Q(t)), \quad t \in [0,b_n].$$

Consequently

$$-u_n''(t) = f(t, Q(t)), \quad t \in [0,b_n]. \tag{3.17}$$

On the other hand, since Q is an upper solution to (1.1) and (1.2), we obviously have

$$-Q''(t) \ge f(t, Q(t)), \quad t \in (0,1). \tag{3.18}$$

Let

$$z(t) = Q(t) - u_n(t), \quad t \in [0,b_n].$$

From (3.17) and (3.18), it follows that $-z''(t) \ge 0,\ t \in [0,b_n],\ z \in C[0,b_n] \bigcap C^2(0, b_n),\ z(0) - \int_0^1 z(t)d\phi(t) \ge 0$, and $z(b_n) \ge 0$. By using Lemma 2.2 we have $z(t) \ge 0,\ t \in [0,b_n]$, a contradiction to the assumption $u_n \not\le Q$. Hence $u_n \not\le Q$ is impossible.

Similarly, suppose by contradiction that $u_n \not\geq H$. Because of the definition of g we have

$$(gu_n)(t) = f(t, H(t)), \quad t \in [0, b_n].$$

Consequently

$$-u_n''(t) = f(t, H(t)), \quad t \in [0, b_n]. \tag{3.19}$$

On the other hand, since H is a lower solution to (1.1) and (1.2), we obviously have

$$-H''(t) \leq f(t, H(t)), \quad t \in (0, 1). \tag{3.20}$$

Let

$$z(t) = u_n(t) - H(t), \quad t \in [0, b_n].$$

From (3.18) and (3.19), it follows that $-z''(t) \geq 0,\ t \in [0, b_n],\ z \in C[0, b_n] \bigcap C^2(0, b_n),\ z(0) - \int_0^1 z(t)d\phi(t) \geq 0$, and $z(b_n) \geq 0$. By using Lemma 2.2 we have $z(t) \geq 0,\ t \in [0, b_n]$, a contradiction to the assumption $u_n \not\geq H$. Hence $u_n \not\geq H$ is impossible.

Consequently (3.15) holds.

Using the method of [8] and [3, Theorem 3.2], we can obtain that there is a $C[0, 1]$ positive solution $\omega(t)$ to (1.1), (1.2) such that $H < \omega < Q$, and a subsequence of $u_n(t)$ converges to $\omega(t)$ on any compact subintervals of $(0, 1)$.

3.3. Uniqueness of the $C[0, 1]$ positive solution and the proof of (1.3)

Suppose that u_1, u_2 are $C[0, 1]$ positive solutions to (1.1) and (1.2). We may assume, without loss of generality, that there exists $t^* \in (0, 1)$ such that $u_2(t^*) - u_1(t^*) = \max(u_2(t) - u_1(t)) > 0$. Let

$$\alpha = \inf\{t_1 | 0 \leq t_1 < t_*, u_2(t) \geq u_1(t),\ t \in (t_1, t^*]\};$$

$$\beta = \sup\{t_2 | t_* < t_2 \leq 1, u_2(t) \geq u_1(t),\ t \in (t^*, t_2\};$$

$$z(t) = u_2(t) - u_1(t),\ t \in [0, 1].$$

Evidently,

$$t^* \in (\alpha, \beta),\ u_2(t) \geq u_1(t),\ f(t, u_2(t)) \leq f(t, u_1(t)),\ t \in [\alpha, \beta].$$

Hence,

$$z''(t) = f(t, u_1(t)) - f(t, u_2(t)) \geq 0, \quad t \in [\alpha, \beta].$$

By using (1.2), it is easy to check that there exist the following two possible cases:

(1) $z(\alpha) = z(\beta) = 0$,

(2) $z(\alpha) > 0,\ z(\beta) = 0$.

Case (1): From $z''(t) \geq 0$ and $z(\alpha) = z(\beta) = 0$ we derive that $z(t) \leq 0,\ t \in [\alpha, \beta]$, which is in contradiction with $u_2(t^*) > u_1(t^*)$.

Case (2): In this case we have $\alpha = 0$, and $z'(t^*) = 0$. Since $z'(t)$ is increasing on $[\alpha, \beta]$, we have $z'(t) \geq 0,\ t \in [t^*, \beta]$, that is, $z(t)$ is increasing on $[t^*, \beta]$. From $z(\beta) = 0$, we see $z(t^*) \leq 0$, which is in contradiction to $u_2(t^*) > u_1(t^*)$. Then it follows from $u(t) \geq H(t) \geq K_H t(1 - t)$ that the inequality(1.3) holds.

4. Further discussions and remarks

Corollary 4.1. *Suppose that in Theorem 1.1 condition* (H_1) *holds and condition* (H_2) *is strengthened and becomes*

$$f(t,\lambda) \not\equiv 0,\ \int_0^1 f(t,\lambda t(1-t))dt < +\infty,\ \lambda > 0. \tag{4.1}$$

Then the problem (1.1), (1.2) *has a unique* $C^1[0,1]$ *positive solution* ω *for which there exist constants* M *and* m *with* $M \geq m \geq 0$ *such that*

$$m(1-t) \leq \omega(t) \leq M(1-t),\ t \in [0,1]. \tag{4.2}$$

Proof. Since $f(t,u)$ is decreasing with respect to u, we obviously have that $f(t,\omega(t)) \leq f(t, mt(1-t))$. From (4.1) it follows that $f(t,\omega(t))$ is integrable on $(0,1)$, that is, $\omega''(t)$ is integrable on $(0,1)$. Thus $\omega'(0+)$ and $\omega'(1-)$ exist, i.e., $\omega(t)$ is a $C^1[0,1]$ positive solution to (1.1) and (1.2).

Since ω is the unique $C[0,1]$ positive solution to (1.1) and (1.2), then $\omega''(t) \leq 0,\ t \in (0,1)$, and so ω is a concave function on [0,1]. From (2.7) we know that $\omega(t)$ can be stated as

$$\omega(t) = \int_0^1 G(t,s)f(s,\omega(s))ds + (1-t)\int_0^1 \omega(s)d\phi(s). \tag{4.3}$$

Therefore,

$$\omega(t) \geq (1-t)\int_0^1 \omega(s)d\phi(s),\ t \in [0,1] \tag{4.4}$$

Since ω is the unique $C^1[0,1]$ positive solution to (1.1) and (1.2), we have

$$\omega(t) = \int_t^1 (-\omega'(s))ds \leq \max_{t\in[0,1]} \mid \omega'(t) \mid (1-t),\ t \in [0,1]. \tag{4.5}$$

From (4.5) and (4.6) it follows that (4.2) holds, which is the required property. This completes the proof of Corollary 4.1. ∎

If $f(t,u)$ is nonsingular at $u=0$, then for all $u \geq 0,\ f(t,u) \leq f(t,0),\ t \in (0,1)$, and then we have the following corollaries.

Corollary 4.2. *Suppose that*

- (H_3) $f \in C((0,1)\times[0,+\infty),[0,+\infty))$, *and* $f(t,u)$ *is decreasing with respect to* u. ϕ *is an increasing nonconstant function defined on [0,1] with* $\phi(0)=0$, $\int_0^1 (1-s)d\phi(s) \in [0,1)$.
- (H_4) $f(t,\lambda) \not\equiv 0$ *for all* $t \in (0,1)$ *and* $\lambda > 0$ *and* $\int_0^1 t(1-t)f(t,0)dt < +\infty$ *for all* $\lambda > 0$.

Then the conclusion of Theorem 1.1 holds.

Corollary 4.3. *Suppose that in Corollary 4.2 the condition* (H_3) *holds, the condition* (H_4) *is strengthened and then become*

- (H_5) $f(t,\lambda) \not\equiv 0$ *for all* $t \in (0,1)$ *and* $\lambda > 0$ *and* $\int_0^1 f(t,0)dt < +\infty$ *for all* $\lambda > 0$.

Then the conclusion of Corollary 4.1 holds.

If $f(t,u)$ is nonsingular at $t,\ u$, then $\int_0^1 f(t,0)dt < +\infty$ holds, therefore we have the following corollary.

Corollary 4.4. *If $f \in C([0,1]\times[0,+\infty),[0,+\infty))$ is decreasing with respect to u and $f(t,\lambda)\not\equiv 0$, for all$t \in (0,1)$ and $\lambda \geq 0$, ϕ is a increasing nonconstant function defined on [0,1] with $\phi(0)=0$, and $\int_0^1(1-s)d\phi(s)\in[0,1)$, then the conclusion of Corollary 4.1 holds.*

Remark 4.1. Consider equation (1.1) and the following singular boundary conditions

$$u(0)=0,\quad u(1)=\int_0^1 u(t)d\phi(t). \tag{4.6}$$

By analogous methods, we have the following.

Assume that u is a $C[0,1]$ positive solution to (1.1) and (4.6), then $u(t)$ can be stated

$$u(t)=\int_0^1 H(t,s)f(s,u(s))ds,\ t\in[0,1], \tag{4.7}$$

where

$$H(t,s)=G(t,s)+\frac{t}{1-\sigma}\int_0^1 G(s,\tau)d\phi(\tau),\quad \sigma=\int_0^1 sd\phi(s), \tag{4.8}$$

and $G(t,s)$ is defined in (2.4).

Theorem 4.1. *Suppose that*

- (H_6) $f\in C((0,1)\times(0,+\infty),[0,+\infty)), f(t,u)$ *is decreasing with respect to u and ϕ is a increasing nonconstant function defined on [0,1] with $\phi(0)=0$ and $\int_0^1(1-s)d\phi(s)\in[0,1)$.*
- (H_7) $f(t,\lambda)\not\equiv 0$ *for all $t\in(0,1)$ and $\lambda>0$ and $\int_0^1 t(1-t)f(t,\lambda t(1-t))dt<+\infty$ for all $\lambda>0$.*

Then the second order singular boundary value problem with integral boundary conditions (1.1), (4.6) has a unique $C[0,1]$ positive solution ω for which there exists constant $m>0$ so that

$$mt(1-t)\leq\omega(t),\ t\in[0,1]. \tag{4.9}$$

Theorem 4.2. *Suppose that in Theorem 4.1 condition (H_1) holds and condition (H_2) is strengthened and becomes*

$$f(t,\lambda)\not\equiv 0,\ \int_0^1 f(t,\lambda t(1-t))dt<+\infty,\ \forall\lambda>0. \tag{4.10}$$

Then the problem (1.1), (4.6) has a unique $C^1[0,1]$ positive solution ω for which there exist constants M and m with $M\geq m\geq 0$ such that

$$m(1-t)\leq\omega(t)\leq M(1-t),\ t\in[0,1]. \tag{4.11}$$

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