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Neighbor Set for the Existence of (g, f, n)-Critical Graphs

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Abstract. Let G be a graph of order p. Let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ with $g(x) \leq f(x)$ for any $x \in V(G)$. A graph G is said to be (g, f, n)-critical if $G - N$ has a (g, f)-factor for each $N \subseteq V(G)$ with $|N| = n$. If $g(x) \equiv a$ and $f(x) \equiv b$ for all $x \in V(G)$, then a (g, f, n)-critical graph is an (a, b, n)-critical graph. In this paper, several sufficient conditions in terms of neighbor set for graphs to be (a, b, n)-critical or (g, f, n)-critical are given.

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1. Introduction

In our daily life many problems on optimization and network design, e.g., coding design, building blocks, the file transfer problems on computer networks, scheduling problems and so on, are related to the factors, factorizations and fractional factors [1]. It is well-known that a network can be represented by a graph. Vertices and edges of the graph correspond to nodes and links between the nodes, respectively. Henceforth, we use the term *graph* instead of *network*.

The graphs considered in this paper will be finite and undirected graphs without loops and multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G and by $N_G(x)$ the set of vertices adjacent to x in G. The *minimum degree* of G is denoted by $\delta(G)$. For a subset $X \subseteq V(G)$, the neighborhood of X is defined as:

$$N_G(X) := \bigcup_{x \in X} N_G(x)$$

For any subset $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and by $G - S$ the subgraph obtained from G by deleting the vertices in S together with

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the edges incident to the vertices in S. If $S, T \subseteq V(G)$, then we write $e_G(S,T)$ for the number of edges in G joining a vertex in S to a vertex in T.

Let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ with $g(x) \leq f(x)$ for any $x \in V(G)$. Then a spanning subgraph F of G is called a (g,f)-factor if $g(x) \leq d_F(x) \leq f(x)$ holds for any $x \in V(G)$. Let a and b be two integers such that $0 \leq a \leq b$. If $g(x) \equiv a$ and $f(x) \equiv b$ for all $x \in V(G)$, then a (g,f)-factor is called an $[a,b]$-factor. An $[a,b]$-factor is called a k-factor if $a = b = k$, which is a regular factor.

A graph G is said to be (g,f,n)-critical if $G - N$ has a (g,f)-factor for each $N \subseteq V(G)$ with $|N| = n$. If $g(x) \equiv a$ and $f(x) \equiv b$ for all $x \in V(G)$, then a (g,f,n)-critical graph is an (a,b,n)-critical graph. That is, a graph G is (a,b,n)-critical if after deleting any n vertices of G the remaining graph of G has an $[a,b]$-factor. If $a = b = k$, then an (a,b,n)-critical graph is called a (k,n)-critical graph. If $k = 1$, then a (k,n)-critical graph is simply called an n-critical graph. The other terminologies and notations can be found in [4, 14].

2. Neighbor set and (a,b,n)-critical graphs

Plummer [10] and Lovász [9] studied the properties of 2-critical graphs. Yu [11] gave a characterization of n-critical graphs. Liu and Yu [8] investigated the characterization of (k,n)-critical graphs. A necessary and sufficient condition for a graph to be (a,b,n)-critical with $a < b$ was due to Liu and Wang [7]. Zhou [12, 13] gave some sufficient conditions for a graph to be (a,b,n)-critical.

Berge and Las Vergnas [3] gave the following sufficient condition in terms of neighbor set of independent subset for the existence of $[1,b]$-factor. This condition was also obtained by Amahashi and Kano [2] independently.

Theorem 2.1. *Let $b \geq 2$ be an integer. Then a graph G has a $[1,b]$-factor if and only if*

$$|N_G(X)| \geq \frac{|X|}{b}$$

for all independent subset X of $V(G)$.

In [5], Kano proved the following result on neighbor set for the existence of $[a,b]$-factors.

Theorem 2.2. *Let a and b be integers such that $2 \leq a < b$, and G be a graph of order p, $p \geq 6a + b$. Put $\lambda = (a-1)/b$. Suppose for any subset $X \subset V(G)$, we have*

$$N_G(X) = V(G) \quad \text{if} \quad |X| \geq \left\lfloor \frac{p}{1+\lambda} \right\rfloor; \quad \text{or}$$

$$|N_G(X)| \geq (1+\lambda)|X| \quad \text{if} \quad |X| < \left\lfloor \frac{p}{1+\lambda} \right\rfloor.$$

Then G has an $[a,b]$-factor.

Furthermore, it was pointed out that the result is best possible in some sense in [5].

We now prove the following result, which is a neighbor set condition for graphs to be (a,b,n)-critical. Our result is an extension of Theorem 2.2.

Theorem 2.3. *Let G be a graph of order p, and a, b and n be nonnegative integers such that $2 \leq a < b$ and $p \geq 6a + b + n$. Let $\lambda = (a-1)/b$. Suppose for any subset $X \subset V(G)$, we have*

$$N_G(X) = V(G) \quad \textit{if} \quad |X| \geq \left\lfloor \frac{p-n}{1+\lambda} \right\rfloor; \quad \textit{or}$$

$$|N_G(X)| \geq (1+\lambda)|X| + n \quad \textit{if} \quad |X| < \left\lfloor \frac{p-n}{1+\lambda} \right\rfloor.$$

Then G is an (a, b, n)-critical graph.

It is easy to see that Theorem 2.2 is a special case of Theorem 2.3 for $n = 0$.

Now we prove Theorem 2.3. The following two lemmas are very useful to our proof.

Lemma 2.1. [7] *Let a, b and n be nonnegative integers such that $1 \leq a < b$, and let G be a graph of order p with $p \geq a + n + 1$. Then G is (a, b, n)-critical if and only if for any $S \subseteq V(G)$ and $|S| \geq n$*

$$\sum_{j=0}^{a-1} (a-j) p_j(G-S) \leq b|S| - bn,$$

or

$$a|T| - d_{G-S}(T) \leq b|S| - bn,$$

where $p_j(G-S) = |\{x : d_{G-S}(x) = j\}|$, $T = \{x : x \in V(G) \backslash S, d_{G-S}(x) \leq a-1\}$.

Lemma 2.2. *Let G be a graph of order p satisfying the assumption of Theorem 2.3. Then $\delta(G) \geq (\lambda p + n + 1)/(1+\lambda)$.*

Proof. Let v be a vertex of G with minimum degree $\delta(G)$. Let $X = V(G) \backslash N_G(v)$. Since $N_G(X)$ does not contain v, we get $(1+\lambda)|X| + n \leq |N_G(X)| \leq p - 1$. Thus $(1+\lambda)(p - \delta(G)) + n \leq p - 1$, and therefore $\delta(G) \geq (\lambda p + n + 1)/(1+\lambda)$. ∎

The proof of Theorem 2.3. We prove the theorem by contradiction. Suppose that G satisfies the hypothesis of Theorem 2.3, but G is not an (a, b, n)-critical graph. Then by Lemma 2.1, there exists a subset S of $V(G)$ with $|S| \geq n$ such that

$$\delta_G(S) = b|S| + d_{G-S}(T) - a|T| \leq bn - 1, \tag{2.1}$$

where $T = \{x : x \in V(G) \backslash S,\ d_{G-S}(x) \leq a-1\}$.

If $T = \emptyset$, then by (2.1), $bn - 1 \geq \delta_G(S, T) = b|S| \geq bn$, which is a contradiction. Hence, $T \neq \emptyset$. Define

$$h = \min\{d_{G-S}(x) \mid x \in T\}.$$

Thus $0 \leq h \leq a - 1$.

We first prove that the following claim holds.

Claim 1. If $a = 2$, then $|N_G(T)| < (1 + \frac{1}{b})|T| + n$.

Proof. By (2.1), $T = \{x : x \in V(G)\backslash S,\ d_{G-S}(x) \le a-1\}$ and $a = 2$, we have

$$\begin{aligned}|N_G(T)| &\le |S| + d_{G-S}(T) < \frac{2}{b}|T| + \left(1 - \frac{1}{b}\right) d_{G-S}(T) + n\\ &\le \frac{2}{b}|T| + \left(1-\frac{1}{b}\right)|T| + n = \left(1+\frac{1}{b}\right)|T| + n.\end{aligned}$$ ∎

In the following we shall consider three cases and derive a contradiction in each case.

Case 1. $|N_G(T)| \ge (1+\lambda)|T| + n$ and $h \ne 1$.

By Claim 1, we have $a \ge 3$. As $(1+\lambda)|T| + n \le |N_G(T)| \le |V(G)| = p$, we have

$$|T| \le \frac{p-n}{1+\lambda}. \tag{2.2}$$

In view of the definition of h and Lemma 2.2, we have

$$|S| \ge \delta(G) - h \ge \frac{\lambda p + n + 1}{1+\lambda} - h. \tag{2.3}$$

Subcase 1.1. $h \ge 2$.

According to (2.1), (2.2) and (2.3), we have

$$\begin{aligned}bn > \delta_G(S,T) &= b|S| - a|T| + d_{G-S}(T)\\ &\ge b|S| - a|T| + h|T|\\ &\ge b\left(\frac{\lambda p + n + 1}{1+\lambda} - h\right) - \frac{(a-h)(p-n)}{1+\lambda}.\end{aligned}$$

Considering $\lambda = (a-1)/b$ and $2 \le h \le a-1$, the above inequality implies

$$\begin{aligned}p &< \frac{(1+\lambda)(bn+bh) - bn - b - an + hn}{h-1}\\ &= b(1+\lambda) + \frac{(1+\lambda)bn + b(1+\lambda) - bn - b - an + hn}{h-1}\\ &= a + b - 1 + \frac{a - 1 - n + hn}{h-1}\\ &= a + b - 1 + \frac{a-1}{h-1} + \frac{n(h-1)}{(h-1)} \le a + b - 1 + a - 1 + n\\ &= 2a + b - 2 + n < 6a + b + n.\end{aligned}$$

This contradicts the assumption $p \ge 6a + b + n$.

Subcase 1.2. $h = 0$.

We define $I = \{x \in T|\ d_{G-S}(x) = 0\}$. Then I is an independent vertex subset of G and $I \ne \emptyset$. Let $|I| = l$, then $l \ge 1$. By Lemma 2.2 and $(1+\lambda)|T| + n \le |N_G(T)| \le p - l$, we have that $|S| \ge \delta(G) \ge \frac{\lambda p+n+1}{1+\lambda}$ and $|T| \le \frac{p-l-n}{1+\lambda}$. As $a \ge 3$ and $\lambda < 1$, we get

$$\begin{aligned}\delta_G(S,T) &= b|S| - a|T| + d_{G-S}(T) \ge b|S| - a|T| + |T| - l\\ &\ge \frac{b(\lambda p + n + 1)}{1+\lambda} - \frac{(a-1)(p-l-n)}{1+\lambda} - l\end{aligned}$$

$$= \frac{b + (a-2-\lambda)l + (a+b-1)n}{1+\lambda}$$
$$\geq \frac{b + (a+b-1)n}{1+\frac{a-1}{b}} \geq bn.$$

Obviously this contradicts (2.1).

Case 2. $|N_G(T)| \geq (1+\lambda)|T| + n$ and $h = 1$.

Let $m = \{x \in T | \ d_{G-S}(x) = 1\} \geq 1$. By $h = 1$ and Lemma 2.2, we have

$$|S| \geq \delta(G) - 1 \geq \frac{\lambda p + n + 1}{1+\lambda} - 1 = \frac{\lambda(p-1)+n}{1+\lambda}. \tag{2.4}$$

Subcase 2.1. $|T| \leq \frac{p-1-n}{1+\lambda}$.

Then by (2.1) and (2.4), we have

$$\begin{aligned} bn - 1 &\geq \delta_G(S,T) = b|S| - a|T| + d_{G-S}(T) \geq b|S| - a|T| + 2|T| - m \\ &\geq b\left(\frac{\lambda(p-1)+n}{1+\lambda}\right) - (a-2)\left(\frac{p-1-n}{1+\lambda}\right) - m = \frac{p-1+(a+b-2)n}{1+\lambda} - m \\ &= \frac{p-1-n}{1+\lambda} - m + \frac{(a+b-1)n}{1+\lambda} = \frac{p-1-n}{1+\lambda} - m + bn \\ &\geq |T| - m + bn \geq bn. \end{aligned}$$

Thus we can get a contradiction.

Subcase 2.2. $|T| > \frac{p-1-n}{1+\lambda}$.

In this case, by (2.4), we have $|S| + |T| > \frac{\lambda(p-1)+n}{1+\lambda} + \frac{p-1-n}{1+\lambda} = p - 1$. Therefore $|S| + |T| = p$. Since $(1+\lambda)|T| + n \leq |N_G(T)| \leq p$, we have $|T| \leq \frac{p-n}{1+\lambda}$. By (2.1) and $h = 1$, we get

$$\begin{aligned} bn - 1 \geq \delta_G(S,T) &= b|S| - a|T| + d_{G-S}(T) \geq b(p - |T|) - a|T| + |T| \\ &= bp - (a+b-1)|T| \geq bp - \frac{(a+b-1)(p-n)}{1+\lambda} = bn. \end{aligned}$$

Hence we obtain a contradiction.

Case 3. $|N_G(T)| < (1+\lambda)|T| + n$.

In this case, we have that $N_G(T) = V(G)$. Combining with (2.1) we can get $1 \leq h \leq a-1$. Since $(1+\lambda)|T| + n > |N_G(T)| = p$, we obtain $|T| > \frac{p-n}{1+\lambda}$. Thus $|T| \geq \lfloor \frac{p-n}{1+\lambda} \rfloor + 1$. If $h = 1$, choose a vertex $v \in T$ such that $d_{G-S}(v) = h = 1$. Thus $N_G(T \backslash N_G(v))$ does not contain v. On the other hand, $|T \backslash N_G(v)| \geq |T| - 1 \geq \lfloor \frac{p-n}{1+\lambda} \rfloor$ implies that $N_G(T \backslash N_G(v)) = V(G)$, which is a contradiction. Therefore we can assume that $h \geq 2$.

Claim 2. $|T| - h \geq \frac{p-n}{1+\lambda}$.

Proof. Since $bn > \delta_G(S,T) = b|S| - a|T| + d_{G-S}(T) \geq b|S| - a|T| + h|T|$, we have $|T| > \frac{b|S|-bn}{a-h}$. By Lemma 2.2, we have

$$|T| - h > \frac{b|S| - bn}{a-h} - h \geq \frac{b(\frac{\lambda p+n+1}{1+\lambda} - h) - bn}{a-h} - h.$$

Therefore in order to prove Claim 2 it suffices to prove that

$$\frac{b(\frac{\lambda p+n+1}{1+\lambda} - h) - bn}{a-h} - h \geq \frac{p-n}{1+\lambda}.$$

This inequality is equivalent to the following one:

$$\begin{aligned} p &\geq \frac{(1+\lambda)h(b+a-h) - bn - b + bn(1+\lambda) - n(a-h)}{b\lambda - a + h} \\ &= \frac{(a+b-h)(1+\lambda)h + nh - b - n}{h-1} \\ &= \frac{-(1+\lambda)h^2 + (a+b)(1+\lambda)h + nh - b - n}{h-1} \\ &= -(1+\lambda)h + (a+b-1)(1+\lambda) + \frac{(a+b-1)(1+\lambda) - b - n + nh}{h-1}. \end{aligned}$$

Define

$$f(h) = -(1+\lambda)h + (a+b-1)(1+\lambda) + \frac{(a+b-1)(1+\lambda) - b - n + nh}{h-1}.$$

In view of $2 \leq h \leq a-1$, the function $f(h)$ attains its maximum value at $h = 2$. Thus

$$f(h) \leq f(2) = 4a + b - 6 + 2a\lambda - 4\lambda + n < 6a + b + n.$$

Since $p \geq 6a + b + n$, we conclude that $|T| - h \geq \frac{p-n}{1+\lambda}$. ∎

Now we proceed to prove the Case 3. We prove it by contradiction. Choose a vertex v of T satisfying $d_{G-S}(v) = h$. Obviously $N_G(T \backslash N_G(v))$ does not contain v. By Claim 2 $|T \backslash N_G(v)| \geq |T| - h \geq \frac{p-n}{1+\lambda}$, which implies that $N_G(T \backslash N_G(v)) = V(G)$. Therefore we get a contradiction.

From all the argument above, we deduce the contradictions. Hence G is an (a,b,n)-critical graph. This completes the proof of Theorem 2.3. ∎

3. Neighbor set and (g,f,n)-critical graphs

We now give a sufficient condition for a graph to be (g,f,n)-critical.

Theorem 3.1. *Let G be a graph of order p, and a, b and n be nonnegative integers such that $2 \leq a < b$ and $p \geq \frac{(a+b-1)^2 + bn(a+b+1)}{a+1}$. Let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ such that $a \leq g(x) < f(x) \leq b$ for any $x \in V(G)$. Let $\lambda' = \frac{b}{a+1}$. Suppose for any subset $X \subset V(G)$, we have*

$$N_G(X) = V(G) \quad \text{if} \quad |X| \geq \left\lfloor \frac{p-n}{1+\lambda'} \right\rfloor; \quad \text{or}$$

$$|N_G(X)| \geq (1+\lambda')|X| + n \quad \text{if} \quad |X| < \left\lfloor \frac{p-n}{1+\lambda'} \right\rfloor.$$

Then G is a (g,f,n)-critical graph.

In Theorem 3.1, if $n = 0$, then we obtain the following corollary.

Corollary 3.1. *Let G be a graph of order p, and a, b be nonnegative integers such that $2 \le a < b$ and $p \ge \frac{(a+b-1)^2}{a+1}$. Let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ such that $a \le g(x) < f(x) \le b$ for any $x \in V(G)$. Let $\lambda' = \frac{b}{a+1}$. Suppose for any subset $X \subset V(G)$, we have*

$$N_G(X) = V(G) \quad \textit{if} \quad |X| \ge \left\lfloor \frac{p}{1+\lambda'} \right\rfloor; \quad \textit{or}$$

$$|N_G(X)| \ge (1+\lambda')|X| \quad \textit{if} \quad |X| < \left\lfloor \frac{p}{1+\lambda'} \right\rfloor.$$

Then G has a (g, f)-factor.

In Theorem 3.1, if $g(x) \equiv a$ and $f(x) \equiv b$, then we obtain the following corollary, which is stronger than Theorem 2.3 in some sense.

Corollary 3.2. *Let G be a graph of order p, and a, b and n be nonnegative integers such that $2 \le a < b$ and $p \ge \frac{(a+b-1)^2+bn(a+b+1)}{a+1}$. Let $\lambda' = \frac{b}{a+1}$. Suppose for any subset $X \subset V(G)$, we have*

$$N_G(X) = V(G) \quad \textit{if} \quad |X| \ge \left\lfloor \frac{p-n}{1+\lambda'} \right\rfloor; \quad \textit{or}$$

$$|N_G(X)| \ge (1+\lambda')|X| + n \quad \textit{if} \quad |X| < \left\lfloor \frac{p-n}{1+\lambda'} \right\rfloor.$$

Then G is an (a, b, n)-critical graph.

Now we prove Theorem 3.1. The following lemma is very useful in our proof. Lemma 3.1 concerning a necessary and sufficient condition for a graph to be (g, f, n)-critical is due to Li and Matsuda [6].

Lemma 3.1. [6] *Let G be a graph, $n \ge 0$ be an integer, and $g, f : V(G) \to Z$ be two functions such that $g(x) < f(x)$ for each $x \in V(G)$. Then G is (g, f, n)-critical if and only if for any $S \subseteq V(G)$ and $|S| \ge n$,*

$$\delta_G(S) = f(S) + d_{G-S}(T) - g(T) \ge \max\{f(N) : N \subseteq S \textit{ and } |N| = n\}$$

where $T = \{x : x \in V(G) \backslash S$ and $d_{G-S}(x) \le g(x)\}$.

Lemma 3.2. *Let G be a graph of order p satisfying the assumption of Theorem 3.1. Then $\delta(G) \ge \frac{\lambda' p+n+1}{1+\lambda'}$.*

Proof. The proof is similar to that of Lemma 2.2. ∎

The proof of Theorem 3.1. We shall use a different technique from Theorem 2.3 to prove Theorem 3.1. We prove the theorem by contradiction. Suppose that G satisfies the assumption of Theorem 3.1, but G is not a (g, f, n)-critical graph. Then by Lemma 3.1, there exists a subset S of $V(G)$ with $|S| \ge n$ such that

$$(3.1) \quad \delta_G(S, T) = f(S) + d_{G-S}(T) - g|T| \le \max\{f(N) : N \subseteq S \text{ and } |N| = n\} - 1,$$

where $T = \{x : x \in V(G) \backslash S,\ d_{G-S}(x) \le g(x)\}$. We choose such subsets which minimizes $|T|$.

We first show that the following claim holds.

Claim 1. $d_{G-S}(x) \le g(x) - 1 \le b - 2$ for each $x \in T$.

Proof. If there exist a vertex $x \in T$ such that $d_{G-S}(x) \ge g(x)$, then the subsets S and $T \setminus \{x\}$ satisfy (3.1), this contradicts the choice of S and T. Therefore

$$d_{G-S}(x) \le g(x) - 1 \le b - 2$$

for all $x \in T$ holds. ∎

If $T = \emptyset$, then by (3.1), $f(S) - 1 \ge \max\{f(N) : N \subseteq S \text{ and } |N| = n\} - 1 \ge \delta_G(S,T) = f(S)$, which is a contradiction. Hence, $T \ne \emptyset$. Define

$$h = \min\{d_{G-S}(x) \mid x \in T\}.$$

By Claim 1, we have

$$0 \le h \le b - 2.$$

In the following we shall consider two cases and derive a contradiction in each case.

Case 1. $h = 0$.

We define $I = \{x \in T \mid d_{G-S}(x) = 0\}$. Then I is an independent vertex subset of G and $I \ne \emptyset$.

Subcase 1.1. $|I| < \left\lfloor \frac{p-n}{1+\lambda'} \right\rfloor$.

In this case, we have $|N_G(I)| \ge (1+\lambda')|I| + n$.

On the other hand, by (3.1) we have

$$\begin{aligned} bn > bn - 1 \ge \delta_G(S,T) &= f(S) + d_{G-S}(T) - g(T) \\ &\ge (a+1)|S| + d_{G-S}(T) - (b-1)|T| \\ &\ge (a+1)|S| + |T - I| - (b-1)|T| \\ &= (a+1)|S| - (b-1)|I| + (2-b)|T - I| \\ &\ge (a+1)|S| - (b-1)|I| + (2-b)(p - |S| - |I|) \\ &= (a+b-1)|S| - |I| + (2-b)p. \end{aligned}$$

Thus

$$|S| < \frac{bn + |I| + (b-2)p}{a+b-1}. \tag{3.2}$$

According to (3.2), Lemma 3.2 and $|I| < \left\lfloor \frac{p-n}{1+\lambda'} \right\rfloor \le \frac{p-n}{1+\lambda'}$, we have

$$\frac{\lambda' p + n + 1}{1+\lambda'} \le \delta(G) \le |S| < \frac{bn + |I| + (b-2)p}{a+b-1} < \frac{bn + \frac{p-n}{1+\lambda'} + (b-2)p}{a+b-1}.$$

This inequality implies

$$p < \frac{bn(a+b+1)}{a+1} - (a+b)n - (a+b-1) \le \frac{(a+b-1)^2 + bn(a+b+1)}{a+1},$$

which contradicts to that $p \ge \frac{(a+b-1)^2 + bn(a+b+1)}{a+1}$.

Subcase 1.2. $|I| \ge \left\lfloor \frac{p-n}{1+\lambda'} \right\rfloor$.

In this case, we have $N_G(I) = V(G)$. By (3.2) we obtain

$$p = N_G(I) \le |S| < \frac{bn + |I| + (b-2)p}{a+b-1} < \frac{bn + (b-1)p}{a+b-1}.$$

It follows that

$$p < \frac{bn}{a} \le \frac{(a+b-1)^2 + bn(a+b+1)}{a+1},$$

which contradicts to the assumption that $p \ge \frac{(a+b-1)^2+bn(a+b+1)}{a+1}$.

Case 2. $1 \le h \le b-2$.

Since

$$\begin{aligned} bn > bn - 1 \ge \delta_G(S,T) &= f(S) + d_{G-S}(T) - g(T) \\ &\ge (a+1)|S| + d_{G-S}(T) - (b-1)|T| \ge (a+1)|S| + (h-b+1)|T| \\ &\ge (a+1)|S| + (h-b+1)(p-|S|) = (a+b-h)|S| - (b-1-h)p, \end{aligned}$$

we obtain

$$|S| < \frac{bn + (b-1-h)p}{a+b-h}.$$

By considering a vertex $v \in T$ with $d_{G-S}(v) = h$, we get

$$\delta(G) \le d_G(v) \le h + |S| < h + \frac{bn + (b-1-h)p}{a+b-h}.$$

Define

$$f(h) = h + \frac{bn + (b-1-h)p}{a+b-h}, \qquad 1 \le h \le b-2.$$

Considering $p \ge \frac{(a+b-1)^2+bn(a+b+1)}{a+1}$ and $h \ge 1$, $f(h)$ attains its maximum value at $h = 1$ since its derivative

$$\begin{aligned} f'(h) &= 1 - \frac{p(a+1) - bn}{(a+b-h)^2} \\ &\le 1 - \frac{p(a+1) - bn}{(a+b-1)^2} \le 0. \end{aligned}$$

Therefore

$$\frac{\lambda' p + n + 1}{1 + \lambda'} \le \delta(G) < f(h) = h + \frac{bn + (b-1-h)p}{a+b-h} \le f(1) = 1 + \frac{bn + (b-2)p}{a+b-1},$$

which implies

$$\begin{aligned} p &< \frac{(a+b+1)(a+b-1) + bn(a+b+1)}{2(a+1)} - \frac{(n+1)(a+b-1)}{2} \\ &< \frac{(a+b-1)^2 + bn(a+b+1)}{a+1}. \end{aligned}$$

That contradicts the assumption that $p \ge \frac{(a+b-1)^2+bn(a+b+1)}{a+1}$.

From all the argument above, we deduce the contradictions. Hence G is a (g, f, n)-critical graph. This completes the proof. ∎

4. Remarks

Remark 4.1. We can show the conditions in Theorem 2.3 are best possible in the following sense. The restrictions on set X in the conditions of Theorems 2.3 can not be eliminated. If we replace the current conditions on neighborhood to "for any subset $X \subset V(G)$, we have either $N_G(X) = V(G)$ or $|N_G(X)| \geq (1+\lambda)|X| + n$", then it is not sufficient. We let $b \geq 3$ be an odd integer. Let $m \geq 5$ be any odd positive integer. Let G_1 be the complete graph $K_{(a-1)m+n}$ and G_2 be $(bm+1)/2$ disjoint copies of K_2. Then let $G = G_1 + G_2$ be the join of G_1 and G_2(that is, $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$). Thus we have $|V(G_1)| = (a-1)m+n$, $|V(G_2)| = bm+1$, and $p = |V(G_1)|+|V(G_2)| = (a+b-1)m + n + 1 \geq 6a + b + n$. We take $S = V(G_1)$, $T = V(G_2)$. Then we can easily prove that G is not an (a,b,n)-critical graph because $\delta_G(S,T) = b|S| - a|T| + d_{G-S}(T) = b((a-1)m+n) - a(bm+1) + bm + 1 = bn - (a-1) < bn$.

We next prove that the condition $N_G(X) = V(G)$ or $|N_G(X)| \geq (1+\lambda)|X| + n$ always holds. If $|X \cap S| \geq 2$, or $|X \cap S| = 1$ and $|X \cap T| = 1$, then $N_G(X) = V(G)$. Obviously if $|X| = 1$ and $X \subseteq S$, then $|N_G(X)| = |V(G)| - 1 \geq 1 + \lambda + n$. Thus we may assume that $X \subseteq T$. Considering $|N_G(X)| = |S| + |X| = (a-1)m + n + |X|$, $|N_G(X)| \geq (1+\lambda)|X| + n$ is equivalent to $(a-1)m + n + |X| \geq (1+\lambda)|X| + n$. This inequality holds if and only if $|X| \leq bm$. Therefore if $X \neq T$ and $X \subset T$, then the condition $N_G(X) = V(G)$ or $|N_G(X)| \geq (1+\lambda)|X| + n$ holds. If $X = T$, then $N_G(X) = V(G)$. This completes the proof of the condition.

If $5 \leq a < b$ and b is even, we can similarly construct $G = G_1 + G_2$, where G_1 is the complete graph $K_{(a-1)m+n}$ and $G_2 = ((bm-2)K_2/2) \cup K_3$(disjoint union). We take $S = V(G_1)$, $T = V(G_2)$. Then we can similarly prove that G is not an (a,b,n)-critical graph and the condition $N_G(X) = V(G)$ or $|N_G(X)| \geq (1+\lambda)|X|+n$ always holds.

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