BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **34***(1) (2011), 165–169*

A Note on $[r, s, c, t]$-Colorings of Graphs

[1]JIAN-TING SHENG AND [2]GUI-ZHEN LIU
School of Mathematics, Shandong University, Jinan, 250100, P. R. China
[1]sjtlzh@163.com, [2]gzliu@sdu.edu.cn

Abstract. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A subset S of $V(G)$ is called an independent set if no two vertices of S are adjacent in G. The minimum number of independent sets which form a partition of $V(G)$ is called chromatic number of G, denoted by $\chi(G)$. A subset S of $E(G)$ is called an edge cover of G if the subgraph induced by S is a spanning subgraph of G. The maximum number of edge covers which form a partition of $E(G)$ is called edge covering chromatic number of G, denoted by $\chi'_c(G)$. Given nonnegative integers r, s, t and c, an $[r, s, c, t]$-coloring of G is a mapping f from $V(G) \bigcup E(G)$ to the color set $\{0, 1, \ldots, k-1\}$ such that the vertices with the same color form an independent set of G, the edges with the same color form an edge cover of G, and $|f(v_i) - f(v_j)| \geq r$ if v_i and v_j are adjacent, $|f(e_i) - f(e_j)| \geq s$ for every e_i, e_j from different edge covers, $|f(v_i) - f(e_j)| \geq t$ for all pairs of incident vertices and edges, respectively, and the number of edge covers formed by the coloring of edges is exactly c. The $[r, s, c, t]$-chromatic number $\chi_{r,s,c,t}(G)$ of G is defined to be the minimum k such that G admits an $[r, s, c, t]$-coloring. In this paper, we present the exact value of $\chi_{r,s,c,t}(G)$ when $\delta(G) = 1$ or G is an even cycle.

2010 Mathematics Subject Classification: 05C15

Keywords and phrases: $[r, s, c, t]$-coloring, edge covering coloring, chromatic number, $[r, s, t]$-coloring.

1. Introduction

In this paper, all graphs are finite, simple and undirected. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of graph G, respectively. In a proper vertex coloring of a graph G, v_i and v_j are colored differently if they are adjacent. In an edge covering coloring of a graph G, $E(G)$ is partitioned into edge covers and different edge covers has different colors. The minimum number of colors such that G admits a proper vertex coloring is the chromatic number $\chi(G)$. The maximum number of colors such that G admits an edge covering coloring is the edge covering coloring chromatic number

Communicated by Rosihan M. Ali, Dato'.
Received: April 22, 2009; *Revised:* October 8, 2009.

$\chi'_c(G)$. It is well known that $\chi(G) \leq \Delta + 1$. For $\chi'_c(G)$, R. P. Gupta first proved the following theorem.

Theorem 1.1. [3] *Let G be a graph. Then*

$$\delta - 1 \leq \chi'_c(G) \leq \delta.$$

Kemnitz and Marangio introduced the $[r, s, t]$-Colorings of graphs in [4]. Given nonnegative integers r, s and t, an $[r, s, t]$-coloring of a graph G is a function c from $V(G) \bigcup E(G)$ to the color set $\{0, 1, \ldots, k\}$ such that $|c(v_i) - c(v_j)| \geq r$ for every two adjacent vertices $v_i, v_j \in V$, $|c(e_i) - c(e_j)| \geq s$ for every two adjacent edges $e_i, e_j \in E$, and $|c(v_i) - c(e_j)| \geq t$ for every vertex v_i and its incident edges e_j. The $[r, s, t]$-chromatic number $\chi_{r,s,t}(G)$ of G is the minimum k such that G admits an $[r, s, t]$-coloring. In [4], the authors also gave exact values and some bounds of $\chi_{r,s,t}(G)$ when at least one among the three parameters is fixed, for example if $\min\{r, s, t\} = 0$ or if two of the three parameters are 1. Similarly, we give the definition of $[r, s, c, t]$-colorings of graphs as follows. Given nonnegative integers r, s, t and c, an $[r, s, c, t]$-coloring of G is a mapping f from $V(G) \bigcup E(G)$ to the color set $\{0, 1, \ldots, k-1\}$ such that the vertices with the same color form an independent set of G, the edges with the same color form an edge cover of G, and $|f(v_i) - f(v_j)| \geq r$ if v_i and v_j are adjacent, $|f(e_i) - f(e_j)| \geq s$ for every e_i, e_j from different edge covers, $|f(v_i) - f(e_j)| \geq t$ for all pairs of incident vertices and edges, respectively, and the number of edge covers formed by the coloring of edges is exactly c. The $[r, s, c, t]$-chromatic number $\chi_{r,s,c,t}(G)$ of G is defined to be the minimum k such that G admits an $[r, s, c, t]$-coloring. It is obvious that we only consider the case that $1 \leq c \leq \chi'_c(G)$, for otherwise either the edges can be colored arbitrarily(if $c = 0$) such that $\chi_{r,s,c,t}(G) = \chi_{r,0,t}(G)$ which has been considered by Kemnitz and Marangio [4] or there is no $[r, s, c, t]$-coloring of G (if $c > \chi'_c(G)$).

2. The proofs of the main results

In this section, we give the $[r, s, c, t]$-chromatic number $\chi_{r,s,c,t}(G)$ if $\delta(G) = 1$ or G is an even cycle. Firstly, we give some basic properties of $[r, s, c, t]$-coloring of graphs.

Lemma 2.1. *Let G be a graph. Given nonnegative integers r, r', s, s', t and t'. If $r' \leq r, s' \leq s, t' \leq t$, then $\chi_{r',s',c,t'}(G) \leq \chi_{r,s,c,t}(G)$ holds for any fixed integer c with $1 \leq c \leq \chi'_c(G)$.*

Proof. An $[r, s, c, t]$-coloring of G is by definition also an $[r', s', c, t']$-coloring of G if $r' \leq r, s' \leq s, t' \leq t$. ∎

Lemma 2.2. *If $a \geq 0$ is an integer, then $\chi_{ar,as,c,at}(G) = a(\chi_{r,s,c,t}(G) - 1) + 1$ holds for any fixed integer c with $1 \leq c \leq \chi'_c(G)$.*

Proof. If $a = 0$ or 1, then the assertion is obvious. Let $a \geq 2$ and f be an $[r, s, c, t]$-coloring of a graph G with $\chi_{r,s,c,t}(G)$ colors. If we multiply all assigned labels by a, then we obtain a coloring f' such that $f'(x) = a \cdot f(x)$ for all elements $x \in V(G) \bigcup E(G)$ and $|f'(v_i) - f'(v_j)| = a|f(v_i) - f(v_j)| \geq ar$ if vertices v_i and v_j are adjacent, $|f'(e_i) - f'(e_j)| = a|f(e_i) - f(e_j)| \geq as$ if edges e_i and e_j belong to different edge covers, and $|f'(v_i) - f'(e_j)| = a|f(v_i) - f(e_j)| \geq at$ if v_i and e_j are incident, respectively. Furthermore, since f forms c edge covers of G, f' also forms c edge covers of G. Therefore, f' is an $[ar, as, c, at]$-coloring of G with colors in $\{0, 1, \ldots, a(\chi_{r,s,c,t}(G) - 1)\}$ which implies that $a(\chi_{r,s,c,t}(G) - 1) + 1$ is an upper bound of $\chi_{as,as,c,at}(G)$.

On the other hand, assume that G has an $[ar, as, c, at]$-coloring f with $a(\chi_{r,s,c,t}(G) - 1)$ colors ($a \geq 2$). Define a coloring f' by $f'(x) = \lfloor f(x)/a \rfloor$. If, for example, x_i and x_j are adjacent vertices, then $|\lfloor f(x_i)/a \rfloor - \lfloor f(x_j)/a) \rfloor| \geq r$ by assumption which implies that $|f'(x_i) - f'(x_j)| = |\lfloor f(x_i)/a \rfloor - \lfloor f(x_j)/a \rfloor| \geq r$. Similar proof can be used when considering s and t. Furthermore, since f forms c edge covers of G, f' also forms c edge covers of G. Therefore, f' is an $[r, s, c, t]$-coloring of G with $\chi_{r,s,c,t}(G) - 1$ colors which contradicts the definition of the $[r, s, c, t]$-chromatic number of G. ∎

Lemma 2.3. *Let G be a graph. Then*

$$\max\{r(\chi(G) - 1) + 1, s(c - 1) + 1, t + 1\} \leq \chi_{r,s,c,t}(G) \leq r(\chi(G) - 1) + s(c - 1) + t + 1.$$

Proof. By Lemma 2.1 and Lemma 2.2, $\chi_{r,s,c,t}(G) \geq \chi_{r,0,c,0}(G) = r(\chi(G) - 1) + 1$ as well as $\chi_{r,s,c,t}(G) \geq \chi_{0,s,c,0}(G) = s(c - 1) + 1$. Obviously, $\chi_{r,s,c,t}(G) \geq t + 1$.

If we color the vertices of G with colors $0, r, \ldots, r(\chi(G) - 1)$ and the edges with colors $r(\chi(G) - 1) + t, r(\chi(G) - 1) + t + s, \ldots, r(\chi(G) - 1) + t + s(c - 1)$. (Note that $1 \leq c \leq \chi'_c(G)$), we obtain an $[r, s, c, t]$-coloring of G. ∎

Next, we give the exact value of $\chi_{r,s,c,t}(G)$ where $\delta(G) = 1$ or G is an even cycle. Given a graph G with $\delta(G) = 1$, it is obvious that $\chi'_c(G) = \delta(G) = 1$. Since $1 \leq c \leq \chi'_c(G) = 1$, we can only let $c = 1$. Thus in order to get a $[r, s, c, t]$-coloring of G, we can only color all the edges of G with the same color. Then we have the following theorem.

Theorem 2.1. *Let $c = 1$ and G be a graph with $\delta(G) = 1$. We have*

$$\chi_{r,s,c,t}(G) = \begin{cases} r(\chi(G) - 1) + 1 & \text{if} \quad r \geq 2t, & \text{(a)} \\ r(\chi(G) - 2) + 2t + 1 & \text{if} \quad t \leq r < 2t, & \text{(b)} \\ r(\chi(G) - 1) + t + 1 & \text{if} \quad r < t. & \text{(c)} \end{cases}$$

Proof. (a) By Lemma 2.2, $\chi_{r,s,c,t}(G) \geq r(\chi(G) - 1) + 1$. On the other hand, color the vertices of G with colors $0, r, 2r, \ldots, r(\chi(G) - 1)$ to get a proper vertex coloring and all the edges of G with color t. It is easy to see that it forms an $[r, s, c, t]$-coloring of G. The color set used by the coloring is $\{0, \ldots, r(\chi(G) - 1)\}$. By definition, $\chi_{r,s,c,t}(G) = r(\chi(G) - 1) + 1$.

(b) Firstly, we prove the lower bound. Given any $[r, s, c, t]$-coloring of G, suppose that the vertices of G are colored with colors $f_1, f_2, \ldots, f_m$ where $m \geq \chi(G)$ and all the edges are colored with color f_0. If $f_i \leq f_0 \leq f_{i+1}$ holds for some $i \in \{1, 2, \ldots, m - 1\}$, we have $f_m - f_1 = f_m - f_{i+1} + f_{i+1} - f_i + f_i - f_1 \geq (m - 2)r + 2t \geq r(\chi(G) - 2) + 2t$. Otherwise, $f_0 \leq f_1$ or $f_0 \geq f_m$ holds. In this case, at least $(m - 1)r + t$ colors are needed which is greater than $r(\chi(G) - 2) + 2t$. Secondly, we color G as follows. Color all vertices in G with color $0, 2t, 2t + r, \ldots, 2t + r(\chi(G) - 2)$ and all edges of G with color t. It is easy to see that it forms an $[r, s, c, t]$-coloring of G with the color set $\{0, \ldots, r(\chi(G) - 2) + 2t\}$. By definition, $\chi_{r,s,c,t}(G) = r(\chi(G) - 2) + 2t$.

(c) The lower bound has already been proved in (b). On the other hand, we color G as follows. Color the vertices of G with color $0, r, \ldots, r(\chi(G) - 1)$ and all edges of G with color $r(\chi(G) - 1) + t$. It is easy to see that it forms an $[r, s, c, t]$-coloring of G with the color set $\{0, \ldots, r(\chi(G) - 1) + t\}$. By definition, $\chi_{r,s,c,t}(G) = r(\chi(G) - 1) + t + 1$. ∎

Given an even cycle C_{2n}, it is known that $\chi(C_{2n}) = 2, \chi'_c(C_{2n}) = 2$. So we must discuss two cases, $c = 1$ and $c = 2$. We have the following theorem.

Theorem 2.2. *Let c be an integer with $1 \le c \le 2$ and C_{2n} be a cycle with $2n$ vertices. If $c = 1$, then*

$$\chi_{r,s,c,t}(C_{2n}) = \begin{cases} r+1 & if \quad r \ge 2t, \\ 2t+1 & if \quad t \le r < 2t, \\ r+t+1 & if \quad r < t. \end{cases}$$

If $c = 2$, then

$$\chi_{r,s_c,t}(C_{2n}) = \begin{cases} r+1 & if \quad r \ge 2t+s, & \text{(a)} \\ 2t+s+1 & if \quad r \ge 2t, t+s \le r < 2t+s, & \text{(b)} \\ r+t+1 & if \quad r \ge 2t, s \le r < t+s, & \text{(c)} \\ s+t+1 & if \quad r \ge 2t, s-t \le r < s, & \text{(d)} \\ 2t+r+1 & if \quad r \ge 2t, s-2t \le r < s-t, & \text{(e)} \\ s+1 & if \quad r \le s-2t, & \text{(f)} \\ 2t+r+1 & if \quad t \le r < 2t, t+r \le s < 2t+r, & \text{(g)} \\ s+t+1 & if \quad t \le r < 2t, 2t \le s < t+r, & \text{(h)} \\ 3t+1 & if \quad t \le r < 2t, t \le s < 2t, & \text{(i)} \\ 2t+s+1 & if \quad t \le r < 2t, s < t, & \text{(j)} \\ 2t+r+1 & if \quad r < t, t \le s < 2t+r, & \text{(k)} \\ r+t+s+1 & if \quad r < t, s < t. & \text{(l)} \end{cases}$$

Proof. If $c = 1$, the proof is the same as Theorem 2.1.

If $c = 2$, the edges of C_{2n} are colored by two colors alternately. Suppose that $C_{2n} = x_1e_1y_1e'_1x_2e_2y_2e'_2 \dots x_ie_iy_ie'_ix_{i+1} \dots x_ne_ny_ne'_nx_1$ and $f_1 = f(e_i) \le f(e'_i) = f_2$, $i = 1, 2, \dots, n$. It is obvious that $f_2 - f_1 \ge s$. Then we give the proof from (a) to (l).

(a) By Lemma 2.3, $\chi_{r,s,c,t}(C_{2n}) \ge r(\chi(C_{2n}) - 1) + 1 = r + 1$. On the other hand, let $f(x_i) = 0$, $f(y_i) = r$, $f_1 = t$, $f_2 = t + s$ for $i = 1, \dots, n$. It is easy to see that f forms an $[r, s, c, t]$-coloring of C_{2n}. Thus $\chi_{r,s_c,t}(C_{2n}) \le r + 1$. Then $\chi_{r,s,c,t}(C_{2n}) = r + 1$.

(b) Firstly we prove that $\chi_{r,s,c,t}(C_{2n}) \ge 2t + s + 1$. Let us pay attention to x_i, y_i for some $i \in \{1, \dots, n\}$. We might as well suppose $f(x_i) \le f(y_i)$. If $f(x_i) \le f(y_i) \le f_1$, then $f_2 - f(x_i) = (f_2 - f_1) + (f_1 - f(y_i)) + (f(y_i) - f(x_i)) \ge r + t + s \ge 2t + s$. If $f(x_i) \le f_1 \le f(y_i) \le f_2$, then $f_2 - f(x_i) = (f_2 - f(y_i)) + (f(y_i) - f(x_i)) \ge t + r \ge 2t + s$. If $f(x_i) \le f_1 \le f_2 \le f(y_i)$, then we have $f(y_i) - f(x_i) = (f(y_i) - f_2) + (f_2 - f_1) + (f_1 - f(x_i)) \ge t + s + t = 2t + s$. If $f_1 \le f(x_i) \le f(y_i) \le f_2$, then we have $f_2 - f_1 = (f_2 - f(y_i)) + (f(y_i) - f(x_i)) + (f(x_i) - f_1) \ge t + r + t \ge 2t + s$. The cases that $f_1 \le f(x_i) \le f_2 \le f(y_i)$ or $f_1 \le f_2 \le f(x_i) \le f(y_i)$ can be proved similarly.

From all the above we can see that $\chi_{r,s,c,t}(C_{2n}) \ge 2t + s + 1$. On the other hand, for $i = 1, \dots, n$, let $f(x_i) = 0$, $f(y_i) = 2t + s$, $f_1 = t, f_2 = t + s$. It is obvious that f is an $[r, s, c, t]$-coloring of C_{2n}. So $\chi_{r,s,c,t}(C_{2n}) = 2t + s + 1$.

The same arguments can be used to prove the lower bounds from (c) to (l) except (f), so we only need to prove the upper bounds. We shall create an $[r,s,c,t]$-coloring of G using given number of colors.

(c) Let $f(x_i)=0,\ f(y_i)=r,\ f_1=t,\ f_2=r+t$. So $\chi_{r,s,c,t}(C_{2n})\le r+t+1$.
(d) Let $f(x_i)=0,\ f(y_i)=r,\ f_1=t,\ f_2=t+s$. So $\chi_{r,s,c,t}(C_{2n})\le s+t+1$.
(e) Let $f(x_i)=t,\ f(y_i)=t+r,\ f_1=0,\ f_2=2t+r$. So $\chi_{r,s,c,t}(C_{2n})\le 2t+r+1$.
(f) By Lemma 2.3, $\chi_{r,s_c,t}(C_{2n})\ge s(c-1)+1=s+1$. On the other hand, let $f(x_i)=t,\ f(y_i)=t+r,\ f_1=0,\ f_2=s$ for $i=1,\dots,n$. It is easy to see that f forms an $[r,s,c,t]$-coloring of C_{2n}. Thus $\chi_{r,s_c,t}(C_{2n})\le s+1$. Then $\chi_{r,s,c,t}(C_{2n})=s+1$.
(g) Let $f(x_i)=t,\ f(y_i)=t+r,\ f_1=0,\ f_2=2t+r$. So $\chi_{r,s,c,t}(C_{2n})\le 2t+r+1$.
(h) Let $f(x_i)=0,\ f(y_i)=2t,\ f_1=t,\ f_2=s+t$. So $\chi_{r,s,c,t}(C_{2n})\le s+t+1$.
(i) Let $f(x_i)=0,\ f(y_i)=2t,\ f_1=t,\ f_2=3t$. So $\chi_{r,s,c,t}(C_{2n})\le 3t+1$.
(j) Let $f(x_i)=0,\ f(y_i)=2t+s,\ f_1=t,\ f_2=t+s$. So $\chi_{r,s,c,t}(C_{2n})\le 2t+s+1$.
(k) Let $f(x_i)=t,\ f(y_i)=t+r,\ f_1=0,\ f_2=2t+r$. So $\chi_{r,s,c,t}(C_{2n})\le 2t+r+1$.
(l) Let $f(x_i)=0,\ f(y_i)=r,\ f_1=r+t,\ f_2=r+t+s$. So $\chi_{r,s,c,t}(C_{2n})\le r+t+s+1$. ∎

Note that $[r,s,c,t]$-coloring of C_{2n} is also an $[r,s,t]$-coloring of C_{2n} when $c=2$, so it gives an upper bound of $\chi_{r,s,t}(C_{2n})$.

Acknowledgement. The authors are indebted to the anonymous referees for their valuable comments and suggestions. This work was supported by NNSF (10871119, 61070230) and RSDP (20080422001) of China.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, American Elsevier Publishing Co., Inc., New York, 1976.
[2] L. Dekar, B. Effantin and H. Kheddouci, $[r,s,t]$-coloring of trees and bipartite graphs, *Discrete Math.* **310** (2010), no. 2, 260–269.
[3] R. P. Gupta, On decompositions of a multi-graph into spanning subgraphs, *Bull. Amer. Math. Soc.* **80** (1974), 500–502.
[4] A. Kemnitz and M. Marangio, $[r,s,t]$-colorings of graphs, *Discrete Math.* **307** (2007), no. 2, 199–207.
[5] H. M. Song and G. Z. Liu, f-edge covering coloring of graphs, *Acta Math. Sinica (Chin. Ser.)* **48** (2005), no. 5, 919–928.
[6] C. Xu and G. Liu, A note on the edge cover chromatic index of multigraphs, *Discrete Math.* **308** (2008), no. 24, 6564–6568.
[7] C. Xu and G. Liu, Edge covered critical multigraphs, *Discrete Math.* **308** (2008), no. 24, 6348–6354.