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On Chebyshev's Polynomials and Certain Combinatorial Identities

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Abstract. Let $T_n(x)$ and $U_n(x)$ be the Chebyshev's polynomial of the first kind and second kind of degree n, respectively. For $n \geq 1$, $U_{2n-1}(x) = 2T_n(x)U_{n-1}(x)$ and $U_{2n}(x) = (-1)^n A_n(x)A_n(-x)$, where $A_n(x) = 2^n \prod_{i=1}^{n}(x-\cos i\theta)$, $\theta = 2\pi/(2n+1)$. In this paper, we will study the polynomial $A_n(x)$. Let $A_n(x) = \sum_{m=0}^{n} a_{n,m}x^m$. We prove that $a_{n,m} = (-1)^k 2^m \binom{l}{k}$, where $k = \lfloor \frac{n-m}{2} \rfloor$ and $l = \lfloor \frac{n+m}{2} \rfloor$. We also completely factorize $A_n(x)$ into irreducible factors over $\mathbb{Z}$ and obtain a condition for determining when $A_r(x)$ is divisible by $A_s(x)$. Furthermore we determine the greatest common divisor of $A_r(x)$ and $A_s(x)$ and also greatest common divisor of $A_r(x)$ and the Chebyshev's polynomials. Finally we prove certain combinatorial identities that arise from the polynomial $A_n(x)$.

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1. Introduction

Chebyshev's polynomials are of great importance in many area of mathematics, particularly approximation theory. Interesting properties of the Chebyshev's polynomials can be found in [9] and [10]. Certain algebraic properties of Chebyshev's polynomials have been studied by Bang [1], Carlitz [3], and Rankin [7]. In 1984, Hsiao [5] gave a complete factorization of Chebyshev's polynomials of the first kind into irreducible factors over the ring of integer $\mathbb{Z}$. Using Hsiao's method, Rivlin [9] extended it to complete factorization of Chebyshev's polynomials of the second kind. Certain decomposition properties of Chebyshev's polynomials including factorization and divisibility have been studied by Rayes, Trevisan, and Wang [8].

The *Chebyshev's polynomials of the first kind* $T_n(x)$ can be defined inductively as follow:

$$T_0(x) = 1, \qquad T_1(x) = x, \qquad \text{and}$$

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$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \qquad n = 2, 3, \ldots \tag{1.1}$$

Alternatively, it may be defined as

$$T_n(x) = \cos n(\arccos x),$$

where $0 \le \arccos x \le \pi$. The roots of $T_n(x)$ are

$$\cos \frac{(2k-1)\pi}{2n}, \qquad k = 1, 2, \ldots, n.$$

The *Chebyshev's polynomials of the second kind* $U_n(x)$ is defined inductively as follow:

$$U_0(x) = 1, \qquad U_1(x) = 2x, \qquad \text{and}$$
$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \qquad n = 2, 3, \ldots \tag{1.2}$$

Alternatively, it may be defined as

$$U_n(x) = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)},$$

where $0 \le \arccos x \le \pi$. The roots of $U_n(x)$ are

$$\cos \frac{k\pi}{n+1}, \qquad k = 1, 2, \ldots, n.$$

Note that the leading coefficients of $T_n(x)$ and $U_n(x)$ are 2^{n-1} and 2^n, respectively, for $n \ge 1$. By looking at the roots of $U_{2n-1}(x)$, we see that

$$U_{2n-1}(x) = 2T_n(x)U_{n-1}(x), \quad n = 1, 2, \ldots. \tag{1.3}$$

For $U_{2n}(x)$, the roots are $\cos(k\pi/(2n+1))$, where $k = 1, 2, \ldots, 2n$. Note that for $1 \le i \le n$, $\cos((2i-1)\pi/(2n+1)) = -\cos(2(n-i+1)\pi/(2n+1))$. Therefore

$$U_{2n}(x) = (-1)^n A_n(x)A_n(-x), \quad n = 1, 2, \ldots, \tag{1.4}$$

where $A_n(x) = 2^n \prod_{i=1}^{n}(x - \cos(2i\pi/(2n+1)))$.

In this paper, we will study the polynomial $A_n(x)$. We will completely factorize $A_n(x)$ into irreducible factors over $\mathbb{Z}$ and prove certain combinatorial identities that arise from the polynomial $A_n(x)$.

2. Properties of $A_n(x)$

Let $\theta = 2\pi/(2n+1)$. The θ will be fixed throughout the paper.

Let us look at the polynomial $T_{n+1}(x) - T_n(x)$. Note that $T_{n+1}(1) - T_n(1) = 0$ and for $i = 1, 2, \ldots, n$,

$$\begin{aligned} T_{n+1}(\cos i\theta) - T_n(\cos i\theta) &= -2\sin\left(\frac{(2n+1)i\theta}{2}\right)\sin\left(\frac{i\theta}{2}\right) \\ &= -2\sin(i\pi)\sin\left(\frac{i\theta}{2}\right) = 0. \end{aligned}$$

This implies Lemma 2.1.

Lemma 2.1. $(x-1)A_n(x) = T_{n+1}(x) - T_n(x)$ *for* $n = 1, 2, \ldots$.

For the sake of completeness, we define $A_0(x) = 1$. This leads (1.4) and Lemma 2.1 to be true even for $n = 0$. Now by Lemma 2.1 and (1.1), $A_1(x) = 2x + 1$. Furthermore one can deduce Lemma 2.2.

Lemma 2.2. *$A_n(x) = 2xA_{n-1}(x) - A_{n-2}(x)$ for $n = 2, 3, \ldots$.*

Recall that a polynomial $p(x) \in \mathbb{Z}[x]$ is said to *divide* $h(x) \in \mathbb{Z}[x]$ or is a *divisor* of $h(x)$ if $h(x) = p(x)l(x)$ for some $l(x) \in \mathbb{Z}[x]$. A polynomial $h(x) \in \mathbb{Z}[x]$ is said to be *irreducible* if the only divisors of $h(x)$ are ± 1 and $\pm h(x)$.

A number $\zeta \in \mathbb{C}$ is said to be an *algebraic number* if there is a $p(x) \in \mathbb{Z}[x]$ with $p(\zeta) = 0$. Furthermore if $p(x)$ is irreducible and of degree k, we say ζ is *algebraic of degree k*. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $a_i \in \mathbb{Z}$ for all i. If $a_n = 1$, we say ζ is an *algebraic integer*. So an algebraic integer is an algebraic number.

Given any $r(x), s(x) \in \mathbb{Z}[x]$, the *greatest common divisor* of $r(x)$ and $s(x)$ will be denoted by $\gcd(r(x), s(x))$. Note that the leading coefficient of the greatest common divisor will be chosen to be positive. Consider a fixed integer $n \geq 1$. Let l_h denote the number of elements in

$$S_h = \{i \ : \ \gcd(i, 2n+1) = h, 1 \leq i \leq n\}.$$

Clearly $l_h = \phi((2n+1)/h)/2$, where ϕ is the *Euler's totient function*. Properties of ϕ can be found in [4, p. 52]. Now let

$$F_h(x) = 2^{l_h} \prod_{\substack{1 \leq i \leq n \\ \gcd(i, 2n+1) = h}} (x - \cos i\theta).$$

Theorem 2.1. *For $n \geq 1$,*

$$A_n(x) = \prod_h F_h(x),$$

where $h \leq n$ runs through all positive divisors of $2n+1$. All the F_h are irreducible over $\mathbb{Z}$.

Proof. Clearly $A_n(x) = \prod_h F_h(x)$. So it is sufficient to show that F_h are irreducible over $\mathbb{Z}$. By Lehmer's Theorem [6, Theorem 1], if $\gcd(i, 2n+1) = 1$ then $2\cos(i\theta)$ is an algebraic integer of degree $\phi(2n+1)/2$. Following the proof of Lehmer's Theorem, we see that all $2\cos(i\theta)$ with $\gcd(i, 2n+1) = 1$ are the roots of the same irreducible polynomial, say $Q(x)$. Note that $Q(2x)$ is also irreducible and $F_1(x) = Q(2x)$. Now if $\gcd(i, 2n+1) = h$ then $\gcd(i/h, (2n+1)/h) = 1$ and $2\cos(i\theta/h)$ is an algebraic integer of degree $\phi((2n+1)/h)/2$. As in the previous paragraph, F_h is irreducible. ∎

An immediate consequence of Theorem 2.1 is the following corollary.

Corollary 2.1. *For all $n \in \mathbb{N}$,*

(a) *$F_1(x)$ is the irreducible factor of $A_n(x)$ of the largest degree $= \phi(2n+1)/2$.*
(b) *The number of irreducible factors of $A_n(x)$ equal to the number of divisors $h \leq n$ of $2n+1$.*

Corollary 2.2. *$A_n(x)$ is irreducible if and only if $n = (p-1)/2$ for some prime p.*

Proof. If $n = (p-1)/2$ for some prime p, then by (b) of Corollary 2.1, $A_n(x)$ is irreducible. Suppose $A_n(x)$ is irreducible. If $2n+1$ is not a prime, then $2n+1 = rs$ for some $r, s \in \mathbb{N}$, $r > s > 1$. This implies that $2n+1 > s^2$ and $s \leq n$. But then by (b) of Corollary 2.1, the number of irreducible factors of $A_n(x)$ is at least 2, a contradiction. Hence $2n+1$ is a prime. ∎

Let $\psi_m(x)$ be the minimal polynomial of $\cos(2\pi/m)$. If $m = 2n+1$, then (see [12, Theorem])

$$T_{n+1}(x) - T_n(x) = 2^n \prod_{d|m} \psi_d(x).$$

Therefore by Lemma 2.1, $A_n(x) = \left(2^n \prod_{d|m} \psi_d(x)\right)/(x-1) = 2^n \prod_{d|m, d\neq 1} \psi_d(x)$. When m is a prime, $A_n(x) = 2^n \psi_m(x)$. The polynomial $\psi_m(x)$ when m is a prime has been studied by Beslin and de Angelis [2], and Surowski and McCombs [11].

Let $A_n(x) = \sum_{m=0}^{n} a_{n,m} x^m$. Given any real number $x \in \mathbb{R}$, we shall denote the greatest integer less than or equal to x by $\lfloor x \rfloor$, and we shall denote the smallest integer greater than or equal to x by $\lceil x \rceil$. As usual, the *binomial coefficient* $\binom{r}{t}$ is the coefficient of x^t in the polynomial expansion of $(1+x)^r$. Recall that $A_0(x) = 1$ and $A_1(x) = 2x+1$. By Lemma 2.2, $A_2(x) = 4x^2 + 2x - 1$.

Theorem 2.2. *Let* $k = \left\lfloor \frac{n-m}{2} \right\rfloor$ *and* $l = \left\lfloor \frac{n+m}{2} \right\rfloor$. *Then*

$$a_{n,m} = (-1)^k 2^m \binom{l}{k} \qquad \text{for } 0 \leq m \leq n.$$

Proof. It can be verified that $a_{n,m} = (-1)^k 2^m \binom{l}{k}$ for all $0 \leq m \leq n$ where $n = 0, 1, 2$. Let $n \geq 3$. Assume that the formula holds for $a_{n',m'}$, for all $0 \leq m' \leq n'$ with $1 \leq n' < n$. Now $A_n(x) = 2xA_{n-1}(x) - A_{n-2}(x)$ (Lemma 2.2) implies that

$$\begin{aligned} a_{n,0} &= -a_{n-2,0}, \\ a_{n,m} &= 2a_{n-1,m-1} - a_{n-2,m} \quad \text{for } 1 \leq m \leq n-2, \\ a_{n,m} &= 2a_{n-1,m-1} \quad \text{for } n-1 \leq m \leq n. \end{aligned}$$

Therefore $a_{n,0} = (-1)^{\lfloor n/2 \rfloor}$, $a_{n,n-1} = 2^{n-1}$, $a_{n,n} = 2^n$ and for all $1 \leq m \leq n-2$,

$$\begin{aligned} a_{n,m} &= 2a_{n-1,m-1} - a_{n-2,m} \\ &= (-1)^k 2^m \binom{l'}{k} + (-1)^k 2^m \binom{l'}{k-1} \\ &= (-1)^k 2^m \binom{l}{k}, \end{aligned}$$

where $l' = \lfloor (n+m-2)/2 \rfloor$. Here we make use of the facts that $\lfloor (t-2)/2 \rfloor = \lfloor t/2 \rfloor - 1$ for all $t \in \mathbb{Z}$, $\binom{r}{s} + \binom{r}{s-1} = \binom{r+1}{s}$ for all $r, s \in \mathbb{N}$, and induction hypothesis. Hence the proof is complete. ∎

Note that $a_{n,0} = (-1)^{\lfloor n/2 \rfloor}$ and $a_{n,n} = 2^n$. Recall that a polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ is said to be *primitive* if $a_n > 0$ and $\gcd(a_n, a_{n-1}, \ldots, a_1, a_0) = 1$. Therefore $A_n(x)$ is primitive.

Corollary 2.3. $A_n(x)$ *is primitive for all integer* $n \geq 0$.

Theorem 2.3. *Let $r \geq s$ be two positive integers. Then $A_s(x)$ divides $A_r(x)$ if and only if $r = (2l+1)s + l$ for some integer $l \geq 0$.*

Proof. Suppose $r = (2l+1)s+l$ for some integer $l \geq 0$. Then the roots of $A_r(x)$ are

$$\cos\left(\frac{2i\pi}{2r+1}\right) = \cos\left(\frac{2i\pi}{(2l+1)(2s+1)}\right) \qquad \text{for } i = 1, 2, \ldots, r.$$

By taking $i_j = (2l+1)j$ for $j = 1, 2, \ldots, s$, we see that $\cos(2j\pi/(2s+1))$ are roots of $A_r(x)$. Note that $\cos(2j\pi/(2s+1))$ are roots of $A_s(x)$. So together with the division algorithm, we have $A_r(x) = H(x)A_s(x)$ for some $H(x) \in \mathbb{Q}[x]$. By Corollary 2.3, $A_r(x)$ and $A_s(x)$ are primitive. Using a standard argument as in [4, Proof of Theorem 237 on p. 205], we may assume that $H(x) \in \mathbb{Z}[x]$. Hence $A_s(x)$ divides $A_r(x)$.

Suppose $A_s(x)$ divides $A_r(x)$. Then $A_s(-x)$ divides $A_r(-x)$. By (1.4), $U_{2s}(x)$ divides $U_{2r}(x)$. Then by [8, Theorem 3], $2r = (l'+1)2s + l'$ for some integer $l' \geq 0$. Clearly, $l' = 2l$ for some integer l. Hence $r = (2l+1)s + l$. ∎

Corollary 2.4. *Let r, s be two nonnegative integers and $\gcd(2r+1, 2s+1) = t$. Then $\gcd(A_r(x), A_s(x)) = A_{(t-1)/2}(x)$.*

Proof. Let $\gcd(A_r(x), A_s(x)) = g(x)$. By Theorem 2.3, $A_{(t-1)/2}(x)$ divides $A_r(x)$ and $A_s(x)$. If $g(x)$ is of degree $(t-1)/2$, then $g(x) = A_{(t-1)/2}(x)$, and we are done. Suppose the degree of $g(x)$ is greater than $(t-1)/2$. Note that $g(-x)$ divides $A_r(-x)$ and $A_s(-x)$. This implies that $g(x)g(-x)$ divides $A_r(x)A_r(-x)$ and $A_s(x)A_s(-x)$. By (1.4), we see that $g(x)g(-x)$ divides $U_{2r}(x)$ and $U_{2s}(x)$. Now $\gcd(U_{2r}(x), U_{2s}(x)) = U_{t-1}(x)$ (see [8, Theorem 4]). But then $g(x)g(-x)$ divides $U_{t-1}(x)$, a contradiction, for the degree of $g(x)g(-x)$ is greater than $t-1$. Hence $\gcd(A_r(x), A_s(x)) = A_{(t-1)/2}(x)$. ∎

Theorem 2.4. *Let r, s be two nonnegative integers. Then* $\gcd(A_r(x), A_s(-x)) = 1$.

Proof. If either $r = 0$ or $s = 0$, we are done. So we may assume $r \geq s \geq 1$. Suppose $\gcd(A_r(x), A_s(-x)) \neq 1$. Then $-\cos(2i'\pi/(2s+1))$ is a root of $A_r(x)$ for some $1 \leq i' \leq s$. This implies that $\cos(2i'\pi/(2s+1)) + \cos(2i\pi/(2r+1)) = 0$ for some $1 \leq i \leq r$. Therefore

$$2\cos\left(\frac{(2r+1)i' + (2s+1)i}{(2s+1)(2r+1)}\pi\right)\cos\left(\frac{(2r+1)i' - (2s+1)i}{(2s+1)(2r+1)}\pi\right) = 0. \tag{2.1}$$

Note that the first term in (2.1) is zero if and only if $2((2r+1)i' + (2s+1)i) = (2s+1)(2r+1)t$ for some odd t. But this is impossible. Now the second term in (2.1) is zero if and only if $2((2r+1)i' - (2s+1)i) = (2s+1)(2r+1)t_1$ for some odd t_1, which is again impossible. Hence $\gcd(A_r(x), A_s(-x)) = 1$. ∎

Corollary 2.5. *For any nonnegative integers r, s,*

(a) $\gcd(A_r(x), A_r(-x)) = 1$.
(b) $\gcd(U_r(x), A_s(x)) = A_{(t-1)/2}$, *where* $t = \gcd(r+1, 2s+1)$.
(c) $\gcd(T_r(x), A_s(x)) = 1$.

Proof. (a) follows from Theorem 2.4.

(b) By [8, Theorem 4], $\gcd(U_r(x), U_{2s}(x)) = U_{t-1}$ where $t = \gcd(r+1, 2s+1)$. Note that $\gcd(A_s(x), A_{(t-1)/2}(-x)) = 1$ and $A_{(t-1)/2}(x)$ divides $A_s(x)$ (see Theorem

2.4 and Theorem 2.3). Recall that $U_{t-1}(x) = (-1)^{(t-1)/2} A_{(t-1)/2}(x) A_{(t-1)/2}(-x)$ (see (1.4)). Therefore $\gcd(U_r(x), A_s(x)) = A_{(t-1)/2}$.

(c) By (1.3), $U_{2r-1}(x) = 2T_r(x)U_{r-1}(x)$. By part (b), $\gcd(U_{2r-1}(x), A_s(x)) = A_{(t-1)/2}(x)$, where $t = \gcd(2r, 2s+1)$, and $\gcd(U_{r-1}(x), A_s(x)) = A_{(t'-1)/2}(x)$, where $t' = \gcd(r, 2s+1)$. Note that $t' = t$. Let $\gcd(T_r(x), A_s(x)) = d(x)$. Then $d(x)$ divides $U_{2r-1}(x)$ and thus $A_{(t-1)/2}(x)$. In fact $d(x)$ divides $U_{2r-1}(x)/A_{(t-1)/2}(x)$. Since all the roots of $U_{2r-1}(x)$ are distinct, we conclude that $\gcd(T_r(x), A_s(x)) = 1$. ∎

3. Certain combinatorial identities

Now if $P(x) = \sum_{i=0}^{n} c_i x^i$ is a polynomial of degree n with roots r_i(not necessarily distinct), $i = 1, 2, \dots, n$, then $P(x) = c_n \prod_{i=1}^{n}(x - r_i)$. By expanding $\prod_{i=1}^{n}(x - r_i)$ and comparing the coefficient of x^{n-m}, we have the following Vieta's formula.

Proposition 3.1. [Vieta's formula]

$$\sum_{1 \le \alpha_1 < \alpha_2 < \dots < \alpha_m \le n} r_{\alpha_1} r_{\alpha_2} \dots r_{\alpha_m} = (-1)^m \frac{c_{n-m}}{c_n}.$$

By Theorem 2.2 and Proposition 3.1, we have Corollary 3.1.

Corollary 3.1.

$$\sum_{1 \le r_1 < r_2 < \dots < r_m \le n} \cos\, r_1\theta \cos\, r_2\theta \dots \cos\, r_m\theta = (-1)^{\overline{m}} \frac{\binom{n-\overline{m}}{\underline{m}}}{2^m},$$

where $\overline{m} = \lceil m/2 \rceil$ *and* $\underline{m} = \lfloor m/2 \rfloor$.

Recall that

$$T_n(x) = \frac{(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n}{2}, \quad x \in \mathbb{R}. \tag{3.1}$$

By Lemma 2.1, we have Proposition 3.2.

Proposition 3.2.

$$A_n(x) = \frac{(x + \sqrt{x^2-1})^n (x - 1 + \sqrt{x^2-1}) + (x - \sqrt{x^2-1})^n (x - 1 - \sqrt{x^2-1})}{2(x-1)}.$$

Corollary 3.2 follows from Theorem 2.2 and Proposition 3.2 (take limit $x \to \pm 1$).

Corollary 3.2.

$$A_n(1) = \sum_{m=0}^{n} (-1)^{\lfloor \frac{n-m}{2} \rfloor} 2^m \binom{\lfloor \frac{n+m}{2} \rfloor}{\lfloor \frac{n-m}{2} \rfloor} = 2n+1 \qquad \text{and}$$

$$A_n(-1) = \sum_{m=0}^{n} (-1)^{m + \lfloor \frac{n-m}{2} \rfloor} 2^m \binom{\lfloor \frac{n+m}{2} \rfloor}{\lfloor \frac{n-m}{2} \rfloor} = (-1)^n.$$

Now

$$\begin{aligned} A_n(-1) &= (-2)^n \prod_{i=1}^{n}(1 + \cos i\theta) \\ &= (-2)^n \left(1 + \sum_{m=1}^{n} \sum_{1 \le r_1 < r_2 < \dots < r_m \le n} \cos\, r_1\theta \cos\, r_2\theta \dots \cos\, r_m\theta\right). \end{aligned}$$

Then using Corollary 3.2, we deduce Corollary 3.3.

Corollary 3.3.

$$\sum_{m=1}^{n} \sum_{1\le r_1<r_2<\ldots<r_m\le n} \cos\ r_1\theta \cos\ r_2\theta \ldots \cos\ r_m\theta = \frac{1-2^n}{2^n}.$$

For $\lfloor n/2\rfloor + 1 \le i \le n$, we have $n+1 \le 2i \le 2n$ and $1 \le 2n+1-2i \le n$. This implies that

$$\{2i\ :\ 1 \le i \le \lfloor n/2\rfloor\} \cup \{2n+1-2i\ :\ \lfloor n/2\rfloor + 1 \le i \le n\} = \{1, 2, \ldots, n\}.$$

Now $\cos((2n+1-2i)\theta) = \cos(2i\theta)$. Therefore

$$A_n(x) = 2^n \prod_{i=1}^{n}(x - \cos i\theta) = 2^n \prod_{i=1}^{n}(x - \cos 2i\theta). \tag{3.2}$$

Let $B_n(x) = A_n(2x-1)$. Then Lemma 3.1 follows from (3.2).

Lemma 3.1. *The roots of $B_n(x)$ are $\cos^2 i\theta$, $i = 1, 2, \ldots, n$.*

Note that $B_0(x) = 1$, $B_1(x) = 4x-1$ and by Lemma 2.2, we have the following recurrence relation for $B_n(x)$.

Lemma 3.2. $B_n(x) = 2(2x-1)B_{n-1}(x) - B_{n-2}(x)$ *for* $n = 2, 3, \ldots$.

Let $B_n(x) = \sum_{m=0}^{n} b_{n,m}x^m$. By mathematical induction and Lemma 3.2 (similar to the proof of Theorem 2.2), one can determine $b_{n,m}$.

Theorem 3.1.

$$b_{n,m} = (-1)^{n-m}4^m \binom{m+n}{2m} \qquad \text{for } 0 \le m \le n.$$

Note that $b_{n,0} = (-1)^n$ and $b_{n,n} = 4^n$. So $B_n(x)$ is primitive.

Corollary 3.4. *$B_n(x)$ is primitive for all integer $n \ge 0$.*

Now Corollary 3.5 follows from Theorem 3.1 and Proposition 3.1, and Corollary 3.6 follows from Proposition 3.2.

Corollary 3.5.

$$\sum_{1\le r_1<r_2<\ldots<r_m\le n} \cos^2\ r_1\theta \cos^2\ r_2\theta \ldots \cos^2\ r_m\theta = (-1)^{n-m}4^{-m}\binom{2n-m}{m}.$$

Corollary 3.6.

$$B_n(x) = \frac{(h(x))^n(h(x)-1) + (g(x))^n(g(x)-1)}{4(x-1)},$$

where $h(x) = 2x - 1 + \sqrt{(2x-1)^2 - 1}$ *and* $g(x) = 2x - 1 - \sqrt{(2x-1)^2 - 1}$.

As in Corollary 3.2, Corollary 3.7 follows from Theorem 3.1 and Corollary 3.6 by taking limit $x \to \pm 1$.

Corollary 3.7.

$$B_n(1) = \sum_{m=0}^{n} (-1)^{n-m} 4^m \binom{m+n}{2m} = 2n+1 \qquad \text{and}$$

$$B_n(-1) = (-1)^n \sum_{m=0}^{n} 4^m \binom{m+n}{2m} = -\frac{(h(-1))^n(h(-1)-1)+(g(-1))^n(g(-1)-1)}{8},$$

where $h(x)$ and $g(x)$ are as in Corollary 3.6.

As in Corollary 3.3, we can deduce Corollary 3.8.

Corollary 3.8.

$$\sum_{m=1}^{n} \sum_{1\le r_1<r_2<\ldots<r_m\le n} \cos^2\ r_1\theta \cos^2\ r_2\theta \ldots \cos^2\ r_m\theta = (-1)^n \frac{B_n(-1)}{4^n} - 1.$$

Recall that $U_{2n}(x) = (-1)^n A_n(x) A_n(-x)$, (see (1.4)). So

$$U_{2n}(x) = 4^n \prod_{i=1}^{n} (x^2 - \cos^2 i\theta) = B_n(x^2) \quad \text{and}$$

$$B_n(x) = (-1)^n A_n(x^{1/2}) A_n(-x^{1/2}), \quad n = 0, 1, \ldots. \tag{3.3}$$

By Theorem 2.2, Theorem 3.1 and (3.3), we have the following corollary.

Corollary 3.9.

$$\sum_{i=0}^{2m} \binom{\lfloor \frac{n+i}{2} \rfloor}{i} \binom{m + \lfloor \frac{n-i}{2} \rfloor}{2m-i} = \binom{m+n}{2m}, \text{ for all } 0 \le m \le n.$$

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