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On f-Edge Cover Coloring of Nearly Bipartite Graphs

[1]JINBO LI AND [2]GUIZHEN LIU
[1,2]School of Mathematics, Shandong University,
Jinan, Shandong, 250100, P. R. China
[1]supremelion@gmail.com, [2]gzliu@sdu.edu.cn

Abstract. Let $G(V,E)$ be a graph, and let f be an integer function on V with $1 \leq f(v) \leq d(v)$ to each vertex $v \in V$. An f-edge cover coloring is an edge coloring C such that each color appears at each vertex v at least $f(v)$ times. The f-edge cover chromatic index of G, denoted by $\chi'_{fc}(G)$, is the maximum number of colors needed to f-edge cover color G. It is well-known that

$$\min_{v\in V}\left\lfloor \frac{d(v)-\mu(v)}{f(v)} \right\rfloor \leq \chi'_{fc}(G) \leq \delta_f,$$

where $\mu(v)$ is the multiplicity of v and $\delta_f = \min\{\lfloor \frac{d(v)}{f(v)} \rfloor : v \in V(G)\}$. If $\chi'_{fc} = \delta_f$, then G is of f_c-class 1, otherwise G is of f_c-class 2. In this paper, we give some new sufficient conditions for a nearly bipartite graph to be of f_c-class 1.

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1. Introduction

Our terminology and notation will be standard. The reader is referred to [1] for the undefined terms. Graphs in this paper allows multiple edges but no loops and has a finite vertex set V and a finite nonempty edge set E. G is simple if it has no multiple edges. Given two vertices $u, v \in V(G)$, the multiplicity $\mu(uv)$ is the number of edges joining u and v in G. The multiplicity of v is $\mu(v) = \max\{\mu(uv) : u \in V(G)\}$. Set $\mu = \max\{\mu(v) : v \in V(G)\}$. When G has no multiple edges (that is $\mu = 1$), G is a simple graph. Let $N_G(v)$ denote the neighborhood of v and let $d(v) = |N_G(v)|$ be the degree of v in G. Let δ denote the minimum degree of G and

$$\delta_f = \min\left\{ \left\lfloor \frac{d(v)}{f(v)} \right\rfloor : v \in V(G) \right\},$$

in which $\lfloor \frac{d(v)}{f(v)} \rfloor$ is the largest integer not greater than $\frac{d(v)}{f(v)}$.

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A nearly bipartite graph is a graph G with a vertex u of G such that $G-u$ is a bipartite graph with bipartition (X,Y), denoted by $G(X,Y;u)$.

Let f be a positive integer function defined on $V(G)$ such that $1 \le f(v) \le d(v)$ for any $v \in V$. A *k-edge coloring* C of a graph G is an assignment of k colors, $\{1,2,\cdots,k\}$, to the edges of G. The coloring C is *proper* if no two adjacent edges have the same color. Unless otherwise stated, the edge coloring of graphs in this paper are not necessarily proper. An f-edge cover coloring is an edge coloring C such that each color appears at each vertex v at least $f(v)$ times. Clearly, the f-edge cover coloring may not be proper. The f-edge cover chromatic index of G, denoted by $\chi'_{fc}(G)$, is the maximum number of colors needed to f-edge cover color G. In our daily life many problems on optimization and network design, for example, coding design, building blocks, the file transfer problem on computer networks, schedule problems and so on [6], are related to the f-edge cover coloring which is firstly presented by Song and Liu [8].

It is easy to verify that $\chi'_{fc}(G) \le \delta_f$. Song and Liu studied the bound of f-edge cover chromatic index of multigraphs and obtained the following result [8].

Theorem 1.1. [8] *Let G be a multigraph. Let $f(v)$, $d(v)$, and δ_f be defined as earlier. Then*

$$\min_{v\in V}\left\lfloor \frac{d(v)-\mu(v)}{f(v)} \right\rfloor \le \chi'_{fc}(G) \le \delta_f.$$

We say that a graph G is of f_c-class 1 if $\chi'_{fc} = \delta_f$, otherwise G is of f_c-class 2. In general, the problem of determining the f-edge cover chromatic index of graphs is NP-hard because deciding edge covering chromatic index of graphs is NP-complete [10], which is the special case of our general problem.

If $f(v) = 1$ for any $v \in V(G)$, then the f-edge cover coloring problem reduces to the classical edge covering coloring of G which is an edge coloring such that the edges assigned the same color form an edge cover of G. Let $\chi'_c(G)$ be the edge cover chromatic index of G. Gupta [2] obtained an important result that $\min\{d(v)-\mu(v) : v \in V\} \le \chi'_c(G) \le \delta$. Similarly, we say a graph G is of $C1$ class if $\chi'_c(G) = \delta$, otherwise G is of $C2$ class. Wang, Zhang and Liu [10] gave some sufficient conditions for a nearly bipartite simple graph to be $C1$ class. One of which is as follows.

Theorem 1.2. [10] *Let $G(X,Y;u)$ be a nearly bipartite simple graph with minimum degree $\delta \ge 3$. If $d(u) \ge 2\delta - 1$, then $G(X,Y;u)$ is of $C1$ class.*

Xu and Jia [12] obtained a sufficient condition for a nearly bipartite graph to be of $C1$ class as an extension of Theorem 1.2.

Theorem 1.3. [12] *Let $G(X,Y;u)$ be a nearly bipartite graph. If there exists a vertex $y \in N_G(u)$, such that $d(u)+d(y) \ge 3\delta - 1$, then $G(X,Y;u)$ is of $C1$ class.*

In this paper, we studied the classification problem of nearly bipartite graphs and obtained the following main results.

Theorem 1.4. *Let $G(X,Y;u)$ be a nearly bipartite graph with f-minimum degree $\delta_f \ge 3$. If $d(u) \ge \delta_f(f(u)+1) - 1$, then $G(X,Y;u)$ is of f_c-class 1.*

Theorem 1.5. *Let $G(X,Y;u)$ be a nearly bipartite graph. If there exists a vertex $y \in N_G(u)$, such that $d(u)+d(y) \geq \delta_f(f(u)+1)+\delta_f f(y)-1$, then $G(X,Y;u)$ is of f_c-class 1.*

2. Main results

Before proving our main results, we need some more preliminary terminologies and results. Given a k-edge coloring C of a graph G, we shall denote by $c_i(v)$ the number of color i represented at v. For any $v \in V$ and $1 \leq i \leq k$, let

$$\sigma_i(v) = \max\{0, f(v)-c_i(v)\}, and\ \sigma(v) = \sum_{i=1}^{k} \sigma_i(v).$$

Clearly, C is an f-edge cover coloring of G if and only if $\sigma(v)=0$ for all vertex $v \in V$.

Let C be a k-edge coloring of G and $\sigma = \Sigma_{v\in V}\sigma(v)$. If we can recolor the edges of G with k colors and get a new coloring $C^{'}$ of G such that $\sigma^{'} < \sigma$, we say that $C^{'}$ is an *improved coloring* of G. If C can not be improved, we call C an *optimal coloring* of G.

A connected graph G is called Eulerian if each vertex of G has even degree. The following lemma is very useful which can be found in [5].

Lemma 2.1. [5] *Let $G(V,E)$ be a connected graph. Then G has a 2-edge coloring C such that*

(a) *If G is Eulerian and $|E|$ is odd, then for an arbitrarily selected $u \in V$, we have $|c_1(u)-c_2(u)|=2$ and $c_1(v)-c_2(v)=0$ for all $v \in V-u$.*
(b) *If G is Eulerian and $|E|$ is even, then $c_1(v)-c_2(v)=0$ for all $v \in V$.*
(c) *If G is not Eulerian, then $|c_1(v)-c_2(v)| \leq 1$ for all $v \in V$.*
(d) *In all cases, we have*

$$||E(1)|-|E(2)|| = \begin{cases} 0 & \text{if } |E| \text{ is even;} \\ 1 & \text{if } |E| \text{ is odd.} \end{cases}$$

Let C be an edge coloring of G and let α and β be two of the used colors. For the sake of simplicity, we write $c_\alpha(v)$ by $\alpha(v)$ sometimes. Let $E(\alpha)$ be the set of edges receiving color α in an edge coloring C of G. The edge induced subgraph $E(\alpha)\bigcup E(\beta)$ is denoted by $G(\alpha,\beta)$. For any $v_0 \in V(G)$, let $G(v_0;\alpha,\beta)$ be the component containing v_0 of the subgraph of $G(\alpha,\beta)$ of G. We call a subgraph H of G an obstruction (about C), if $H=G(v_0;\beta,\alpha)$ is Eulerian with $|E(H)|$ is odd and $\alpha(x)=\beta(x)=f(x)$ for all $x \in V(H)-\{v_0\}$, $\alpha(v_0)=\beta(v_0)+2=f(v_0)+1$.

Let C be an optimal edge coloring of G with δ_f colors. If G is not of f_c-class 1. Then there exists $v \in V(G)$, $i,j \in \{1,2,\cdots,\delta_f\}$, such that $i(v) \geq f(v)+1$ and $j(v) \leq f(v)-1$. If $H=G(v;i,j)$ is not an obstruction, by Lemma 2.1, we can get an improved coloring, which contradicts to the fact that C is optimal. So $H=G(v;i,j)$ is an obstruction. Since $G(X,Y;u)$ is a nearly bipartite graph, so $u \in V(H)$. By Lemma 2.1, we can recolor $G(v;i,j)$ such that $i(u)=f(u)+1$, $j(u)=f(u)-1$ and $i(v)=j(v)=f(v)$ for all $v \in V(H)-\{u\}$. By iterating this process, we can get an optimal edge coloring $C^{'}$ of G with δ_f colors such that $i(v) \geq f(v)$ and $j(v) \geq f(v)$

for any $v \in V(G) - \{u\}$ and $i \in \{1, 2, \cdots, \delta_f\}$. Such a coloring is called a *standard optimal δ_f-edge coloring.*

By Lemma 2.1, we have:

Lemma 2.2. *Let G be a graph and $\delta_f(G) \geq 2$. There exists an f-edge cover coloring of G with two colors if and only if each component of G is not an obstruction.*

Proof. Without loss of generality, suppose that G is connected. Assume first that G is not an obstruction. By Lemma 2.1, we may assume that G is Eulerian and $|E(G)|$ is odd. Since G is not an obstruction, there is a vertex $x \in V(G)$ with $d(x) \geq 2f(x) + 2$. We choose x as u in Lemma 2.1. By Lemma 2.1, G has a 2-edge-coloring C such that $|1(x) - 2(x)| = 2$ and $1(v) - 2(v) = 0$ for all $v \in V - \{x\}$. It follows that $1(x) \geq f(x)$ and $2(x) \geq f(x)$ since $d(x) \geq 2f(x) + 2$. Note that $\delta_f \geq 2$, this implies that $1(v) \geq f(v)$ and $2(v) \geq f(v)$ for all $v \in V - \{x\}$. Thus, G has an f-edge cover coloring with 2 colors. The necessity is obvious. ∎

By Lemma 2.1, we obtained the following useful lemmas.

Lemma 2.3. *Let C be a k-edge-coloring of $G (2 \leq k \leq \delta_f)$. If there is a vertex u and two colors i and j such that $c_i(u) \geq f(u) + 1$, $c_j(u) \leq f(u) - 1$, then there is an improved coloring by recoloring the edges of $G(u; i, j)$ using i and j if and only if $G(u; i, j)$ is not an obstruction.*

Lemma 2.4. *Let C be a δ_f-edge coloring of graph G. If $|c_\alpha(v_0) - c_\beta(v_0)| > 2$ for some pair α, β of colors and some vertex v_0, then G has a δ_f-edge coloring C' such that*

$$\max_{1 \leq i < j \leq \delta_f} |c_i'(v_0) - c_j'(v_0)| \leq 2$$

and

$$\sigma(v) \geq \sigma'(v)$$

for all $v \in V(G)$.

The following concept is given in [3]. Let G be edge colored and let α and β be two of the used colors. Let $G(v; \beta, \alpha)$ be the component containing v of the subgraph of G induced by the edges colored α and β. An (β, α) exchange chain L of G is a sequence $(v_0, e_1, v_1, e_2, \cdots, v_{r-1}, e_r, v_r)$ of vertices and edges of G in which

(a) For $1 \leq i \leq r$, the vertices v_{i-1} and v_i are distinct and both incident with the edge e_i,
(b) The edges are all distinct and are colored alternately by α and β,
(c) e_1 is colored by α and $\alpha(v_0) > \beta(v_0)$. Similarly, let γ denote the color of e_r and $\overline{\gamma}$ denote the other color of $\{\alpha, \beta\}$. When $v_0 \neq v_r$, $\gamma(v_r) > \overline{\gamma}(v_r)$; when $v_0 = v_r$, then $\gamma = \alpha$ and $c_\alpha(v_0) > c_\beta(v_0 + 1)$.

An (β, α) exchange chain L is called *minimal* if there exists no other (β, α) exchange chain L' which is starting at the same vertex as L and $L' \subset L$.

If $c_\alpha(v_0) > c_\beta(v_0)$, the existence of an (β, α) exchange chain starting at v_0 is proved in [7].

Lemma 2.5. *Let C be an edge coloring of G, and $G(u; i, j)$ be an obstruction, where $e = uy$ has color i. We can get an improved coloring if one of the following is satisfied.*

(a) $\alpha(u) > f(u)+1$ *for some color* α *of* C;
(b) $\alpha(y) > f(y)+1$ *for some color* α *of* C;
(c) $\alpha(u) = f(u)+1$ *and* $\alpha(y) = f(y)+1$ *for some color* α *of* C.

Proof. If (a) is satisfied, that is $\alpha(u) > f(u)+1$ for some color α of C, then $G(u;\alpha,j)$ is not an obstruction. By Lemma 2.3, we can get an improved coloring.

If (b) happens, that is $\alpha(y) > f(y)+1$ for some color α of C, recolor e with j. Then $G(y;\alpha,i)$ is not an obstruction. By Lemma 2.3, we can get an improved coloring.

If (c) is satisfied, then $\alpha(u) = f(u)+1$ and $\alpha(y) = f(y)+1$ for some color α of C. In $G(y;\alpha,i)-e$, choose a minimal (i,α) exchange chain L starting at y. L must be in one of the following three cases.

(1) L does not stop at u or y.
(2) L stops at u with an edge colored α.
(3) L stops at y with an edge colored α.

We first exchange the two colors on L. If one of the cases (1) and (2) happens, recolor e with j. It is easy to verify that the resulting coloring is an improved coloring. When (3) happens, recolor e with α. Then in the new coloring, $G(u;\alpha,j)$ is not an obstruction. By Lemma 2.3, we obtained an improved coloring. ∎

Proof of Theorem 1.4. Let C be an optimal δ_f-edge coloring of $G(X,Y;u)$. If $\sigma(v) = 0$ holds for all vertex v of $G(X,Y;u)$, then $G(X,Y;u)$ is of f_c-class 1. Now we assume that $\sigma(v_0) > 0$ for some vertex v_0 of $G(X,Y;u)$. Since $d(v_0) \geq f(v_0)\delta_f$, there must be some color, say α, is represented at most $f(v_0)-1$ times at v_0 and some other color, say β is represented at least $f(v_0)+1$ times at v_0, that is $c_\alpha(v_0) \leq f(v_0)-1$ and $c_\beta(v_0) \geq f(v_0)+1$. By Lemma 2.3, then the component of $G[E_\alpha \bigcup E_\beta]$ that contains v_0, denoted by $G(v_0;\alpha,\beta)$, is Eulerian which has odd number of edges. So $G(v_0;\alpha,\beta)$ must contain an odd cycle, denote by $H(v_0;\alpha,\beta)$. Since $G(X,Y;u)$ is a nearly bipartite graph, $G(X,Y;u)-u$ contain no odd cycle, it follows that any odd cycle in $G(X,Y;u)$ must contain vertex u, i.e., u is a vertex of $H(v_0;\alpha,\beta)$. If $v_0 \neq u$, we can recolor $G(v_0;\alpha,\beta)$ with colors α, β such that $\sigma(v_0)$ increasing and $\sigma(u)$ may be reduced, but for other vertices, whose σ has no change. Repeating this process for the vertices which satisfy $\sigma(v) > 0$ $(\forall v \in X \bigcup Y)$. We finally obtain a δ_f-edge coloring C' such that $\sigma'(v) = 0$ for any $v \neq u$ and may be $|c_i'(u) - c_j'(u)| > 2$ for some pair of color i and j. By Lemma 2.4, $G(X,Y;u)$ has a δ_f-edge coloring C'' such that $|c_i''(u) - c_j''(u)| \leq 2$ for any $i,j \in \{1,\cdots,\delta_f\}$ and for each vertex $v \neq u$, $\sigma''(v) = 0$. Now we consider the value of $\sigma(u)$. Since $d(u) \geq \delta_f(f(u)+1)-1$ and $|c_i''(u) - c_j''(u)| \leq 2$ for any $i,j \in \{1,\cdots,\delta_f\}$. It follows that $\sigma''(u) = 0$. As we have known, C'' is an f-edge cover coloring with δ_f colors if and only if $\sigma''(v) = 0$ holds for all vertices v of $G(X,Y;u)$. The proof is completed. ∎

Proof of Theorem 1.5. It is well-known that all bipartite graphs are of f_c-class 1. So we can suppose that $G(X,Y;u)$ is connected. For $\delta_f = 1$, the assertion is trivial. For $\delta_f = 2$, by Lemma 2.2, connected graph is of f_c-class 1 if and only if it is not an obstruction. So from the condition that there exits a vertex $y \in N_G(u)$ such that $d(u)+d(y) \geq \delta_f(f(u)+1)+\delta_f f(y)-1$, we got that $G(X,Y;u)$ is not an obstruction, which means G is of f_c-class 1.

Now we assume that $\delta_f \geq 3$ and C is a standard optimal δ_f-edge coloring of G. By contradiction, we suppose that C is not an f-edge cover coloring of G. Then there exists a color $\beta \in \{1, 2, \cdots, \delta_f\}$ such that $\beta(u) = f(u) - 1$. If there exists a color $i \neq \beta$ with $i(u) > f(u) + 1$, by Lemma 2.5, we can get an improved coloring, which induces a contradiction to the choice of C. So for any $i \in \{1, 2, \cdots, \delta_f\} - \{\beta\}$, $i(u) \leq f(u) + 1$ and $\beta(u) = f(u) - 1$. Thus if $d(u) \geq \delta_f(f(u) + 1) - 1$, we get a contradiction. So G is of f_c-class 1 if $d(u) \geq \delta_f(f(u) + 1) - 1$.

Now we suppose that $d(u) \leq \delta_f(f(u)+1)-2$. Set $d(u) = \delta_f f(u)+p$, $0 \leq p \leq \delta_f-2$. Thus there are at least $p+1$ colors, each of which appears $f(u)+1$ times at u. The set of such colors is denoted by M_u. Clearly, $\beta \notin M_u$. For $y \in N_G(u)$, suppose that one of the edge $e = uy$ is colored by α. Without loss of generality, we can suppose that $\alpha(u) = f(u) + 1$. In fact, since $|M_u| \geq p + 1$, if $\alpha(u) = f(u)$, there is an other color γ such that $\gamma(u) = f(u) + 1$. Let L be a minimum (α, γ) exchange chain starting at u. Since $\gamma(u) = f(u) + 1$, $\alpha(u) = f(u)$ and L is minimal, so L doesn't contain the edge e and doesn't end at u. Exchanging the colors on L, we get an other standard optimal δ_f-edge coloring $C^{'}$ such that $\alpha(u) = f(u) + 1$, and e is still colored by α. From the above analysis, we can assume that $\alpha(u) = f(u) + 1$ in coloring C.

Since C is optimal, $H = G(u; \alpha, \beta)$ is an obstruction. Since $d(u) + d(y) \geq \delta_f(f(u)+1)+\delta_f f(y)-1$, we have $d(y) \geq \delta_f(f(y)+1)-p-1$. From C is a standard optimal δ_f-edge coloring and by Lemma 2.5, we have $f(y) \leq i(y) \leq f(y) + 1$ for any $i \in \{1, 2, \cdots, \delta_f\}$. So there are $\delta_f - p - 1$ colors each of which appears $f(y) + 1$ times at y. Set the set of such colors be M_y. From the analysis above, we know that $|M_u| + |M_y| \geq \delta_f$ and $M_u \cup M_y \subseteq \{1, 2, \cdots, \delta_f\}$. Now we will consider the relations of the two colors set M_u and M_y.

Case 1. $\alpha \in M_u \cap M_y$. We have $\alpha(u) = f(u)+1$ and $\alpha(y) = f(y)+1$. Since $e = uy$ is colored by α, we can recolor e by color β. Hence we get an improved coloring, which induces a contradiction to the choice of C.

Case 2. $\alpha \notin M_u \cap M_y$.

Since C is optimal, we know that $H = G(u; \alpha, \beta)$ is an obstruction. Thus, $\alpha(y) = \beta(y) = f(y)$, that is to say, $\beta \notin M_u \cup M_y$, so we have $|M_u \cup M_y| \leq \delta_f - 1$. Since $|M_u| + |M_y| \geq \delta_f$, so we have that there exists a color $\gamma \in M_u \cap M_y$. That is $\gamma(u) = f(u) + 1$ and $\gamma(y) = f(y) + 1$, by Lemma 2.5, we get an improved coloring, which induces a contradiction to the choice of C.

From the analysis of each case above, we get a contradiction. So C is an f-edge cover coloring of G with δ_f colors. The proof is completed. ∎

Remark 2.1. Theorem 1.4 is best possible in the following sense. The condition that $d(u) \geq \delta_f(f(u) + 1) - 1$ cannot be replaced by $d(u) \geq \delta_f(f(u) + 1) - 2$. For example, let $K_{n,n}$ denote a complete bipartite graph. Construct a nearly bipartite graph G by adding a new vertex u and joining u to each vertex of $K_{n,n}$. Let $f(v) = 1$ for any $v \in V(G)$. Clearly, $|V(G)| = 2n+1$, $|E(G)| = n^2+2n$ and $\delta_f = n+1$, but the cardinality of each f-edge cover of G is at least $n + 1$, so $|E(G)| < \lceil \frac{|V(G)|}{2} \rceil \cdot \delta_f$, and then G is of f_c-class 2. So we can say that the condition that $d(u) \geq \delta_f(f(u)+1)-1$

is sharp in this sense. The same example also shows that the bound in Theorem 1.5 is sharp.

Other interesting results on the f-edge cover coloring and f-edge coloring can be found in [4, 9, 11, 13].

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