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Prime Ideals in Semirings

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Abstract. In this paper, we prove the following theorems:
(1) A nonzero ideal I of $(\mathbb{Z}^+, +, \cdot)$ is prime if and only if $I = \langle p \rangle$ for some prime number p or $I = \langle 2, 3 \rangle$.
(2) Let R be a reduced semiring. Then a prime ideal P of R is minimal if and only if $P = A_P$ where $A_P = \{r \in R : \exists\ a \notin P \text{ such that } ra = 0\}$.

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1. Introduction

There are many characterizations of prime ideals in semirings in the literature (e.g. cf. [2]). In this paper, we give a characterization of prime ideals to be minimal in reduced semirings.

Definition 1.1. *A nonempty set R with two binary operations $+$ and $\cdot$ is called a semiring if*

(1) $(R, +)$ *is a commutative monoid with identity element* 0.
(2) $(R, \cdot)$ *is a monoid with identity element* $1 \neq 0$.
(3) *Both the distributive laws hold in* R.
(4) $a \cdot 0 = 0 \cdot a = 0$ *for all* $a \in R$.

We assume that all ideals are proper. $\mathbb{Z}^+$ will denote the set of all nonnegative integers. Let I be an ideal of a semiring R and let $a, b \in R$. Define $a \sim b$ if and only if there exist $x, y \in I$ such that $a + x = b + y$. Then $\sim$ is an equivalence relation on R. Let $[a]_I^R$ or $[a]$ be the equivalence class of $a \in R$. Then $R/I = \{[a]_I^R : a \in R\}$ is a semiring under the binary operations defined as follows: $[a] + [b] = [a + b]$, $[a][b] = [ab]$ for all $a, b \in R$. This semiring is called the Bourne factor semiring of R by I. We assume that $[0] \neq R$. An ideal I of a semiring R is called subtractive if whenever a, $a + b \in I$, $b \in R$, we have $b \in I$. An ideal I of a semiring R is called

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prime (completely prime) if whenever $aRb \subseteq I$ ($ab \in I$) where $a, b \in R$ implies $a \in I$ or $b \in I$. A semiring R is called reduced if it has no nonzero nilpotent elements.

Let R be a semiring and let $x \in R$. Then $r(x) = \{a \in R : xa = 0\}$ is called the right annihilator of x. Similarly we can define $\ell(x)$, the left anihilator of x. We write $RxR = \langle x \rangle$. Let $X(R)$ be the set of all prime ideals of a semiring R. Let I be an ideal of R. Define support of I as follows: $\operatorname{supp} I = \{P \in X(R) : I \not\subseteq P\}$. Let $\tau = \{\operatorname{supp} I : I \text{ is an ideal of } R\}$. Then $(X(R), \tau)$ is a topological space.

2. Prime Ideals in $\mathbb{Z}^+$

Now we give a short and elementary proof of the following lemma which will be used in the subsequent theorem.

Lemma 2.1. [1, Lemma 7] *Let $a, b \in \mathbb{Z}^+$, $b > a > 1$ and let $(a, b) = 1$. Then there exists $n \in \mathbb{Z}^+$ such that $t \in \langle a, b \rangle$ for all $t \geq n$.*

Proof. Since $(a, b) = 1$, there exist $p, q \in \mathbb{Z}^+$ such that $qa = pb + 1$. Clearly $p, q \neq 0$. Let us write $n = paqb \in \langle a, b \rangle$. Let $t = n + r$ where $r \geq 0$. If $r = 0$ then $t = n \in \langle a, b \rangle$. If $0 < r < a$ then $t = n + r = (pa - r)pb + pa + rqa \in \langle a, b \rangle$. If $r \geq a$ then by the division algorithm, $r = au + v$, $0 \leq v < a$. Now $t = n + v + au \in \langle a, b \rangle$. ∎

Theorem 2.1. *A nonzero ideal I of $(\mathbb{Z}^+, +, \cdot)$ is prime if and only if $I = \langle p \rangle$ for some prime number p or $I = \langle 2, 3 \rangle$.*

Proof. Let I be a nonzero prime ideal of $(\mathbb{Z}^+, +, \cdot)$. If $I = \langle p \rangle$ then p is a prime number. Suppose I is not a principal ideal. Let a be the nonzero smallest element of I. Then a is a prime number. Also there exists the smallest $b \in I$ such that $b > a$ and $(a, b) = 1$. By Lemma 2.1, there exists $n \in \mathbb{Z}^+$ such that $t \in I$ for all $t \geq n$. If $a > 2$ then choose the smallest $j \in \mathbb{Z}^+$, $j > 1$ such that $2^j \in I$. So $2^{j-1} \in I$ or $2 \in I$, a contradiction. Hence $a = 2$. If $b > 3$ then choose the smallest $k \in \mathbb{Z}^+$, $k > 1$ such that $3^k \in I$. So $3^{k-1} \in I$ or $3 \in I$, a contradiction. Hence $b = 3$. Now $I = \langle 2, 3 \rangle = \mathbb{Z}^+ - \{1\}$. The converse is obvious. ∎

The following result follows directly from Theorem 2.1.

Corollary 2.1. *A nonzero ideal I of $(\mathbb{Z}^+, +, \cdot)$ is subtractive prime if and only if $I = \langle p \rangle$ for some prime number p.*

3. Prime ideals in semirings

We give the following lemmas and proposition which will be used in the subsequent study.

Lemma 3.1. *Let R be reduced semiring and let $0 \neq x \in R$. Then*

(1) *$r(x) = \ell(x)$ and it is an ideal of R*
(2) *$x \notin r(x)$*
(3) *$r(x)$ is subtractive*
(4) *$R/r(x)$ is reduced*
(5) *If $xa \in r(x)$ or $ax \in r(x)$ then $a \in r(x)$*

Proof. (1), (2), (3) and (5) are obvious. For (4), let $[a] \in R/r(x)$ such that $[a]^n = 0$ for some n. Then $a^n + u = v$ for some $u, v \in r(x)$. Since $r(x)$ is subtractive, $a^n \in r(x)$. Since R is reduced, we have $xa = 0$. Now $a \in r(x)$. Hence $[a] = [0]$. ∎

Let P be a prime ideal of semiring R. Denote $A_P = \{s \in R : \exists\ a \notin P$ such that $sa = 0\}$.

We give the following examples of a subset A_P in a semiring.

Example 3.1. Let $\mathbb{Z}^+ = (\mathbb{Z}^+, +, \dot{})$. Then the prime ideal P of $\mathbb{Z}^+$ is 0 or $\langle p \rangle$ for some prime p or $\langle 2, 3 \rangle$ by Theorem 2.1. We have $A_P = 0$.

Example 3.2. Let $\mathbb{Z}_2^+$ be the full matrix semiring of order 2 over the semiring $\mathbb{Z}^+$. Then $P = 0$ is a prime ideal of $\mathbb{Z}_2^+$ (the proof is similar as for an arbitrary prime ring with identity element). From this we have $A_P = \left\{ \begin{bmatrix} \mathbb{Z}^+ & 0 \\ \mathbb{Z}^+ & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathbb{Z}^+ \\ 0 & \mathbb{Z}^+ \end{bmatrix} \right\}$.

Now we have the following:

Lemma 3.2. *Let R be a reduced semiring and let P be a prime ideal of R. Then*

(1) *A_P is a subtractive (proper) ideal of R*
(2) *$A_P \subseteq P$*
(3) *R/A_P is reduced*

Proof. (1) Let $r_1, r_2 \in A_P$. Then $\exists\ a_1, a_2 \notin P$ such that $r_1 a_1 = r_2 a_2 = 0$. Hence $(r_1 + r_2) a_1 y a_2 = 0$ where $y \in R$. Since $a_1 y a_2 \notin P$ for some $y \in R$, we have $(r_1 + r_2) \in A_P$. Also $rt, tr \in A_P$ for all $r \in A_P$ and $t \in R$. Let $r, r + s \in A_P$, $s \in R$. Then $(r + s)a = rb = 0$ for some $a, b \notin P$. Since $0 = (r + s)ayb = sayb$ and $ayb \notin P$ for some $y \in R$, we get $s \in A_P$.

(2) It is obvious.

(3) Suppose $[s]^n = [0]$ for some n. Then $s^n \in A_P$. So $s^n a = 0$ for some $a \notin P$. Since R is reduced, we get $sa = 0$. Now $s \in A_P$. Hence $[s] = [0]$. ∎

Lemma 3.3. *Let $I \subseteq H$ be ideals of a semiring R and let H be subtractive. Then $R/H \cong (R/I)/(H/I)$.*

Proof. Since H is subtractive, we have $H = [0]_H^R$. Now it follows by [2, Proposition 10.20]. ∎

Proposition 3.1. *Let R be a reduced semiring and let $0 \neq x \in R$. Then $G(x) = \{I : I$ is a subtractive ideal of R, $x \notin I$, $rx \in I$ implies that $r \in I$, R/I is reduced$\}$ has a maximal element. Moreover every maximal member of $G(x)$ is a completely prime ideal of R.*

Proof. Since $r(x) \in G(x)$, so it is a nonempty partially ordered set under $\subseteq$, in which every totally ordered subset has an upper bound. By Zorn's Lemma, $G(x)$ has a maximal element K.

Let $ab \in K$. Suppose $a \notin K$. Let us write $N = \{y \in R : ay \in K\}$. Then $K \subseteq N$. Since K is subtractive and R/K is reduced, we see that N is a subtractive ideal of R and $x \notin N$. If $rx \in N$ then $r \in N$. Easily, $N/K = r([a]_K^R)$. By Lemmas 3.3 and 3.1, $R/N \cong (R/K)/(N/K)$ is reduced. Thus $N \in G(x)$ and so $K = N$. Now $b \in K$. ∎

Theorem 3.1. *Let R be a reduced semiring. Then prime ideal P of R is minimal if and only if $P = A_P$. Moreover if $P = A_P$ then P is a completely prime ideal of R.*

Proof. Let P be a minimal prime ideal of R. We have $A_P \subseteq P$. Suppose $A_P \neq P$. Then there exists a nonzero element $a \in P$ such that $a \notin A_P$. Let $M = R - P$. Then M is an m-system. Let $S = \{a, a^2, a^3, \ldots\}$ and let $T = \{r \in R : r \neq 0, r = a^{i_0}x_0a^{i_1}x_1 \ldots a^{i_n}x_na^{i_{n+1}}$, where $i_0, i_{n+1}, n \geq 0$; $i_1, i_2, \ldots, i_n \geq 1$, $x_j \in M$ for all $j, 1 \leq j \leq n\}$.

Then $\delta = M \cup S \cup T$ is an m-system: Clearly $0 \notin \delta$. Let $x, y \in \delta$.

(i) Let $x \in M$. If $y \in M$ then there exists $r \in R$ such that $xry \in M \subseteq \delta$. If $y \in S$ then $y = a^n$ for some $n \geq 1$. Let $r = a$. Suppose $xry = 0$. Hence $ax = 0$. Since $x \notin P$, $a \in A_P$, a contradiction. Hence $xry \neq 0$. Now $xry \in T \subseteq \delta$. If $y \in T$ then $y = a^{i_0}x_0a^{i_1}x_1 \ldots a^{i_n}x_na^{i_{n+1}}$. Let $r = a$. Suppose $xry = xaa^{i_0}x_0a^{i_1}x_1a^{i_2} \ldots a^{i_n}xa^{i_{n+1}} = 0$. Since R is reduced, we get

$$ax x_0 x_1 \ldots x_n = 0 \tag{3.1}$$

Now $x, x_0, x_1, \ldots, x_n \in M$ and M is a m-system. Hence there exist $r_0, r_1, \ldots, r_n \in R$ such $w = xr_0x_0r_1x_1 \ldots x_{n-1}r_nx_n \in M$. Inserting $r_0, r_1, \ldots r_n$ in (3.1), we get $aw = axr_0x_0r_1x_1 \ldots x_{n-1}r_nx_n = 0$ where $w \notin P$. Hence $a \in A_P$, a contradiction. So $xry \neq 0$. Now $xry \in T \subseteq \delta$.

(ii) Let $x \in S$. Then $x = a^n$ for some $n \geq 1$. If $y \in S$ then $y = a^m$ for some $m \geq 1$. Let $r = a$. Then $xry = a^{n+m+1} \in S \subseteq \delta$. If $y \in T$ then $y = a^{i_0}x_0a^{i_1}x_1 \ldots a^{i_n}x_na^{i_{n+1}}$. Let $r = a$. Suppose $xry = a^{n+1+i_0}x_0a^{i_1}x_1 \ldots a^{i_n}x_n a^{i_{n+1}} = 0$. Since R is reduced, we have

$$ax_0x_1 \ldots x_n = 0. \tag{3.2}$$

Now $x_0, x_1, \ldots, x_n \in M$ and M is an m-system. Hence there exist $r_0, r_1, \ldots, r_n \in R$ such that $w = xr_0x_0r_1x_1 \ldots r_{n-1}x_{n-1}r_nx_n \in M$. Inserting $r_0, r_1, \ldots, r_n$ in (3.2), we get $aw = 0$ where $w \notin P$. Hence $a \in A_P$, a contradiction. So $xry \neq 0$. Now $xry \in T \subseteq \delta$.

(iii) Let $x, y \in T$. Then

$$x = a^{i_0}x_0a^{i_1}x_1 \ldots a^{i_n}x_na^{i_{n+1}} \quad \text{and} \quad y = a^{j_0}y_0a^{j_1}y_1 \ldots a^{j_m}y_ma^{j_{m+1}}.$$

Let $r = a$. Suppose

$$xry = a^{i_0}x_0a^{i_1}x_1 \ldots a^{i_n}x_na^{i_{n+1}}aa^{j_0}y_0a^{j_1}y_1 \ldots a^{j_m}y_ma^{j_{m+1}} = 0.$$

Since R is reduced, we get

$$ax_0x_1 \ldots x_ny_0y_1 \ldots y_m = 0. \tag{3.3}$$

Now $x_0, x_1, \ldots, x_n, y_0, y_1, \ldots y_m \in M$ and M is an m-system. Hence there exist $r_0, r_1, \ldots, r_{n-1}, t, s_0, \ldots, s_{m-1} \in R$ such that

$$w = x_0r_0x_1r_1x_2 \ldots x_{n-1}r_{n-1}x_nty_0s_0y_1s_1y_2 \ldots y_{m-1}s_{m-1}y_m \in M.$$

Inserting $r_0, r_1, \ldots, r_{n-1}, t, s_0, s_1, \ldots s_{m-1}$ in (3.3), we get $aw = 0$ where $w \notin P$. Hence $a \in A_P$, a contradiction. So $xry \neq 0$. Now $xry \in T$.

Let Q be an ideal of R maximal with respect to the property that $\delta \cap Q = \emptyset$. Then Q is a prime ideal of R and $Q \subsetneq P$, a contradiction to the minimality of P. Hence $A_P = P$. Conversely, let $A_P = P$. Let P' be a prime ideal of R such that $P' \subseteq P$. Let $x \in P = A_P$. Then$xb = 0$ for some $b \notin P$. Hence $xRb = 0 \subseteq P'$ implies that $x \in P'$. So $P = P'$. Now P is a minimal prime ideal of R. Easily, if $A_P = P$ then by Lemma 3.2, P is a completely prime ideal of R. ∎

Motivated by the above theorem, as a consequence, we obtain the following result.

Proposition 3.2. *Let R be a reduced semiring and let α be a subspace of $X(R)$ which consists of all minimal prime ideals of R. Then α is a Hausdorff space having a base of open and closed sets.*

Proof. Let $P_1, P_2 \in \alpha$ such that $P_1 \neq P_2$. By Theorem 3.1, $P_1 = A_{P_1}$ and $P_2 = A_{P_2}$. Hence $A_{P_1} \nsubseteq P_2$. Then there exists $x \in A_{P_1}$ such that $x \notin P_2$. Hence $\exists\ s \notin P_1$ such that $xs = 0$. Since R is reduced, we get $\langle x\rangle\langle s\rangle = 0$. Suppose $\mathrm{supp}\langle x\rangle \cap \mathrm{supp}\langle s\rangle \neq \phi$. Let $P \in \mathrm{supp}\langle x\rangle \cap \mathrm{supp}\langle s\rangle$. Then $\langle x\rangle \nsubseteq P$ and $\langle s\rangle \nsubseteq P$, a contradiction. Hence $\mathrm{supp}\langle x\rangle \cap \mathrm{supp}\langle s\rangle = \phi$. Since $\langle s\rangle \nsubseteq P_1$ and $\langle x\rangle \nsubseteq P_2$, we have $P_1 \in \mathrm{supp}\langle s\rangle$ and $P_2 \in \mathrm{supp}\langle x\rangle$. For any nonzero element $a \in R$, we have $\mathrm{supp}\langle a\rangle = \alpha - \mathrm{supp}(r(a))$:

Let $P \in \mathrm{supp}\langle a\rangle$. Then $\langle a\rangle \nsubseteq P$. Hence $a \notin P$. Thus $r(a) \subseteq P$. Hence $P \notin \mathrm{supp}(r(a))$. Otherway, let $P \in \alpha - \mathrm{supp}(r(a))$. Then $P \notin \mathrm{supp}(r(a))$. Hence $r(a) \subseteq P$. Since P is a minimal prime ideal, we have $P = A_P$. Suppose $a \in P$. Then $a \in A_P$. Hence $\exists\ b \notin P$ such that $ab = 0$. Then $b \in r(a) \subseteq P$, a contradiction. Hence $a \notin P$. Now $\langle a\rangle \nsubseteq P$. So $P \in \mathrm{supp}\langle a\rangle$. ∎

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