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Weighted Sharp Inequality for Vector-Valued Multilinear Integral Operator

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Abstract. In this paper, we prove the sharp inequality for some vector-valued multilinear integral operators. The operators include Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator. By using this inequality, we obtain the weighted L^p-norm inequality and $L \log L$-type inequality for the vector-valued multilinear operators.

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1. Introduction and theorems

In this paper, we shall study some vector-valued multilinear integral operators which are defined as following.

Suppose m_j are the positive integers ($j = 1, \cdots, l$), $m_1 + \cdots + m_l = m$ and A_j are the functions on $R^n (j = 1, \cdots, l)$. Let $F_t(x, y)$ be the function defined on $R^n \times R^n \times [0, +\infty)$. Set

$$F_t(f)(x) = \int_{R^n} F_t(x, y) f(y) dy$$

and

$$F_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j; x, y)}{|x - y|^m} F_t(x, y) f(y) dy$$

for every bounded and compactly supported function f, where

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\alpha| \le m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x - y)^\alpha.$$

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Let H be the Banach space $H = \{h : ||h|| < \infty\}$ such that, for each fixed $x \in R^n$, $F_t(f)(x)$ and $F_t^A(f)(x)$ may be viewed as mappings from $[0, +\infty)$ to H. For $1 < s < \infty$, the vector-valued multilinear operator related to F_t is defined by

$$|T^A(f)(x)|_s = \left(\sum_{i=1}^{\infty}(T^A(f_i)(x))^s\right)^{1/s},$$

where

$$T^A(f_i)(x) = ||F_t^A(f_i)(x)||,$$

and F_t satisfies: for fixed $\varepsilon > 0$,

$$||F_t(x, y)|| \le C|x - y|^{-n}$$

and

$$||F_t(y, x) - F_t(z, x)|| \le C|y - z|^{\varepsilon}|x - z|^{-n-\varepsilon}$$

if $2|y - z| \le |x - z|$. We also denote

$$|T(f)(x)|_s = \left(\sum_{i=1}^{\infty}|T(f_i)(x)|^s\right)^{1/s} \quad \text{and} \quad |f(x)|_s = \left(\sum_{i=1}^{\infty}|f_i(x)|^s\right)^{1/s},$$

where

$$T(g)(x) = ||F_t(g)(x)||,$$

and suppose that $|T|_s$ is bounded on $L^p(R^n)$ for $1 < p < \infty$ and and weak (L^1, L^1)-bounded.

Note that when $m = 0$, T^A is just the vector-valued multilinear commutator of T and A (see [9, 10, 12]). While when $m > 0$, T^A is non-trivial generalizations of the commutator. It is well-known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [1–5]). Hu and Yang (see [7]) proved a variant sharp estimate for the multilinear singular integral operators. In [16] and [17], Pérez and Trujillo-Gonzalez prove a sharp estimate for the multilinear commutator. The main purpose of this paper is to prove a sharp inequality for the vector-valued multilinear integral operators. As the application, we obtain the weighted L^p-norm inequality and $L \log L$-type inequality for the vector-valued multilinear operators. In Section 4, we shall give some applications.

First, let us introduce some notations. Throughout this paper, Q will denote a cubes of R^n with sides parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{x \in Q} \frac{1}{|Q|}\int_Q |f(y) - f_Q|dy,$$

where, and in what follows, $f_Q = |Q|^{-1}\int_Q f(x)dx$. It is well-known that (see [6])

$$f^{\#}(x) \approx \sup_{x \in Q} \inf_{c \in C} \frac{1}{|Q|}\int_Q |f(y) - c|dy.$$

We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^{\infty}(R^n)$ and define $||f||_{BMO} = ||f^{\#}||_{L^{\infty}}$. For $0 < r < \infty$, we denote $f_r^{\#}$ by

$$f_r^{\#}(x) = [(|f|^r)^{\#}(x)]^{1/r}.$$

Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x)=\sup_{x\in Q}|Q|^{-1}\int_Q |f(y)|dy.$$

For $k\in N$, we denote by M^k the operator M iterated k times, i.e., $M^1(f)(x)=M(f)(x)$ and

$$M^k(f)(x)=M(M^{k-1}(f))(x) \text{ when } k\geq 2.$$

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ, we denote the Φ-average by, for a function f,

$$||f||_{\Phi,Q}=\inf\left\{\lambda>0:\frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(y)|}{\lambda}\right)dy\leq 1\right\}$$

and the maximal function associated to Φ by

$$M_\Phi(f)(x)=\sup_{x\in Q}||f||_{\Phi,Q};$$

The Young functions to be using in this paper are $\Phi(t)=t(1+\log t)^r$ and $\tilde{\Phi}(t)=\exp(t^{1/r})$, the corresponding average and maximal functions denoted by $||\cdot||_{L(\log L)^r,Q}$, $M_{L(\log L)^r}$ and $||\cdot||_{\exp L^{1/r},Q}$, $M_{\exp L^{1/r}}$. Following [13–16], we know the generalized Hölder's inequality:

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|dy\leq ||f||_{\Phi,Q}||g||_{\tilde{\Phi},Q}$$

and the following inequality, for $r,r_j\geq 1, j=1,\cdots,l$ with $1/r=1/r_1+\cdots+1/r_l$, and any $b\in BMO(R^n)$,

$$\begin{aligned}
&||f||_{L(\log L)^{1/r},Q}\leq M_{L(\log L)^{1/r}}(f)\leq CM_{L(\log L)^l}(f)\leq CM^{l+1}(f),\\
&||b-b_Q||_{\exp L^r,Q}\leq C||b||_{BMO},\\
&|b_{2^{k+1}Q}-b_{2Q}|\leq Ck||b||_{BMO}.
\end{aligned}$$

The Muckenhoupt A_p weight is defined by (see [6], pp. 389–390)

$$A_p=\left\{w:\sup_Q\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1}<\infty\right\},\ 1<p<\infty,$$

and

$$A_1=\{w:M(w)(x)\leq Cw(x), a.e.\}.$$

We shall prove the following theorems.

Theorem 1.1. *Let $1<s<\infty$, $D^\alpha A_j\in BMO(R^n)$ for all α with $|\alpha|=m_j$ and $j=1,\cdots,l$. Then there exists a constant $C>0$ such that for any $f=\{f_i\}\in C_0^\infty(R^n)$, $0<r<1$ and $x\in R^n$,*

$$(|T^A(f)|_s)_r^\#(x)\leq C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M^{l+1}(|f|_s)(x).$$

Theorem 1.2. *Let $1 < s < \infty$, $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Then $|T^A|_s$ is bounded on $L^p(w)$ for any $1 < p < \infty$ and $w \in A_p$, that is*

$$|||T^A(f)|_s||_{L^p(w)} \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{BMO} \right) |||f|_s||_{L^p(w)},$$

Theorem 1.3. *Let $1 < s < \infty$, $w \in A_1$, $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Then there exists a constant $C > 0$ such that for all $\lambda > 0$,*

$$w(\{x \in R^n : |T^A(f)(x)|_s > \lambda\}) \le C \int_{R^n} \frac{|f(x)|_s}{\lambda} \left[1 + \log^+ \left(\frac{|f(x)|_s}{\lambda}\right)\right]^l w(x)dx.$$

2. Proof of theorem

To prove the theorems, we need the following lemmas.

Lemma 2.1. [3] *Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \le C|x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

Lemma 2.2. [6, p.485] *Let $0 < p < q < \infty$ and for any function $f \ge 0$. We define that, for $1/r = 1/p - 1/q$*

$$||f||_{WL^q} = \sup_{\lambda>0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q}, \quad N_{p,q}(f) = \sup_E ||f\chi_E||_{L^p}/||\chi_E||_{L^r},$$

where the sup is taken for all measurable sets E with $0 < |E| < \infty$. Then

$$||f||_{WL^q} \le N_{p,q}(f) \le (q/(q - p))^{1/p} ||f||_{WL^q}.$$

Lemma 2.3. [17] *Let $r_j \ge 1$ for $j = 1, \cdots, l$, we denote that $1/r = 1/r_1 + \cdots + 1/r_l$. Then*

$$\frac{1}{|Q|} \int_Q |f_1(x) \cdots f_l(x) g(x)| dx \le ||f||_{\exp L^{r_1}, Q} \cdots ||f||_{\exp L^{r_l}, Q} ||g||_{L(\log L)^{1/r}, Q}.$$

Proof of Theorem 1.1. It suffices to prove for $f = \{f_i\} \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|} \int_Q ||T^A(f)(x)|_s^r - C_0| dx \right)^{1/r} \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{BMO} \right) M^{l+1}(|f|_s)(x).$$

Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m_j} \frac{1}{\alpha!} (D^\alpha A_j)_{\tilde{Q}} x^\alpha$, then $R_{m_j}(A_j; x, y) = R_{m_j}(\tilde{A}_j; x, y)$ and $D^\alpha \tilde{A}_j = D^\alpha A_j - (D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We split $f = g + h = \{g_i\} + \{h_i\}$ for $g_i = f_i \chi_{\tilde{Q}}$ and $h_i = f_i \chi_{R^n \setminus \tilde{Q}}$. Write

$F_t^A(f_i)(x)$

$$
\begin{aligned}
&= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m} F_t(x,y) f_i(y) dy \\
&= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m} F_t(x,y) h_i(y) dy \\
&\quad + \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m} F_t(x,y) g_i(y) dy \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1}\tilde{A}_1(y) F_t(x,y) g_i(y) dy \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2}\tilde{A}_2(y) F_t(x,y) g_i(y) dy \\
&\quad + \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{A}_1(y) D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m} F_t(x,y) g_i(y) dy,
\end{aligned}
$$

then, by Minkowski's inequality,

$$
\begin{aligned}
&\left[\frac{1}{|Q|}\int_Q \left| |T^A(f)(x)|_s^r - |T^{\tilde{A}}(h)(x_0)|_s^r \right| dx\right]^{1/r} \\
&\le \left[\frac{1}{|Q|}\int_Q \left(\sum_{i=1}^\infty |T^A(f_i)(x) - T^{\tilde{A}}(h_i)(x_0)|^s dx\right)^{r/s}\right]^{1/r} \\
&\le \left[\frac{C}{|Q|}\int_Q \left(\sum_{i=1}^\infty \left|\left|\int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m} F_t(x,y) g_i(y) dy\right|\right|^s\right)^{r/s} dx\right]^{1/r} \\
&\quad + \left[\frac{C}{|Q|}\int_Q \left(\sum_{i=1}^\infty \left|\left|\sum_{|\alpha_1|=m_1} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}\right.\right.\right.\right. \\
&\qquad \left.\left.\left.\left. \times D^{\alpha_1}\tilde{A}_1(y) F_t(x,y) g_i(y) dy\right|\right|^s\right)^{r/s} dx\right]^{1/r} \\
&\quad + \left[\frac{C}{|Q|}\int_Q \left(\sum_{i=1}^\infty \left|\left|\sum_{|\alpha_2|=m_2} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}\right.\right.\right.\right. \\
&\qquad \left.\left.\left.\left. \times D^{\alpha_2}\tilde{A}_2(y) F_t(x,y) g_i(y) dy\right|\right|^s\right)^{r/s} dx\right]^{1/r} \\
&\quad + \left[\frac{C}{|Q|}\int_Q \left(\sum_{i=1}^\infty \left|\left|\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{A}_1(y) D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}\right.\right.\right.\right. \\
&\qquad \left.\left.\left.\left. \times F_t(x,y) g_i(y) dy\right|\right|^s\right)^{r/s} dx\right]^{1/r}
\end{aligned}
$$

$$+\left[\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}|T_{\tilde{A}}(h_i)(x)-T_{\tilde{A}}(h_i)(x_0)|^s\right)^{r/s}dx\right]^{1/r}$$
$$:= I_1+I_2+I_3+I_4+I_5.$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, for $x\in Q$ and $y\in\tilde{Q}$, by Lemma 2.1, we get

$$R_m(\tilde{A}_j;x,y)\le C|x-y|^m\sum_{|\alpha_j|=m}||D^{\alpha_j}A_j||_{BMO},$$

thus, by Lemma 2.2 and the weak type (1,1) of $|T|_r$, we obtain

$$\begin{aligned}
I_1 &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\left(\frac{1}{|Q|}\int_Q|T(g)(x)|_s^r dx\right)^{1/r}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|Q|^{-1}\frac{|||T(g)|_s\chi_Q||_{L^r}}{|Q|^{1/r-1}}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|Q|^{-1}|||T(g)|_s||_{WL^1}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|\tilde{Q}|^{-1}|||g|_s||_{L^1}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M(|f|_s)(\tilde{x}).
\end{aligned}$$

For I_2, we get, by Lemma 2.2 and generalized Hölder's inequality,

$$\begin{aligned}
I_2 &\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1g)(x)|_s^r dx\right)^{1/r}\\
&\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}|Q|^{-1}\frac{|||T(D^{\alpha_1}\tilde{A}_1g)|_s\chi_Q||_{L^r}}{|Q|^{1/r-1}}\\
&\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}|Q|^{-1}|||T(D^{\alpha_1}\tilde{A}_1g)|_s||_{WL^1}\\
&\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}|\tilde{Q}|^{-1}||D^{\alpha_1}\tilde{A}_1|g|_s||_{L^1}\\
&\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}||D^{\alpha_1}A_1-(D^{\alpha_1}A_1)_{\tilde{Q}}||_{\exp L,\tilde{Q}}|||f|_s||_{L(\log L),\tilde{Q}}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{BMO}\right)M^2(|f|_s)(\tilde{x}).
\end{aligned}$$

For I_3, similar to the proof of I_2, we get

$$I_3 \le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) M^2(|f|_s)(\tilde{x}).$$

Similarly, for I_4, taking $r, r_1, r_2 \ge 1$ such that $1/r = 1/r_1 + 1/r_2$, we obtain, by Lemma 2.3,

$$\begin{aligned}
I_4 &\le C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 g)(x)|_s^r dx \right)^{1/r} \\
&\le C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} |Q|^{-1} \frac{|||T(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 g)|_s \chi_Q||_{L^r}}{|Q|^{1/r-1}} \\
&\le C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} |Q|^{-1} |||T(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 g)|_s||_{WL^1} \\
&\le C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} |Q|^{-1} ||D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 |g|_s||_{L^1} \\
&\le C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \prod_{j=1}^{2} \left|\left| D^{\alpha_j} A_j - (D^{\alpha_j} A_j)_{\tilde{Q}} \right|\right|_{\exp L^{r_j}, \tilde{Q}} \cdot |||f|_s||_{L(\log L)^{1/r}, \tilde{Q}} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) M^3(|f|_s)(\tilde{x}).
\end{aligned}$$

For I_5, we write

$$\begin{aligned}
& F_t^{\tilde{A}}(h_i)(x) - F_t^{\tilde{A}}(h_i)(x_0) \\
&= \int_{R^n} \left(\frac{F_t(x,y)}{|x-y|^m} - \frac{F_t(x_0,y)}{|x_0-y|^m} \right) \prod_{j=1}^{2} R_{m_j}(\tilde{A}_j; x, y) h_i(y) dy \\
&\quad + \int_{R^n} \left(R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; x_0, y) \right) \frac{R_{m_2}(\tilde{A}_2; x, y)}{|x_0-y|^m} F_t(x_0,y) h_i(y) dy \\
&\quad + \int_{R^n} \left(R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x_0, y) \right) \frac{R_{m_1}(\tilde{A}_1; x_0, y)}{|x_0-y|^m} F_t(x_0,y) h_i(y) dy \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \left[\frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} F_t(x,y) - \frac{R_{m_2}(\tilde{A}_2; x_0, y)(x_0-y)^{\alpha_1}}{|x_0-y|^m} F_t(x_0,y) \right] \\
&\quad \times D^{\alpha_1}\tilde{A}_1(y) h_i(y) dy \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \left[\frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} F_t(x,y) - \frac{R_{m_1}(\tilde{A}_1; x_0, y)(x_0-y)^{\alpha_2}}{|x_0-y|^m} F_t(x_0,y) \right] \\
&\quad \times D^{\alpha_2}\tilde{A}_2(y) h_i(y) dy \\
&\quad + \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} F_t(x,y) - \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m} F_t(x_0,y) \right] \\
&\quad \times D^{\alpha_1}\tilde{A}_1(y) D^{\alpha_2}\tilde{A}_2(y) h_i(y) dy
\end{aligned}$$

$= I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)}.$

By Lemma 2.1 and the following inequality (see [18]):

$$|b_{Q_1} - b_{Q_2}| \le Clog(|Q_2|/|Q_1|)||b||_{BMO} \text{ for } Q_1 \subset Q_2,$$

we know that, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$\begin{aligned}|R_m(\tilde{A}; x, y)| &\le C|x-y|^m \sum_{|\alpha|=m} (||D^\alpha A||_{BMO} + |(D^\alpha A)_{\tilde{Q}(x,y)} - (D^\alpha A)_{\tilde{Q}}|) \\ &\le Ck|x-y|^m \sum_{|\alpha|=m} ||D^\alpha A||_{BMO}.\end{aligned}$$

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain, by the conditions on F_t,

$$\begin{aligned}||I_5^{(1)}|| &\le C \int_{R^n} \left(\frac{|x-x_0|}{|x_0-y|^{m+n+1}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon}}\right) \prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)||h_i(y)|dy \\ &\le C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO}\right) \\ &\quad \times \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k^2 \left(\frac{|x-x_0|}{|x_0-y|^{n+1}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon}}\right) |f_i(y)|dy \\ &\le C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO}\right) \sum_{k=1}^\infty k^2(2^{-k} + 2^{-\varepsilon k}) \frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |f_i(y)|dy,\end{aligned}$$

thus, by Minkowski's inequality,

$$\begin{aligned}\left(\sum_{i=1}^\infty ||I_5^{(1)}||^s\right)^{1/s} &\le C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO}\right) \\ &\quad \times \sum_{k=1}^\infty k^2(2^{-k} + 2^{-\varepsilon k}) \frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |f(y)|_s dy \\ &\le C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO}\right) M(|f|_s)(\tilde{x}).\end{aligned}$$

For $I_5^{(2)}$, by the formula (see [3]):

$$R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y) = \sum_{|\beta|<m} \frac{1}{\beta!} R_{m-|\beta|}(D^\beta \tilde{A}; x, x_0)(x-y)^\beta$$

and Lemma 2.1, we have

$$|R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y)| \le C \sum_{|\beta|<m} \sum_{|\alpha|=m} |x-x_0|^{m-|\beta|}|x-y|^{|\beta|}||D^\alpha A||_{BMO},$$

thus

$$\left(\sum_{i=1}^{\infty}||I_5^{(2)}||^s\right)^{1/s} \le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right) \times \sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k\frac{|x-x_0|}{|x_0-y|^{n+1}}|f(y)|_s dy \le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M(|f|_s)(\tilde{x}).$$

Similarly,

$$\left(\sum_{i=1}^{\infty}||I_5^{(3)}||^r\right)^{1/r} \le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M(|f|_s)(\tilde{x}).$$

For $I_5^{(4)}$, recall that $|b_{2^{k+1}Q}-b_{2Q}|\le Ck||b||_{BMO}$, we get

$$\begin{aligned}
&\left(\sum_{i=1}^{\infty}||I_5^{(4)}||^s\right)^{1/s} \\
&\le C\sum_{|\alpha_1|=m_1}\int_{R^n}\left|\left|\frac{(x-y)^{\alpha_1}F_t(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}F_t(x_0,y)}{|x_0-y|^m}\right|\right| \\
&\quad\times|R_{m_2}(\tilde{A}_2;x,y)||D^{\alpha_1}\tilde{A}_1(y)||h(y)|_s dy \\
&\quad+C\sum_{|\alpha_1|=m_1}\int_{R^n\setminus\tilde{Q}}|R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)|\frac{|(x_0-y)^{\alpha_1}|||F_t(x_0,y)||}{|x_0-y|^m} \\
&\quad\times|D^{\alpha_1}\tilde{A}_1(y)||h(y)|_s dy \\
&\le C\sum_{|\alpha|=m_2}||D^\alpha A_2||_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|_s|D^{\alpha_1}\tilde{A}_1(y)|dy \\
&\le C\sum_{|\alpha|=m_2}||D^\alpha A_2||_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})||D^{\alpha_1}A_1 \\
&\quad-(D^{\alpha_1}A_1)_{\tilde{Q}}||_{\exp L,2^k\tilde{Q}}\cdot|||f|_s||_{L(\log L),2^k\tilde{Q}} \\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)\sum_{k=1}^{\infty}k^2(2^{-k}+2^{-\varepsilon k})|||f|_s||_{L(\log L),2^k\tilde{Q}} \\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M^2(|f|_s)(\tilde{x}).
\end{aligned}$$

Similarly,

$$\left(\sum_{i=1}^{\infty}||I_5^{(5)}||^s\right)^{1/s} \le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M^2(|f|_s)(\tilde{x}).$$

For $I_5^{(6)}$, similarly to the proof of I_4, we get

$$\begin{aligned}
&\left(\sum_{i=1}^{\infty}||I_5^{(6)}||^s\right)^{1/s}\\
&\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{R^n\setminus\tilde{Q}}\left|\left|\frac{(x-y)^{\alpha_1+\alpha_2}F_t(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1+\alpha_2}F_t(x_0,y)}{|x_0-y|^m}\right|\right|\\
&\quad\times|D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||h(y)|_s dy\\
&\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|_s|D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)|dy\\
&\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\prod_{j=1}^{2}\left|\left|D^{\alpha_j}A_j-(D^{\alpha_j}A_j)_{\tilde{Q}})\right|\right|_{\exp L^{r_j},2^k\tilde{Q}}\cdot|||f|_s||_{L(\log L)^{1/r},2^k\tilde{Q}}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M^3(|f|_s)(\tilde{x}).
\end{aligned}$$

Thus

$$I_5 \le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M^3(|f|_s)(\tilde{x}).$$

This completes the proof of Theorem 1.1. ∎

By Theorem 1.1 and the $L^p(w)$-boundedness of M^3, we may obtain the conclusions of Theorem 1.2. By Theorem 1.1 and [8, 10], we may obtain the conclusions of Theorem 1.3.

3. Applications

Now we give some applications of theorems in this paper.

Application 1. Littlewood-Paley operators.

Fixed $\varepsilon > 0$ and $\mu > (3n+2)/n$. Let ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n}\psi(x)dx = 0$,
(2) $|\psi(x)| \le C(1+|x|)^{-(n+1)}$,
(3) $|\psi(x+y)-\psi(x)| \le C|y|^\varepsilon(1+|x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$;

We denote $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley multilinear operators are defined by

$$g_\psi^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$S_\psi^A(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2}$$

and

$$g_\mu^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} \psi_t(x-y) f(y)dy,$$

$$F_t^A(f)(x,y) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,z)}{|x-z|^m} f(z)\psi_t(y-z)dz$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define that

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$S_\psi(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}$$

and

$$g_\mu(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

which are the Littlewood-Paley operators (see [19]). Let H be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t\right)^{1/2} < \infty\right\}$$

or

$$H = \left\{h : ||h|| = \left(\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+1}\right)^{1/2} < \infty\right\},$$

then, for each fixed $x \in R^n$, $F_t^A(f)(x)$ and $F_t^A(f)(x,y)$ may be viewed as the mapping from $[0,+\infty)$ to H, and it is clear that

$$g_\psi^A(f)(x) = ||F_t^A(f)(x)||, \quad g_\psi(f)(x) = ||F_t(f)(x)||,$$

$$S_\psi^A(f)(x) = \left||\chi_{\Gamma(x)} F_t^A(f)(x,y)\right||, \quad S_\psi(f)(x) = \left||\chi_{\Gamma(x)} F_t(f)(y)\right||$$

and

$$g_\mu^A(f)(x) = \left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t^A(f)(x,y)\right|\right|,$$

$$g_\mu(f)(x) = \left\| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t(f)(y) \right\|.$$

It is easily to see that g_ψ, S_ψ and g_μ satisfy the conditions of Theorems 1.1, 1.2 and 1.3 (see [8–10]), thus Theorems 1.1, 1.2 and 1.3 hold for g_ψ^A, S_ψ^A and g_μ^A.

Application 2. Marcinkiewicz operators.

Fix $\lambda > \max(1, 2n/(n+2))$ and $0 < \gamma \le 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$. The Marcinkiewicz multilinear operators are defined by

$$\mu_\Omega^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

$$\mu_S^A(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2}$$

and

$$\mu_\lambda^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{|x-y|\le t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, y)}{|x-y|^m} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy$$

and

$$F_t^A(f)(x,y) = \int_{|y-z|\le t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; y, z)}{|y-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz;$$

Set

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy;$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

$$\mu_S(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2}$$

and

$$\mu_\lambda(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2},$$

which are the Marcinkiewicz operators (see [20]). Let H be the space

$$H = \left\{ h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3 \right)^{1/2} < \infty \right\}$$

or

$$H = \left\{ h : ||h|| = \left(\int \int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+3} \right)^{1/2} < \infty \right\}.$$

Then, it is clear that

$$\mu_\Omega^A(f)(x) = ||F_t^A(f)(x)||, \quad \mu_\Omega(f)(x) = ||F_t(f)(x)||,$$

$$\mu_S^A(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t^A(f)(x,y)\right|\right|, \quad \mu_S(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t(f)(y)\right|\right|$$

and

$$\mu_\lambda^A(f)(x) = \left|\left| \left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} F_t^A(f)(x,y) \right|\right|,$$

$$\mu_\lambda(f)(x) = \left|\left| \left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} F_t(f)(y) \right|\right|.$$

It is easily to see that μ_Ω, μ_S and μ_λ satisfy the conditions of Theorems 1.1, 1.2 and 1.3 (see [11] and [20]), thus Theorems 1.1, 1.2 and 1.3 hold for μ_Ω^A, μ_S^A and μ_λ^A.

Application 3. Bochner-Riesz operator .

Let $\delta > (n-1)/2$, $B_t^\delta(f)\hat{}(\xi) = (1 - t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n}B^\delta(z/t)$ for $t > 0$. Set

$$F_{\delta,t}^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, y)}{|x-y|^m} B_t^\delta(x-y) f(y) dy,$$

The maximal Bochner-Riesz multilinear operator are defined by

$$B_{\delta,*}^A(f)(x) = \sup_{t>0} |B_{\delta,t}^A(f)(x)|.$$

We also define that

$$B_{\delta,*}(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)|$$

which is the maximal Bochner-Riesz operator (see [13]). Let H be the space $H = \{h : ||h|| = \sup_{t>0} |h(t)| < \infty\}$, then

$$B_{\delta,*}^A(f)(x) = ||B_{\delta,t}^A(f)(x)||, \quad B_*^\delta(f)(x) = ||B_t^\delta(f)(x)||.$$

It is easily to see that $B_{\delta,*}^A$ satisfies the conditions of Theorems 1.1, 1.2 and 1.3 (see [21]), thus Theorems 1.1, 1.2 and 1.3 hold for $B_{\delta,*}^A$.

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