**BULLETIN** *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

*Bull. Malays. Math. Sci. Soc. (2)* **34**(1) (2011), 181–188

# Bipolar Fuzzy Finite State Machines

[1]YOUNG BAE JUN AND [2]JACOB KAVIKUMAR
[1]Department of Mathematics Education (and RINS), Gyeongsang National University, Chinju 660–701, Republic of Korea
[2]Faculty of Science, Arts and Heritage, Universiti Tun Hussein Onn Malaysia, 86400 Batu Pahat, Johor, Malaysia
[1]skywine@gmail.com, [2]kaviphd@gmail.com

**Abstract.** As a generalization of a fuzzy finite state machine, the notion of a bipolar fuzzy finite state machine is introduced. Also, the concepts of a bipolar (immediate) successor, a bipolar exchange property and a bipolar subsystem are introduced. Some related properties are discussed. A condition for a bipolar fuzzy finite state machine to satisfy the bipolar exchange property is established. A characterization of a bipolar subsystem is initiated.

2010 Mathematics Subject Classification: 68Q70, 68Q45, 18B20, 03E72

Keywords and phrases: Bipolar fuzzy finite state machine, bipolar (immediate) successor, bipolar exchange property, bipolar subsystem.

## 1. Introduction

Fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets etc. Bipolar-valued fuzzy sets, which are introduced by Lee [5, 6], are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. Using the notion of bipolar-valued fuzzy sets, Jun *et al.* [2] dealt with subalgebras and closed ideals of BCH-algebras based on bipolar-valued fuzzy sets. Lee [4] discussed bipolar fuzzy subalgebras and bipolar fuzzy ideals in BCK/BCI-algebras. Malik *et al.* [7] introduced the notions of submachine of a fuzzy finite state machine, retrievable, separated and connected fuzzy finite state machines and discussed their basic properties. They also initiated a decomposition theorem for fuzzy finite state machines in terms of primary submachines. Kumbhojkar and Chaudhari [3] provided several ways of constructing products of fuzzy finite state machines and their mutual relationship, through isomorphism and coverings. In this paper, using the notion of bipolar-valued fuzzy sets, we introduce the concepts of bipolar fuzzy finite state machines (bffsm) as a generalization of fuzzy finite state

*Communicated by* Lee See Keong.
*Received:* July 8, 2009; *Revised:* October 19, 2009.

machines, bipolar (immediate) successors, bipolar subsystems, and study related properties. We establish a condition for a bffsm to satisfy the bipolar exchange property. We initiate a characterization of a bipolar subsystem.

## 2. Preliminaries

In the traditional fuzzy sets, the membership degrees of elements range over the interval $[0, 1]$. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set, and the membership degree 0 indicates that an element does not belong to the fuzzy set. The membership degrees on the interval $(0, 1)$ indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set (see [1, 8]) In the viewpoint of satisfaction degree, the membership degree 0 is assigned to elements which do not satisfy some property. The elements with membership degree 0 are usually regarded as having the same characteristics in the fuzzy set representation. By the way, among such elements, some have irrelevant characteristics to the property corresponding to a fuzzy set and the others have contrary characteristics to the property. The traditional fuzzy set representation cannot tell apart contrary elements from irrelevant elements. Consider a fuzzy set "*young*" defined on the *age* domain $[0, 100]$ (see Figure 1). Now consider two ages 50 and 95 with membership degree 0. Although both of them do not satisfy the property "*young*", we may say that age 95 is more apart from the property rather than age 50 (see [5]).

Only with the membership degrees ranged on the interval [0, 1], it is difficult to express the difference of the irrelevant elements from the contrary elements in fuzzy sets. If a set representation could express this kind of difference, it would be more informative than the traditional fuzzy set representation. Based on these observations, Lee [5] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets.

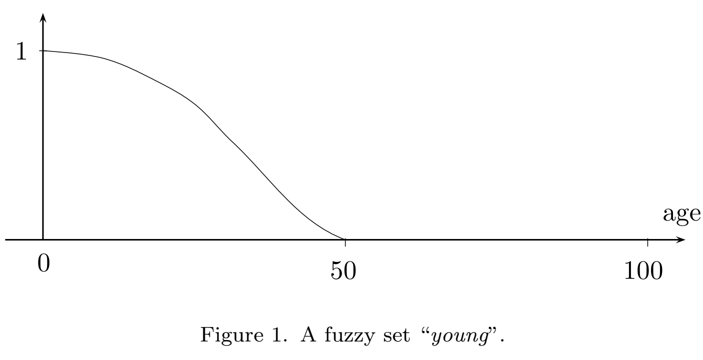


Figure 1. A fuzzy set "*young*".

Bipolar-valued fuzzy sets have membership degrees that represent the degree of satisfaction to the property corresponding to a fuzzy set and its counter-property. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degrees on $(0,1]$ indicate that elements somewhat satisfy the property, and the membership degrees on $[-1,0)$ indicate that elements somewhat satisfy the implicit counter-property (see [5]). Figure 2 shows a bipolar-valued fuzzy set redefined for the fuzzy set "*young*" of Figure 1. The negative membership degrees indicate the satisfaction extent of elements to an implicit counter-property (e.g., old against the property young). This kind of bipolar-valued fuzzy set representation enables the elements with membership degree 0 in traditional fuzzy sets, to be expressed into the elements with membership degree 0 (irrelevant elements) and the elements with negative membership degrees (contrary elements). The age elements 50 and 95, with membership degree 0 in the fuzzy sets of Figure 1, have 0 and a negative membership degree in the bipolar-valued fuzzy set of Figure 2, respectively. Now it is manifested that 50 is an irrelevant age to the property young and 95 is more apart from the property young than 50, i.e., 95 is a contrary age to the property young (see [5]). Let $X$ be the universe of discourse.

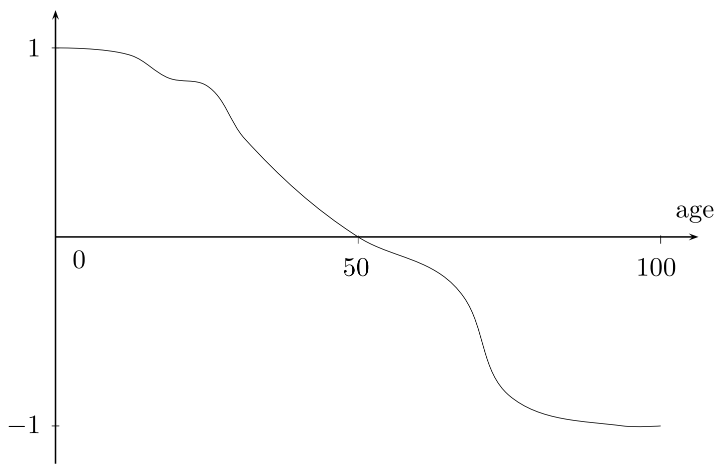


Figure 2. A bipolar fuzzy set "*young*".

A *bipolar-valued fuzzy set* $\varphi$ in $X$ is an object having the form

$$\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$$

where $\varphi^- : X \to [-1,0]$ and $\varphi^+ : X \to [0,1]$ are mappings. The positive membership degree $\varphi^+(x)$ denotes the satisfaction degree of an element $x$ to the property corresponding to a bipolar-valued fuzzy set $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$, and the negative membership degree $\varphi^-(x)$ denotes the satisfaction degree of $x$ to some implicit counter-property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$. If $\varphi^+(x) \neq 0$ and $\varphi^-(x) = 0$, it is the situation that $x$ is regarded as having only positive satisfaction for $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$. If $\varphi^+(x) = 0$ and $\varphi^-(x) \neq 0$, it is the situation that $x$ does not satisfy the property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$ but somewhat satisfies the counter-property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$. It is possible for an element $x$ to be $\varphi^+(x) \neq 0$ and $\varphi^-(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain (see [6]). For the sake of simplicity, we shall use the symbol $\varphi = \langle \varphi^-, \varphi^+ \rangle$ for the bipolar-valued fuzzy set $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

## 3. Bipolar fuzzy finite state machines

**Definition 3.1.** *A bipolar fuzzy finite state machine (bffsm, for short) is a triple $\mathcal{M} = (Q, X, \varphi)$, where $Q$ and $X$ are finite nonempty sets, called the set of states and the set of input symbols, respectively, and $\varphi = \langle \varphi^-, \varphi^+ \rangle$ is a bipolar fuzzy set in $Q \times X \times Q$.*

Let $X^*$ denote the set of all words of elements of $X$ of finite length. Let $\lambda$ denote the empty word in $X^*$ and $|x|$ denote the length of $x$ for every $x \in X^*$.

**Definition 3.2.** *Let $\mathcal{M} = (Q, X, \varphi)$ be a bffsm. Define a bipolar fuzzy set $\varphi_* = \langle \varphi_*^-, \varphi_*^+ \rangle$ in $Q \times X^* \times Q$ by*

$$\varphi_*^-(q, \lambda, p) := \begin{cases} -1 & \text{if } q = p, \\ 0 & \text{if } q \neq p, \end{cases} \qquad \varphi_*^+(q, \lambda, p) := \begin{cases} 1 & \text{if } q = p, \\ 0 & \text{if } q \neq p, \end{cases}$$

$$\varphi_*^-(q, xa, p) = \inf_{r \in Q} [\varphi_*^-(q, x, r) \vee \varphi^-(r, a, p)]$$

$$\varphi_*^+(q, xa, p) = \sup_{r \in Q} [\varphi_*^+(q, x, r) \wedge \varphi^+(r, a, p)]$$

*for all $p, q \in Q$, $x \in X^*$ and $a \in X$.*

**Lemma 3.1.** *Let $\mathcal{M} = (Q, X, \varphi)$ be a bffsm. Then*

$$\varphi_*^-(q, xy, p) = \inf_{r \in Q} [\varphi_*^-(q, x, r) \vee \varphi_*^-(r, y, p)]$$

*and*

$$\varphi_*^+(q, xy, p) = \sup_{r \in Q} [\varphi_*^+(q, x, r) \wedge \varphi_*^+(r, y, p)]$$

*for all $p, q \in Q$ and $x, y \in X^*$.*

*Proof.* Let $p, q \in Q$ and $x, y \in X^*$. We prove the result by induction on $|y| = n$. If $n = 0$, then $y = \lambda$ and so $xy = x\lambda = x$. Hence

$$\begin{aligned} \inf_{r \in Q} [\varphi_*^-(q, x, r) \vee \varphi_*^-(r, y, p)] &= \inf_{r \in Q} [\varphi_*^-(q, x, r) \vee \varphi_*^-(r, \lambda, p)] \\ &= \varphi_*^-(q, x, p) = \varphi_*^-(q, xy, p) \end{aligned}$$

and

$$\begin{aligned} \sup_{r \in Q} [\varphi_*^+(q, x, r) \wedge \varphi_*^+(r, y, p)] &= \sup_{r \in Q} [\varphi_*^+(q, x, r) \wedge \varphi_*^+(r, \lambda, p)] \\ &= \varphi_*^+(q, x, p) = \varphi_*^+(q, xy, p), \end{aligned}$$

and thus the result holds for $n = 0$. Suppose that the result is true for all $u \in X^*$ such that $|u| = n - 1$, $n > 0$. Let $y = ua$ where $u \in X^*$ and $a \in X$, and $|u| = n - 1$. Then

$$\varphi_*^-(q, xy, p) = \varphi_*^-(q, xua, p) = \inf_{r \in Q} [\varphi_*^-(q, xu, r) \vee \varphi^-(r, a, p)]$$

$$
\begin{aligned}
&= \inf_{r\in Q}\big[\inf_{s\in Q}[\varphi_*^-(q,x,s)\vee\varphi_*^-(s,u,r)]\vee\varphi^-(r,a,p)\big]\\
&= \inf_{r,s\in Q}[\varphi_*^-(q,x,s)\vee\varphi_*^-(s,u,r)\vee\varphi^-(r,a,p)]\\
&= \inf_{s\in Q}[\varphi_*^-(q,x,s)\vee\big(\inf_{r\in Q}[\varphi_*^-(s,u,r)\vee\varphi^-(r,a,p)]\big)]\\
&= \inf_{s\in Q}[\varphi_*^-(q,x,s)\vee\varphi_*^-(s,ua,p)]\\
&= \inf_{s\in Q}[\varphi_*^-(q,x,s)\vee\varphi_*^-(s,y,p)]
\end{aligned}
$$

and

$$
\begin{aligned}
\varphi_*^+(q,xy,p)=\varphi_*^+(q,xua,p) &= \sup_{r\in Q}[\varphi_*^+(q,xu,r)\wedge\varphi^+(r,a,p)]\\
&= \sup_{r\in Q}[\sup_{s\in Q}[\varphi_*^+(q,x,s)\wedge\varphi_*^+(s,u,r)]\wedge\varphi^+(r,a,p)]\\
&= \sup_{r,s\in Q}[\varphi_*^+(q,x,s)\wedge\varphi_*^+(s,u,r)\wedge\varphi^+(r,a,p)]\\
&= \sup_{s\in Q}[\varphi_*^+(q,x,s)\wedge\big(\sup_{r\in Q}[\varphi_*^+(s,u,r)\wedge\varphi^+(r,a,p)]\big)]\\
&= \sup_{s\in Q}[\varphi_*^+(q,x,s)\wedge\varphi_*^+(s,ua,p)]\\
&= \sup_{s\in Q}[\varphi_*^+(q,x,s)\wedge\varphi_*^+(s,y,p)].
\end{aligned}
$$

Hence the result is valid for $|y|=n$. This completes the proof. ∎

**Definition 3.3.** *Let $\mathcal{M}=(Q,X,\varphi)$ be a bffsm and let $p,q\in Q$. Then $p$ is called a bipolar immediate successor of $q$ if the following condition holds:*

$$(\exists a\in X)\ (\varphi^-(q,a,p)<0,\ \varphi^+(q,a,p)>0).$$

*We say that $p$ is a bipolar successor of $q$ if the following condition holds:*

$$(\exists x\in X^*)\ (\varphi_*^-(q,x,p)<0,\ \varphi_*^+(q,x,p)>0).$$

*We denote by $\mathcal{S}(q)$ the set of all bipolar successors of $q$. For any subset $T$ of $Q$, the set of all bipolar successors of $T$, denoted by $\mathcal{S}(T)$, is defined to be the set*

$$\mathcal{S}(T):=\cup\{\mathcal{S}(q)\mid q\in T\}.$$

**Proposition 3.1.** *For any bffsm $\mathcal{M}=(Q,X,\varphi)$, we have the following properties:*

(1) $(\forall q\in Q)\ (q\in\mathcal{S}(q))$.

(2) $(\forall p,q,r\in Q)\ (p\in\mathcal{S}(q),\ r\in\mathcal{S}(p)\ \Rightarrow\ r\in\mathcal{S}(q))$.

*Proof.* (1) Since $\varphi_*^-(q,\lambda,q)=-1<0$ and $\varphi_*^+(q,\lambda,q)=1>0$, we have $q\in\mathcal{S}(q)$.
(2) Let $p\in\mathcal{S}(q)$ and $r\in\mathcal{S}(p)$. Then there exist $x,y\in X^*$ such that $\varphi_*^-(q,x,p)<0$, $\varphi_*^+(q,x,p)>0$, $\varphi_*^-(p,y,r)<0$ and $\varphi_*^+(p,y,r)>0$. Using Lemma 3.1, we have

$$
\begin{aligned}
\varphi_*^-(q,xy,r) &= \inf_{s\in Q}[\varphi_*^-(q,x,s)\vee\varphi_*^-(s,y,r)]\\
&\le \varphi_*^-(q,x,p)\vee\varphi_*^-(p,y,r)<0
\end{aligned}
$$

and

$$\varphi_*^+(q,xy,r)=\sup_{s\in Q}[\varphi_*^+(q,x,s)\wedge\varphi_*^+(s,y,r)]$$

$$\geq \varphi_*^+(q,x,p) \wedge \varphi_*^+(p,y,r) > 0.$$

Hence $r \in \mathcal{S}(q)$. ∎

**Proposition 3.2.** *Let $\mathcal{M} = (Q, X, \varphi)$ be a bffsm. For any subsets $A$ and $B$ of $Q$, the following assertions hold.*

(1) $A \subseteq B \Rightarrow \mathcal{S}(A) \subseteq \mathcal{S}(B)$.
(2) $A \subseteq \mathcal{S}(A)$.
(3) $\mathcal{S}(\mathcal{S}(A)) = \mathcal{S}(A)$.
(4) $\mathcal{S}(A \cup B) = \mathcal{S}(A) \cup \mathcal{S}(B)$.
(5) $\mathcal{S}(A \cap B) \subseteq \mathcal{S}(A) \cap \mathcal{S}(B)$.

*Proof.* The proofs of (1), (2), (4) and (5) are straightforward.

For (3), obviously $\mathcal{S}(A) \subseteq \mathcal{S}(\mathcal{S}(A))$. If $q \in \mathcal{S}(\mathcal{S}(A))$, then $q \in \mathcal{S}(p)$ for some $p \in \mathcal{S}(A)$. From $p \in \mathcal{S}(A)$, there exists $r \in A$ such that $p \in \mathcal{S}(r)$. It follows from Proposition 3.1 (2) that $q \in \mathcal{S}(r) \subseteq \mathcal{S}(A)$ so that $\mathcal{S}(\mathcal{S}(A)) \subseteq \mathcal{S}(A)$. Thus (3) is valid. ∎

**Definition 3.4.** *Let $\mathcal{M} = (Q, X, \varphi)$ be a bffsm. We say that $\mathcal{M}$ satisfies the bipolar exchange property if the following condition holds:*

$$(\forall p, q \in Q)\ (\forall T \subseteq Q)\ (p \in \mathcal{S}(T \cup \{q\}) \setminus \mathcal{S}(T) \ \Rightarrow\ q \in \mathcal{S}(T \cup \{p\})).$$

**Theorem 3.1.** *Let $\mathcal{M} = (Q, X, \varphi)$ be a bffsm. Then the following assertions are equivalent.*

(1) *$\mathcal{M}$ satisfies the bipolar exchange property.*
(2) $(\forall p, q \in Q)\ (p \in \mathcal{S}(q) \iff q \in \mathcal{S}(p))$.

*Proof.* Assume that $\mathcal{M}$ satisfies the bipolar exchange property. Let $p, q \in Q$ be such that $p \in \mathcal{S}(q) = \mathcal{S}(\emptyset \cup \{q\})$. Note that $p \notin \mathcal{S}(\emptyset)$ and so $q \in \mathcal{S}(\emptyset \cup \{p\}) = \mathcal{S}(p)$. Similarly if $q \in \mathcal{S}(p)$ then $p \in \mathcal{S}(q)$. Conversely suppose that (2) is valid. Let $p, q \in Q$ and $T \subseteq Q$. If $p \in \mathcal{S}(T \cup \{q\}) \setminus \mathcal{S}(T)$, then $p \in \mathcal{S}(q)$. It follows from (2) that $q \in \mathcal{S}(p) \subseteq \mathcal{S}(T \cup \{p\})$. Hence $\mathcal{M}$ satisfies the bipolar exchange property. ∎

**Definition 3.5.** *Let $\mathcal{M} = (Q, X, \varphi)$ be a bffsm. Let $\varphi_Q = (\varphi_Q^-, \varphi_Q^+)$ be a bipolar fuzzy set in $Q$. Then $(Q, \varphi_Q, X, \varphi)$ is called a bipolar subsystem of $\mathcal{M}$ if for all $p, q \in Q$ and $a \in X$,*

$$\varphi_Q^-(q) \leq \varphi_Q^-(p) \vee \varphi^-(p, a, q),$$

$$\varphi_Q^+(q) \geq \varphi_Q^+(p) \wedge \varphi^+(p, a, q).$$

**Example 3.1.** Let $Q = \{p, q\}$, $X = \{a\}$, $\varphi^-(r, a, t) = -\frac{1}{2}$ and $\varphi^+(r, a, t) = \frac{1}{2}$ for all $r, t \in Q$. Let $\varphi_Q = (\varphi_Q^-, \varphi_Q^+)$ be given by $\varphi_Q^-(q) = -\frac{7}{8}$, $\varphi_Q^+(q) = \frac{3}{4}$, $\varphi_Q^-(p) = -\frac{2}{3}$. and $\varphi_Q^+(p) = \frac{1}{2}$. Then

$$\varphi_Q^-(q) \vee \varphi^-(q, a, p) = (-\tfrac{7}{8}) \vee (-\tfrac{1}{2}) = -\tfrac{1}{2} > -\tfrac{2}{3} = \varphi_Q^-(p),$$

$$\varphi_Q^+(q) \wedge \varphi^+(q, a, p) = \tfrac{3}{4} \wedge \tfrac{1}{2} = \tfrac{1}{2} = \varphi_Q^+(p).$$

Hence $(Q, \varphi_Q, X, \varphi)$ is a bipolar subsystem.

**Theorem 3.2.** *Let $\mathcal{M} = (Q, X, \varphi)$ be a bffsm and let $\varphi_Q = (\varphi_Q^-, \varphi_Q^+)$ be a bipolar fuzzy set in $Q$. Then $(Q, \varphi_Q, X, \varphi)$ is a bipolar subsystem of $\mathcal{M}$ if and only if*

$$\varphi_Q^-(q) \leq \varphi_Q^-(p) \vee \varphi_*^-(p, x, q)$$
$$\varphi_Q^+(q) \geq \varphi_Q^+(p) \wedge \varphi_*^+(p, x, q)$$

*for all $p, q \in Q$ and $x \in X^*$.*

*Proof.* Suppose that $(Q, \varphi_Q, X, \varphi)$ is a bipolar subsystem of $\mathcal{M}$. Let $p, q \in Q$ and $x \in X^*$. The proof is by induction on $|x| = n$. If $n = 0$, then $x = \lambda$. Now if $p = q$, then

$$\varphi_Q^-(q) \vee \varphi_*^-(q, \lambda, q) = \varphi_Q^-(q)$$

and

$$\varphi_Q^+(q) \wedge \varphi_*^+(q, \lambda, q) = \varphi_Q^+(q).$$

If $q \neq p$, then

$$\varphi_Q^-(p) \vee \varphi_*^-(p, \lambda, q) = 0 \geq \varphi_Q^-(q)$$

and

$$\varphi_Q^+(p) \wedge \varphi_*^+(p, \lambda, q) = 0 \leq \varphi_Q^+(q).$$

Thus the result is true for $n = 0$. Suppose the result is valid for all $y \in X^*$ with $|y| = n - 1$, $n > 0$. For the $y$ above, let $x = ya$ where $a \in X$. Then

$$\begin{aligned}
\varphi_Q^-(p) \vee \varphi_*^-(p, x, q) &= \varphi_Q^-(p) \vee \varphi_*^-(p, ya, q) \\
&= \varphi_Q^-(p) \vee \big(\inf_{r \in Q}[\varphi_*^-(p, y, r) \vee \varphi^-(r, a, q)]\big) \\
&= \inf_{r \in Q}[\varphi_Q^-(p) \vee \varphi_*^-(p, y, r) \vee \varphi^-(r, a, q)] \\
&\geq \inf_{r \in Q}[\varphi_Q^-(r) \vee \varphi^-(r, a, q)] \geq \varphi_Q^-(q)
\end{aligned}$$

and

$$\begin{aligned}
\varphi_Q^+(p) \wedge \varphi_*^+(p, x, q) &= \varphi_Q^+(p) \wedge \varphi_*^+(p, ya, q) \\
&= \varphi_Q^+(p) \wedge \big(\sup_{r \in Q}[\varphi_*^+(p, y, r) \wedge \varphi^+(r, a, q)]\big) \\
&= \sup_{r \in Q}[\varphi_Q^+(p) \wedge \varphi_*^+(p, y, r) \wedge \varphi^+(r, a, q)] \\
&\leq \sup_{r \in Q}[\varphi_Q^+(r) \wedge \varphi^+(r, a, q)] \leq \varphi_Q^+(q).
\end{aligned}$$

The converse is trivial, completing the proof. ∎

## 4. Conclusion

In this paper we have considered a new generalization of a fuzzy finite state machine. We have introduced the concept of a bffsm, a bipolar (immediate) successor, a bipolar exchange property and a bipolar subsystem. We have investigated several related properties, and established a condition for a bffsm to satisfy the bipolar exchange property. We have also initiated a characterization of a bipolar subsystem. Based on these results, we will study retrievable (resp. separated and connected) bffsms, bipolar fuzzy transformation semigroups and bipolar fuzzy topology associated with a bffsm. We also would like to try to find an example of real life problem in respect to philosophical study.

**Acknowledgement.** The authors are highly grateful to the referees for their valuable comments and suggestions in improving the paper. This research is partially supported by FRGS grant Vot No: 0555, Ministry of Higher Education, Malaysia.

## References

[1] D. Dubois and H. Prade, *Fuzzy Sets and Systems*, Mathematics in Science and Engineering, 144, Academic Press, New York, 1980.
[2] Y. B. Jun and S. Z. Song, Subalgebras and closed ideals of BCH-algebras based on bipolar-valued fuzzy sets, *Sci. Math. Jpn.* **68** (2008), no. 2, 287–297.
[3] H. V. Kumbhojkar and S. R. Chaudhari, On covering of products of fuzzy finite state machines, *Fuzzy Sets and Systems* **125** (2002), no. 2, 215–222.
[4] K. J. Lee, Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras, *Bull. Malays. Math. Sci. Soc. (2)* **32** (2009), no. 3, 361–373.
[5] K. M. Lee, *Bipolar-valued Fuzzy Sets and Their Operations,* Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand (2000), 307–312.
[6] K. M. Lee, Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets, *J. Fuzzy Logic Intelligent Systems* **14** (2004), 125–129.
[7] D. S. Malik, J. N. Mordeson and M. K. Sen, Submachines of fuzzy finite state machines, *J. Fuzzy Math.* **2** (1994), no. 4, 781–792.
[8] H.-J. Zimmermann, *Fuzzy Set Theory—and Its Applications*, International Series in Management Science/Operations Research, Kluwer·Nijhoff, Boston, MA, 1985.