On Conharmonic Curvature Tensor in K-contact and Sasakian Manifolds

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Abstract. Some necessary and/or sufficient condition(s) for K-contact and/or Sasakian manifolds to be quasi conharmonically flat, ξ-conharmonically flat and φ-conharmonically flat are obtained. In last, it is proved that a compact φ-conharmonically flat K-contact manifold with regular contact vector field is a principal S^1-bundle over an almost Kaehler space of constant holomorphic sectional curvature $\left(3 - \frac{2}{2n-1}\right)$.

1. Introduction

Let M be an almost contact metric manifold equipped with an almost contact metric structure (φ, ξ, η, g). At each point $p \in M$, decompose the tangent space T_pM into the direct sum $T_pM = \varphi(T_pM) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the 1-dimensional linear subspace of T_pM generated by ξ_p. Thus the conformal curvature tensor $\mathcal{C}$ is a map

$$\mathcal{C} : T_pM \times T_pM \times T_pM \to \varphi(T_pM) \oplus \{\xi_p\}, \qquad p \in M.$$

An almost contact metric manifold M is said to be

(1) *conformally symmetric* [6] if the projection of the image of $\mathcal{C}$ in $\varphi(T_pM)$ is zero,
(2) *ξ-conformally flat* [13] if the projection of the image of $\mathcal{C}$ in $\{\xi_p\}$ is zero, and
(3) *φ-conformally flat* [4] if the projection of the image of $\mathcal{C}|_{\varphi(T_pM) \times \varphi(T_pM) \times \varphi(T_pM)}$ in $\varphi(T_pM)$ is zero.

In [6], it is proved that a conformally symmetric K-contact manifold is locally isometric to the unit sphere. In [13], it is proved that a K-contact manifold is ξ-conformally flat if and only if it is an η-Einstein Sasakian manifold. In [1], some results for φ-conformally flat, φ-conharmonically flat and φ-concircularly flat on (k, μ)-contact manifolds are given. In [10], Weyl conformal curvature tensor, conharmonic curvature tensor and projective curvature tensor are discussed on Lorentzian para-Sasakian manifolds. In [4], some necessary conditions for a K-contact manifold to be φ-conformally flat are proved. In [5], a necessary and sufficient condition for a Sasakian manifold to be φ-conformally flat is obtained. In [12], projective curvature tensor in K-contact and Sasakian manifolds is studied. Moreover, the author [11] considered some conditions on conharmonic curvature tensor K, which has many applications in physics and mathematics, on a hypersurface in the semi-Euclidean space E_s^{n+1}. He proved that every conharmonicaly Ricci-symmetric hypersurface M satisfying the condition $K \cdot R = 0$ is pseudosymmetric. He also considered the condition $K \cdot K = L_K Q(g, K)$ on hypersurfaces of the semi-Euclidean space E_s^{n+1}.

On the other hand in a Riemannian manifold M of dimension $m \geq 3$, the *conharmonic curvature tensor* $\mathcal{K}$ is defined by [7]

$$\mathcal{K}(X, Y) Z = R(X, Y) Z - \frac{1}{m-2} \{ S(Y, Z) X - S(X, Z) Y \\ + g(Y, Z) QX - g(X, Z) QY \}$$

for $X, Y, Z \in TM$, where R is the curvature tensor and Q is the Ricci operator.

Motivated by the studies of conformal curvature tensor in [6, 13, 4, 5], and the studies of projective curvature tensor in K-contact and Sasakian manifolds [12] and and Lorentzian para-Sasakian manifolds in [10], in this paper we study conharmonic curvature tensor in K-contact and Sasakian manifolds. The paper is organized as follows. Section 2 contains some preliminaries. In Section 3, in an almost contact metric manifold we consider three cases of conharmonic curvature tensor, analogous to conformally symmetric, ξ-conformally flat and φ-conformally flat conformal curvature tensor, and give definitions of quasi conharmonically flat, ξ-conharmonically flat and φ-conharmonically flat almost contact metric manifolds. It is proved that if a K-contact manifold is quasi conharmonically flat then the scalar curvature vanishes. We also prove that a Sasakian manifold is ξ-conharmonically flat if and only if it is η-Einstein. Necessary and sufficient conditions for a K-contact manifold and Sasakian manifold to be φ-conharmonically flat are obtained. In the last section, it is established that a φ-conharmonically flat compact regular K-contact manifold is a principal S^1-bundle over an almost Kaehler space of constant holomorphic sectional curvature $\left(3 - \frac{2}{2n-1} \right)$.

2. Preliminaries

Let M be an *almost contact metric manifold* equipped with an almost contact metric structure (φ, ξ, η, g) consisting of a $(1, 1)$ tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g. Then

(2.1) $\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0,$

(2.2) $g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y), \qquad X, Y \in TM.$

From (2.1) and (2.2) we easily get

(2.3) $g\left(X, \varphi Y\right) = -g\left(\varphi X, Y\right), \qquad g\left(X, \xi\right) = \eta(X), \qquad X, Y \in TM.$

An almost contact metric manifold is

(1) a *contact metric manifold* if $g\left(X, \varphi Y\right) = d\eta\left(X, Y\right)$ for all $X, Y \in TM$;
(2) a *K-contact manifold* if $\nabla \xi = -\varphi$, where ∇ is Levi-Civita connection; and
(3) a *Sasakian manifold* if $(\nabla_X \varphi)Y = g\left(X, Y\right)\xi - \eta(Y)X$ for all $X, Y \in TM$.

A K-contact manifold is a contact metric manifold, while converse is true if the Lie derivative of φ in the characteristic direction ξ vanishes. A Sasakian manifold is always a K-contact manifold. A 3-dimensional K-contact manifold is a Sasakian manifold. A contact metric manifold is Sasakian if and only if

(2.4) $R\left(X, Y\right)\xi = \eta\left(Y\right)X - \eta\left(X\right)Y, \qquad X, Y \in TM.$

In a Sasakian manifold M equipped with a Sasakian structure (φ, ξ, η, g), the following relations are well known.

(2.5) $R\left(\xi, X\right)Y = g\left(X, Y\right)\xi - \eta\left(Y\right)X, \qquad X, Y \in TM,$

(2.6) $S\left(X, \xi\right) = 2n\eta\left(X\right), \qquad X \in TM,$

where $\dim(M) = 2n + 1$. For more details we refer to [2].

The following equations of this section are taken from [12]. In a $(2n + 1)$-dimensional almost contact metric manifold M, if $\{e_1, \ldots, e_{2n}, \xi\}$ is a local orthonormal basis of vector fields in M, then $\{\varphi e_1, \ldots, \varphi e_{2n}, \xi\}$ is also a local orthonormal basis and

(2.7) $$\sum_{i=1}^{2n} g\left(e_i, e_i\right) = \sum_{i=1}^{2n} g\left(\varphi e_i, \varphi e_i\right) = 2n,$$

(2.8) $$\sum_{i=1}^{2n} g\left(e_i, Z\right)S(Y, e_i) = \sum_{i=1}^{2n} g\left(\varphi e_i, Z\right)S(Y, \varphi e_i) = S(Y, Z) - S(Y, \xi)\eta(Z)$$

for all $Y, Z \in TM$. In particular, in view of $\eta \circ \varphi = 0$ we get

(2.9) $$\sum_{i=1}^{2n} g\left(e_i, \varphi Z\right)S(Y, e_i) = \sum_{i=1}^{2n} g\left(\varphi e_i, \varphi Z\right)S(Y, \varphi e_i) = S(Y, \varphi Z)$$

for all $Y, Z \in TM$. If M is a K-contact manifold then it is known that

(2.10) $R\left(X, \xi\right)\xi = X - \eta\left(X\right)\xi, \qquad X \in TM.$

and

(2.11) $S\left(\xi, \xi\right) = 2n.$

Moreover, M is Einstein if and only if

(2.12) $S = 2ng.$

From (2.11) we get

(2.13) $$\sum_{i=1}^{2n} S(e_i, e_i) = \sum_{i=1}^{2n} S(\varphi e_i, \varphi e_i) = r - 2n.$$

In a K-contact manifold we also get

(2.14) $$R\left(\xi, Y, Z, \xi\right) = g\left(\varphi Y, \varphi Z\right), \qquad Y, Z \in TM.$$

Consequently,

(2.15) $$\sum_{i=1}^{2n} R(e_i, Y, Z, e_i) = \sum_{i=1}^{2n} R(\varphi e_i, Y, Z, \varphi e_i) = S(Y, Z) - g(\varphi Y, \varphi Z).$$

for all $Y, Z \in TM$.

3. Some structure theorems

In a $(2n+1)$-dimensional almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ the conharmonic curvature tensor $\mathcal{K}$ is given by

(3.1) $$\mathcal{K}\left(X, Y\right) Z = R\left(X, Y\right) Z - \frac{1}{2n-1}\left\{S\left(Y, Z\right) X - S\left(X, Z\right) Y \right.$$
$$\left. + g\left(Y, Z\right) QX - g\left(X, Z\right) QY \right\},$$

where $X, Y, Z \in TM$.

Analogous to the considerations of conformal curvature tensor, we give the following.

Definition 3.1. *An almost contact metric manifold M is said to be* quasi conharmonically flat *if*

(3.2) $$g\left(\mathcal{K}(X, Y)Z, \varphi W\right) = 0, \qquad X, Y, Z, W \in TM,$$

ξ-conharmonic flat *if*

(3.3) $$\mathcal{K}(X, Y)\xi = 0, \qquad X, Y \in TM,$$

and φ-conharmonically flat *if*

(3.4) $$g\left(\mathcal{K}(\varphi X, \varphi Y)\varphi Z, \varphi W\right) = 0, \qquad X, Y, Z, W \in TM.$$

We begin with the following:

Theorem 3.1. *If a $(2n+1)$-dimensional K-contact manifold is quasi conharmonically flat then*

(3.5) $$r = 0,$$

(3.6) $$S(Y, Z) = -\, g(Y, Z) - (2n-1)\eta(Y)\eta(Z) + \eta(Y)S(Z, \xi) + \eta(Z)S(Y, \xi)$$

for all $Y, Z \in TM$.

Proof. From (3.1) we get

$$g\left(\mathcal{K}\left(X, Y\right) Z, \varphi W\right) = g\left(R\left(X, Y\right) Z, \varphi W\right)$$

(3.7) $$-\frac{1}{2n-1}\left\{S\left(Y, Z\right) g\left(X, \varphi W\right) -\ S\left(X, Z\right) g\left(Y, \varphi W\right)\right.$$
$$\left. + g(Y, Z)S(X, \varphi W) - g\left(X, Z\right) S(Y, \varphi W)\right\}$$

for $X, Y, Z, W \in TM$. For a local orthonormal basis $\{e_1, \ldots, e_{2n}, \xi\}$ of vector fields in M, putting $X = \varphi e_i$ and $W = e_i$ in (3.7) we get

$$\sum_{i=1}^{2n} g\left(\mathcal{K}\left(\varphi e_i, Y\right) Z, \varphi e_i\right) = \sum_{i=1}^{2n} R\left(\varphi e_i, Y, Z, \varphi e_i\right)$$

$$- \frac{1}{2n-1} \sum_{i=1}^{2n} \{S(Y, Z)g(\varphi e_i, \varphi e_i) - S(\varphi e_i, Z)g(Y, \varphi e_i)$$

$$+ g(Y, Z)S(\varphi e_i, \varphi e_i) - g(\varphi e_i, Z)S(Y, \varphi e_i)\}$$

for $Y, Z \in TM$. Using (2.15), (2.7), (2.8) and (2.13) in the above equation we get

$$\sum_{i=1}^{2n} g\left(\mathcal{K}\left(\varphi e_i, Y\right) Z, \varphi e_i\right) = S\left(Y, Z\right) - g(\varphi Y, \varphi Z)$$

(3.8)
$$- \frac{1}{2n-1} \{(2n-2)S(Y, Z) + (r - 2n)g(Y, Z)$$
$$+ S(Z, \xi)\eta\left(Y\right) + S(Y, \xi)\eta\left(Z\right)\}$$

for $Y, Z \in TM$. In particular, if M is quasi conharmonically flat then (??) reduces to

(3.9) $\quad S\left(Y, Z\right) = (r - 1)g(Y, Z) - (2n - 1)\eta\left(Y\right)\eta\left(Z\right) + \eta\left(Y\right)S(Z, \xi) + \eta\left(Z\right)S(Y, \xi)$

for $Y, Z \in TM$. Putting $Z = \xi$ in (3.9) and using (2.11) and $\eta(\xi) = 1$ we get (3.5) and consequently (3.6). ∎

Corollary 3.1. *If a $(2n+1)$-dimensional K-contact manifold is quasi conharmonically flat then*

(3.10) $$S(\varphi X, \varphi Y) = - g(\varphi X, \varphi Y),$$

for all $X, Y \in TM$.

Remark 3.1. In [12, Theorem 3.3], it is proved that a quasi projectively flat K-contact manifold is Einstein. But from equations (3.6) and (3.10), it seems that the same result is not true for a quasi conharmonically flat K-contact manifold.

Next, we prove the following:

Lemma 3.1. *A $(2n+1)$-dimensional quasi conharmonically flat Sasakian manifold M is η-Einstein.*

Proof. Let M be a $(2n+1)$-dimensional Sasakian manifold. Using (2.6) in (3.6) we get

(3.11) $$S = - g + (2n+1)\eta \otimes \eta. \qquad \blacksquare$$

Theorem 3.2. *A Sasakian manifold M is quasi conharmonically flat if and only if*

$$R(X, Y)Z = - \frac{2}{2n-1} \{g\left(Y, Z\right) X - g\left(X, Z\right) Y\}$$

(3.12)
$$+ \frac{2n+1}{2n-1} \{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}$$

$$+ \frac{2n+1}{2n-1} \left\{ g\left(Y,Z\right)\eta(X)\xi - g\left(X,Z\right)\eta(Y)\xi \right\}$$

for all $X, Y, Z \in TM$.

Proof. Let M is quasi conharmonically flat using (3.2), (3.11) and replacing W by φW in (3.7), we get

$$\begin{aligned} g\left(R\left(X,Y\right)Z, \varphi^2 W\right) = \ &\frac{1}{2n-1} \left\{ 2g\left(Y,Z\right)g\left(\varphi X, \varphi W\right) - 2g\left(X,Z\right)g\left(\varphi Y, \varphi W\right) \right. \\ &- (2n+1)\eta(Y)\eta(Z)g\left(\varphi X, \varphi W\right) \\ &\left. + (2n+1)\eta(X)\eta(Z)g\left(\varphi Y, \varphi W\right) \right\}, \end{aligned}$$

where $X, Y, Z, W \in TM$, now using (2.1), (2.2) and (2.4) in above equation we get (3.12). The converse is straightforward. ∎

In [12, Theorem 3.5], it is proved that a K-contact manifold is ξ-projectively flat if and only if it is Einstein Sasakian. Unlike to this result, here we have the following:

Theorem 3.3. *If a K-contact manifold is ξ-conharmonically flat then*

$$\begin{aligned} R(X,Y)\xi = \ &\frac{1}{2n-1} \left\{ S(Y,\xi)R(X,\xi)\xi - S(X,\xi)R(Y,\xi)\xi \right. \\ &\left. + \eta\left(X\right)Y - \eta\left(Y\right)X \right\} \end{aligned}$$
(3.13)

for all $X, Y \in TM$.

Proof. Putting $Z = \xi$ in (3.1) and $g\left(X,\xi\right) = \eta\left(X\right)$ we get

$$\begin{aligned} g\left(\mathcal{K}\left(X,Y\right)\xi, W\right) = \ &g\left(R\left(X,Y\right)\xi, W\right) \\ &- \frac{1}{2n-1} \left\{ S\left(Y,\xi\right)g\left(X,W\right) - S\left(X,\xi\right)g\left(Y,W\right) \right. \\ &\left. + \eta\left(Y\right)S\left(X,W\right) - \eta\left(X\right)S\left(Y,W\right) \right\} \end{aligned}$$
(3.14)

for all $X, Y, W \in TM$. For a local orthonormal basis $\{e_1, \ldots, e_{2n}, \xi\}$ of vector fields in M, from (3.14) we get

$$\begin{aligned} \sum_{i=1}^{2n} g\left(\mathcal{K}\left(e_i, Y\right)\xi, e_i\right) = \ &\sum_{i=1}^{2n} g\left(R\left(e_i, Y\right)\xi, e_i\right) - \frac{1}{2n-1} \sum_{i=1}^{2n} \left\{ S\left(Y,\xi\right)g\left(e_i, e_i\right) \right. \\ &\left. - S\left(e_i, \xi\right)g\left(Y, e_i\right) + \eta\left(Y\right)S\left(e_i, e_i\right) \right\} \end{aligned}$$

for all $Y \in TM$. If M is ξ-conharmonically flat using (3.3), (2.7), (2.8), (2.11), (2.15) and (2.13) in above equation we get (3.5). Now putting $Y = \xi$ in (3.14) and using (3.3), (2.1), (2.3), (2.10) and (2.11) we get

$$\begin{aligned} S\left(X,W\right) = \ &- g\left(X,W\right) + S\left(X,\xi\right)\eta\left(W\right) + \eta\left(X\right)S\left(\xi, W\right) \\ &- (2n-1)\eta\left(X\right)\eta\left(W\right) \end{aligned}$$
(3.15)

using (3.15) in (3.14) we get (3.13). ∎

Now, we have the following

Theorem 3.4. *A $(2n+1)$-dimensional Sasakian manifold M is ξ-conharmonically flat if and only if it is η-Einstein.*

Proof. For a $(2n + 1)$-dimensional ξ-conharmonically flat Sasakian manifold M, in view of (2.6) and (3.15) we get

$$S = - \, g + (2n + 1)\, \eta \otimes \eta,$$

that is M is η-Einstein. The converse is easy to follow. ∎

Remark 3.2. In [8], it is shown that a conharmonically flat (that is, $\mathcal{K} = 0$) Einstein Sasakian manifold of dimension $(2n + 1)$ is locally isometric to the unit sphere $S^{2n+1}(1)$. However, in view of Theorem 3.4, it follows that a conharmonically flat Sasakian manifold can not be Einstein.

4. φ-conharmonic flatness

Theorem 4.1. *A $(2n+1)$-dimensional K-contact manifold M is φ-conharmonically flat if and only if*

$$g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) = -\frac{2}{2n-1}\{g(\varphi Y, \varphi Z)g(\varphi X, \varphi W)$$

(4.1)
$$- g(\varphi X, \varphi Z)g(\varphi Y, \varphi W)\}$$

for all $X, Y, Z, W \in TM$.

Proof. Let M be a K-contact manifold of dimension $(2n + 1)$. From (3.1) we get

$$g\left(\mathcal{K}\left(\varphi X, \varphi Y\right)\varphi Z, \varphi W\right) = \; g\left(R\left(\varphi X, \varphi Y\right)\varphi Z, \varphi W\right)$$

$$- \frac{1}{2n-1}\{S\left(\varphi Y, \varphi Z\right)g\left(\varphi X, \varphi W\right) - S\left(\varphi X, \varphi Z\right)g\left(\varphi Y, \varphi W\right)$$

(4.2)
$$+ S\left(\varphi X, \varphi W\right)g\left(\varphi Y, \varphi Z\right) - S\left(\varphi Y, \varphi W\right)g\left(\varphi X, \varphi Z\right)\}$$

for all $X, Y, Z, W \in TM$. Let $\{e_1, \ldots, e_{2n}, \xi\}$ be an orthonormal basis then $\{\varphi e_1, \ldots, \varphi e_{2n}, \xi\}$ is also an orthononmal basis. Putting $X = W = e_i$ and taking summation over i in (4.2) we get

$$\sum_{i=1}^{2n} g(\mathcal{K}(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) = \sum_{i=1}^{2n} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i)$$

$$- \frac{1}{2n-1}\{S(\varphi Y, \varphi Z)g(\varphi e_i, \varphi e_i) - S(\varphi e_i, \varphi Z)g(\varphi Y, \varphi e_i)$$

$$+ S(\varphi e_i, \varphi e_i)g(\varphi Y, \varphi Z) - S(\varphi Y, \varphi e_i)g(\varphi e_i, \varphi Z)\}$$

for all $Y, Z \in TM$. Suppose M is φ-conharmonically flat. Then using (3.4), (2.15), (2.7), (2.9) and (2.13) in the previous equation we get

$$S\left(\varphi Y, \varphi Z\right) = (r - 1)\, g\left(\varphi Y, \varphi Z\right), \qquad Y, Z \in TM.$$

Putting $Y = Z = e_i$ and taking summation over i and using (2.13) and (2.7) we get (3.5) therefore from above equation we get (3.10). Now using (3.10) and (3.4) in (4.2) we get (4.1). The converse is straightforward. ∎

Theorem 4.2. *Let M be a $(2n + 1)$-dimensional Sasakian manifold. Then the following statements are equivalent:*

   (1) *M is conharmonically flat (that is, $\mathcal{K} = 0$).*
   (2) *M is φ-conharmonically flat.*

(3) *The curvature tensor of M is given by*

$$R(X,Y)Z = -\frac{2}{2n-1}\{g\,(Y,Z)\,X - g\,(X,Z)\,Y\}$$
$$-\frac{2n+1}{2n-1}\{g\,(X,Z)\,\eta(Y)\xi - g\,(Y,Z)\,\eta(X)\xi$$
(4.3)
$$-\,\eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y\}$$

       for all $X, Y, Z \in TM$.

Proof. The statement (2) follows from the statement (1) obviously. In a Sasakian manifold, in view of (2.5) and (2.4) we can verify

$$R\left(\varphi^2 X, \varphi^2 Y, \varphi^2 Z, \varphi^2 W\right) = R\,(X,Y,Z,W) - g\,(Y,Z)\,\eta\,(X)\,\eta\,(W)$$
$$+ g\,(X,Z)\,\eta\,(Y)\,\eta\,(W) + g\,(Y,W)\,\eta\,(X)\,\eta\,(Z)$$
(4.4)
$$-\,g\,(X,W)\,\eta\,(Y)\,\eta\,(Z)$$

for all $X, Y, Z, W \in TM$. Replacing X, Y, Z, W by $\varphi X, \varphi Y, \varphi Z, \varphi W$ respectively in (4.1) and using (2.3), (2.1) and (4.4) we get (4.3). Hence, the statement (2) implies the statement (3). Now, we assume the the statement (3). From (4.3) it follows that

(4.5)
$$S = -\,g + (2n+1)\,\eta \otimes \eta.$$

Using (4.5) and (4.3) in (3.1) we get the statement (1).   ∎

5. Compact regular K-contact manifolds

A $(2n+1)$-dimensional K-contact manifold M is said to be regular if for each point $p \in M$ there is a cubical coordinate neighborhood U of p such that the integral curves of ξ in U pass through U only once. Moreover, if M is compact also, the orbits of ξ are closed curves. Let the space of orbits of ξ be denoted by B. Then we have the natural projection $\pi : M \to B$ and B is a $2n$-dimensional differentiable manifold such that π is a differentiable map. In [3], Boothby and Wang proved that if M is a $(2n+1)$-dimensional compact regular contact manifold, then M is a principal S^1-bundle over B, where S^1 is a 1-dimensional compact Lie group which is isomorphic to the 1-parameter group of global transformations generated by ξ.

    Now, we prove the following:

Theorem 5.1. *A φ-conharmonically flat compact regular K-contact manifold is a principal S^1-bundle over an almost Kaehler space of constant holomorphic sectional curvature $\left(3 - \frac{2}{2n-1}\right)$.*

Proof. Let M be a compact regular K-contact manifold. Since in a K-contact manifold ξ is a Killing vector field, the metric g is invariant under the action of the group S^1. Hence a metric $\tilde{g}$ and a $(1,1)$ tensor field J on B can be defined by

(5.1)
$$\tilde{g}\,(X,Y) = g\,(X^*, Y^*),$$

(5.2)
$$JX = \pi_* \varphi X^*$$

for any vector fields $X, Y \in TB$, where $*$ denotes the horizontal lift with respect to η. It is well known that $(J, \tilde{g})$ is an almost Kaehler structure on B [9]. Let $\tilde{R}$ denote the Riemann curvature tensor on B. Then we have [4]

$$\tilde{R}(X, Y, Z, W) = R(X^*, Y^*, Z^*, W^*) + 2g(X^*, \varphi Y^*) g(\varphi Z^*, W^*)$$
$$- g(Z^*, \varphi X^*) g(\varphi Y^*, W^*) + g(Z^*, \varphi Y^*) g(\varphi X^*, W^*)$$

for all $X, Y, Z, W \in TB$. So from (5.2), we obtain [4]

$$\tilde{R}(JX, JY, JZ, JW) = R(\varphi X^*, \varphi Y^*, \varphi Z^*, \varphi W^*) + 2g(X^*, \varphi Y^*) g(\varphi Z^*, W^*)$$
$$(5.3) \qquad\qquad - g(Z^*, \varphi X^*) g(\varphi Y^*, W^*) + g(Z^*, \varphi Y^*) g(\varphi X^*, W^*).$$

Moreover, if M is φ-conharmonically flat then from Theorem 4.1 and the identity (5.3) we have

$$\tilde{R}(JX, JY, JZ, JW) = -\frac{2}{(2n-1)} \{g(\varphi Y^*, \varphi Z^*) g(\varphi X^*, \varphi W^*)$$
$$- g(\varphi X^*, \varphi Z^*) g(\varphi Y^*, \varphi W^*)\}$$
$$+ 2g(X^*, \varphi Y^*) g(\varphi Z^*, W^*)$$
$$- g(Z^*, \varphi X^*) g(\varphi Y^*, W^*)$$
$$+ g(Z^*, \varphi Y^*) g(\varphi X^*, W^*).$$

In the above equation, replacing X and W by JX and JW respectively, we get

$$\tilde{R}(X, JY, JZ, W) = -\frac{2}{(2n-1)} \{g(Y^*, Z^*) g(X^*, W^*) - g(\varphi X^*, Z^*) g(Y^*, \varphi W^*)\}$$
$$+ 2g(X^*, Y^*) g(Z^*, W^*) + g(X^*, Z^*) g(Y^*, W^*)$$
$$+ g(\varphi Y^*, Z^*) g(\varphi X^*, W^*),$$

which for a unit vector field $X \in TB$ gives

$$\tilde{R}(X, JX, JX, X) = \left(3 - \frac{2}{2n-1}\right).$$

Thus the base manifold B is of constant holomorphic sectional curvature

$$\left(3 - \frac{2}{2n-1}\right). \qquad\blacksquare$$

Remark 5.1. In [12, Theorem 4.1], it is proved that a φ-projectively flat compact regular K-contact manifold is a principal S^1-bundle over an almost Kaehler space of constant holomorphic sectional curvature 4. Comparing this fact with Theorem 5.1, we observe that for a compact regular K-contact manifold the conditions of being φ-projectively flat and φ-conharmonically flat are quite different.

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