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On Axiom Systems of Pseudo-BCK Algebras

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Abstract. A symplified axiomatization of pseudo-BCK algebras is given. In addition, we obtain a system of axioms defining bounded commutative pseudo-BCK algebras and show that these axioms are independent (that is, none follows from the others).

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1. Introduction

In 1958, C. C. Chang [1] introduced MV (Many Valued) algebras. In 1966, Y. Imai and K. Iséki [8] introduced the notion of BCK algebra. A *BCK algebra* [2] is an algebra $(A; *, 0)$ of type $(2, 0)$ satisfying the following axioms:
(BCK-1) $((x * y) * (x * z)) * (z * y) = 0$,
(BCK-2) $(x * (x * y)) * y = 0$,
(BCK-3) $x * x = 0$,
(BCK-4) $0 * x = 0$,
(BCK-5) $x * y = 0$ and $y * x = 0$ imply $x = y$.

In 1996, P. Hájek ([6, 7]) invented Basic Logic (BL for short) and BL algebras, structures that correspond to this logical system. The class of BL algebras contains the MV algebras. G. Georgescu and A. Iorgulescu [3] (1999), and independently J. Rachůnek [15] introduced pseudo-MV algebras which are a non-commutative generalization of MV algebras. After pseudo-MV algebras, the pseudo-BL algebras [4] (2000), and the pseudo-BCK algebras [5] (2001) were introduced and studied. The paper [5] contains basic properties of pseudo-BCK algebras and their connections with pseudo-MV algebras and with pseudo-BL algebras. Y. B. Jun [12] obtained some characterizations of pseudo-BCK algebras. A. Iorgulescu [10, 11] studied particular classes of pseudo-BCK algebras. There exist several generalizations of BCK algebras such as BCI algebras [19] and BE algebras [18].

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In this paper, we give a system of axioms defining pseudo-BCK algebras. We provide conditions for a pseudo-BCK algebra to be commutative and bounded. It is also shown that the question of [5] (see Open problem 23) has a negative answer.

2. An axiomatization of pseudo-BCK algebras

The notion of pseudo-BCK algebras is defined by Georgescu and Iorgulescu [5] as follows:

Definition 2.1. *A pseudo-BCK algebra is a structure* $\mathcal{A} = (A; \leq, *, \circ, 0)$*, where for any* $x, y, z \in A$ *the following conditions are satisfied:*
(pBCK-1) $(x * y) \circ (x * z) \leq z * y, \quad (x \circ y) * (x \circ z) \leq z \circ y,$
(pBCK-2) $x * (x \circ y) \leq y, \quad x \circ (x * y) \leq y,$
(pBCK-3) $x \leq x,$
(pBCK-4) $0 \leq x,$
(pBCK-5) $(x \leq y \text{ and } y \leq x) \Rightarrow x = y,$
(pBCK-6) $x \leq y \Leftrightarrow x * y = 0 \Leftrightarrow x \circ y = 0.$

Remark 2.1. An equivalent definition of pseudo-BL algebras can be found (in a dual form) in [9], namely as a structure $(A; \leq, *, \circ, 0)$ such that for all $x, y, z \in A$:
(R0) $(A; \leq, 0)$ is a poset with smallest element 0,
(R1) $[(x * y) \circ (x * z)] \circ (z * y) = 0, \quad [(x \circ y) * (x \circ z)] * (z \circ y) = 0,$
(R3) $x * 0 = x = x \circ 0;$
(R4) $y * x = 0 \Leftrightarrow y \circ x = 0 \Leftrightarrow x \geq y,$
(R5) $x \leq y \Rightarrow x * z \leq y * z, \ x \circ z \leq y \circ z.$

Remark 2.2. If $(A; \leq, *, \circ, 0)$ is a pseudo-BCK algebra verifying $x * y = x \circ y$ for all $x, y \in A$, then $(A; *, 0)$ is a BCK algebra.

Proposition 2.1. [12, Theorem 3.8] *A structure* $(A; \leq, *, \circ, 0)$ *is a pseudo* BCK-*algebra if and only if it satisfies* (pBCK-1), (pBCK-5), (pBCK-6), *and*

$$x * (0 \circ y) = x = x \circ (0 * y) \quad \text{for all} \quad x, y \in A.$$

Proposition 2.2. *Let*$(A; *, \circ, 0)$ *be an algebra of type* $(2, 2, 0)$ *satisfying the following*
(A1) $[(x * y) \circ (x * z)] \circ (z * y) = 0,$
(A2) $[(x \circ y) * (x \circ z)] * (z \circ y) = 0,$
(A3) $x \circ (0 * y) = x,$
(A4) $x * (0 \circ y) = x.$
Then for all $x, y \in A$:
(a) $0 * x = 0 = 0 \circ x,$
(b) $x * 0 = x = x \circ 0,$
(c) $[x \circ (x * y)] \circ y = 0, \ [x * (x \circ y)] * y = 0,$
(d) $x * y = 0 \Leftrightarrow x \circ y = 0.$

Proof. (a) Substituting $x = z = 0$, $y = x$, (A1) becomes

$$0 = [(0 * x) \circ (0 * 0)] \circ (0 * x).$$

Hence, applying (A3) we get $0 = (0 * x) \circ (0 * 0) = 0 * x$, that is,

$$0 * x = 0 \tag{2.1}$$

for all $x \in A$. Using (A2) and (A4) we obtain

$$0 = [(0 \circ x) * (0 \circ 0)] * (0 \circ x) = (0 \circ x) * (0 \circ 0) = 0 \circ x.$$

From this and (2.1) we see that (a) holds.

(b) By (A3), $x = x \circ (0 * y) = x \circ 0$. Applying (A4) gives $x = x * (0 \circ y) = x * 0$.

(c) (A1) yields

$$[(x * 0) \circ (x * y)] \circ (y * 0) = 0.$$

Then $[x \circ (x * y)] \circ y = 0$. Similarly, $[x * (x \circ y)] * y = 0$.

(d) Follows from (c). ∎

Remark 2.3. Note that (A1) & (A2) = (R1).

The following theorem gives a simplified axiomatization of pseudo-BCK algebras.

Theorem 2.1. *Let*$(A; *, \circ, 0)$ *be an algebra of type* $(2, 2, 0)$ *obeying* (A1)–(A4) *and the following axiom:*
(A5) $x * y = y \circ x = 0 \;\Rightarrow\; x = y$.

Let us define

$$x \leq y \Leftrightarrow x * y = 0.$$

Then the structure $\mathcal{A} = (A; \leq, *, \circ, 0)$ *is a pseudo*-BCK *algebra. Conversely, if* $\mathcal{A}$ *is a pseudo*-BCK *algebra, then it satisfies the axioms* (A1)–(A5)*.*

Proof. We conclude (pBCK-1) from (A1), (A2), and Proposition 2.2 (d). By Proposition 2.2 (c, d) we have (pBCK-2). Using (A2) we get

$$x * x = [(x \circ 0) * (x \circ 0)] * (0 \circ 0) = 0,$$

which is (pBCK-3). From Proposition 2.2(a) we obtain (pBCK-4). Moreover, (A5) and Proposition 2.2(d) clearly force (pBCK-5) and (pBCK-6). Hence the structure $\mathcal{A}$ is a pseudo-BCK algebra.

The converse is obvious. ∎

According to the above theorem, we say that $\mathbf{A} = (A; *, \circ, 0)$ is a pseudo-BCK algebra if (A1)–(A5) are true in $\mathbf{A}$.

Remark 2.4. If Ψ is a statement expressed in terms $*$ and $\circ$, then we denote by Ψ' the statement we get from Ψ by interchanging $*$ and $\circ$. Obviously (A2) = (A1'), (A4) = (A3'), and (A5) = (A5').

Hence:

Corollary 2.1. *If* $\mathbf{A} = (A; *, \circ, 0)$ *is a pseudo*-BCK *algebra, then* $\mathbf{A}^d = (A; \circ, *, 0)$ *is also a pseudo*-BCK *algebra.*

Remark 2.5. Applying Proposition 2.2 we can assert that the axioms (A1)–(A5) are equivalent to the conditions (1.1.7)–(1.1.12) from Theorem 1.1.10 of [13].

Lemma 2.1. [5] *Let* $(A; *, \circ, 0)$ *be a pseudo*-BCK *algebra. Then for all* $x, y, z \in A$*:*

$$(x * y) \circ z = (x \circ z) * y.$$

Definition 2.2. [5, 10] *In a pseudo*-BCK *algebra* $\mathbf{A} = (A; *, \circ, 0)$ *we define, for all* $x, y \in A$*:*

$$x \wedge y = y \circ (y * x), \quad x \cap y = y * (y \circ x).$$

We say that $\mathbf{A}$ *is:*

(a) *$\wedge$-commutative if $x \wedge y = y \wedge x$ for all $x, y \in A$,*
(b) *$\cap$-commutative if $x \cap y = y \cap x$ for all $x, y \in A$.*

Proposition 2.3. [5, Theorem 18] *Let $\mathbf{A} = (A; *, \circ, 0)$ be a pseudo-*BCK *algebra. Then:*

(i) *$\mathbf{A}$ is $\wedge$ commutative if and only if $(A; \wedge)$ is a semilattice,*
(i') *$\mathbf{A}$ is $\cap$ commutative if and only if $(A; \cap)$ is a semilattice.*

Proposition 2.4. *Let $\mathbf{A} = (A; *, \circ, 0)$ be an algebra of type $(2, 2, 0)$. Then:*

(i) *$\mathbf{A}$ is a $\wedge$ commutative pseudo-*BCK *algebra if and only if $\mathbf{A}$ satisfies* (A1)–(A4) *and*
(B1) $y \circ (y * x) = x \circ (x * y)$;
(i') *$\mathbf{A}$ is a $\cap$ commutative pseudo-*BCK *algebra if and only if $\mathbf{A}$ satisfies* (A1)–(A4) *and*
(B1') $y * (y \circ x) = x * (x \circ y)$.

Proof. (i) If $\mathbf{A}$ is a $\wedge$-commutative pseudo-BCK algebra, then it obviously satisfies (A1)–(A4) and (B1). Conversely, let $\mathbf{A}$ satisfy the axioms (A1)–(A4) and (B1). By Proposition 2.2 (b),

$$x \circ 0 = x \quad \text{for all } x \in A. \tag{2.2}$$

To prove (A5), let $x * y = y \circ x = 0$. Using Proposition 2.2 (d) we get $y * x = 0$. From (2.2) and (B1) we obtain

$$x = x \circ 0 = x \circ (x * y) = y \circ (y * x) = y \circ 0 = y.$$

Consequently, (A5) holds in $\mathbf{A}$. Thus $\mathbf{A}$ is a pseudo-BCK algebra. By (B1), $\mathbf{A}$ is $\wedge$-commutative.

(i') has a similar proof. ∎

Following a suggestion of [13] (see also [14]), we call a pseudo-BCK algebra $(A; *, \circ, 0)$ *commutative* if and only if it satisfies both (B1) and (B1'). In [5] these algebras were considered under name "semilattice-ordered" pseudo-BCK algebras.

Corollary 2.2. *An algebra $\mathbf{A} = (A; *, \circ, 0)$ of type $(2, 2, 0)$ is a commutative pseudo-*BCK *algebra if and only if $\mathbf{A}$ satisfies* (A1)–(A4), (B1)*, and* (B1').

Remark 2.6. Another axiomatization of commutative pseudo-BCK algebras can be found in [13]. The axioms are: (B1), (B1'), and (SK1)–(SK4), where

(SK1) $y * (y \circ x) = x \circ (x * y)$,
(SK2) $(z \circ y) * x = (z * x) \circ y$,
(SK3) $0 * x = 0 = 0 \circ x$,
(SK4) $x * 0 = x = x \circ 0$.

Proposition 2.5. [5, Corollary 22] *If $\mathbf{A}$ is a commutative pseudo-*BCK *algebra, then*

$$x \wedge y = x \cap y \quad \textit{for all } x, y \in A. \tag{2.3}$$

Remark 2.7. Is every pseudo-BCK algebra verifying the condition (2.3) commutative? (Open problem 23 of [5]). The answer is in the negative. Indeed, let $(A; *, 0)$ be a non-commutative BCK algebra (that is, $y * (y * x) \neq x * (x * y)$ for some

$x, y \in A$). Then $\mathbf{A} = (A; *, *, 0)$ is a pseudo-BCK algebra satisfying (2.3), but it is not commutative since (B1) does not hold in $\mathbf{A}$.

3. Bounded pseudo-BCK algebras

Definition 3.1. *If there is an element* 1 *of a pseudo*-BCK *algebra* $\mathbf{A}$*, satisfying*
(B2) $x * 1 = 0$ *for all* $x \in A$*, then* 1 *is called a unit of* $\mathbf{A}$*. We say that* $(A; *, \circ, 0, 1)$ *is a bounded pseudo*-BCK *algebra if* $(A; *, \circ, 0)$ *is a pseudo*-BCK *algebra and* 1 *is a unit of* $(A; *, \circ, 0)$.

Theorem 3.1. *Let* $\mathbf{A} = (A; *, \circ, 0, 1)$ *be an algebra of type* $(2, 2, 0, 0)$. $\mathbf{A}$ *is a bounded commutative pseudo*-BCK *algebra if and only if it satisfies* (B1), (B2)*, and the following axioms:*
(B3) $(x * y) \circ z = (x \circ z) * y$,
(B4) $y \circ (y * x) = y * (y \circ x)$
(B5) $x * 0 = x$.

Proof. Let $\mathbf{A}$ be a bounded commutative pseudo-BCK algebra. It is obvious that $\mathbf{A}$ satisfies (B1), (B2), and (B5). The axiom (B3) follows from Lemma 2.1. By (2.3) we obtain (B4).

Conversely, let $\mathbf{A}$ satisfy (B1)–(B5). Using (B4) and (B1) we have

$$y * (y \circ x) = y \circ (y * x) = x \circ (x * y) = x * (x \circ y),$$

which is (B1'). Applying (B2) and (B3) we get

$$0 \circ x = (x * 1) \circ x = (x \circ x) * 1 = 0,$$

i.e., for all $x \in A$,

$$0 \circ x = 0. \tag{3.1}$$

We have

$$\begin{aligned} x \circ x &= x \circ (x * 0) \quad \text{(by (B5))} \\ &= x * (x \circ 0) \quad \text{(by (B4))} \\ &= 0 * (0 \circ x) \quad \text{(by (B1}'))} \\ &= 0 * 0 = 0 \quad \text{(by (3.1) and (B5))}. \end{aligned}$$

Then

$$x \circ x = 0 \tag{3.2}$$

for all $x \in A$. Observe that

$$x \circ y = 0 \Rightarrow x * y = 0. \tag{3.3}$$

Indeed, let $x \circ y = 0$. Using (B5), (B4), (B3), and (3.2) we get

$$x * y = (x * 0) * y = [x * (x \circ y)] * y = [x \circ (x * y)] * y = (x * y) \circ (x * y) = 0.$$

Thus (3.3) is satisfied. Now applying (3.2) and (3.3) we have

$$x * x = 0 \tag{3.4}$$

for all $x \in A$. From (B5), (3.2), (B4), and (3.4) we obtain

$$x = x * 0 = x * (x \circ x) = x \circ (x * x) = x \circ 0,$$

i.e., (B5') holds in **A**. Since

$$x \circ 1 = x \circ (1 * 0) = x * (1 \circ 0) = x * 1 = 0,$$

we see that (B2') also holds in **A**. Moreover, (B3) = (B3') and (B4) = (B4'). Thus the identities (B1')–(B5') are true in **A**. From this and (3.3) we deduce that

$$x * y = 0 \Rightarrow x \circ y = 0 \tag{3.5}$$

for all $x, y \in A$. By (B3) and (B1),

$$\begin{aligned}(x * y) \circ (x * z) &= [x \circ (x * z)] * y \\ &= [z \circ (z * x)] * y \\ &= (z * y) \circ (z * x).\end{aligned}$$

Hence, applying (B3), (3.4), and (3.1), we obtain

$$\begin{aligned}[(x * y) \circ (x * z)] * (z * y) &= [(z * y) \circ (z * x)] * (z * y) \\ &= [(z * y) * (z * y)] \circ (z * x) \\ &= 0 \circ (z * x) = 0.\end{aligned}$$

Using (3.5) we get (A1). Similarly, since the identities (B1')–(B5') are true in **A**, we conclude that (A2) = (A1') holds in **A**. It is easy to see that (A3) and (A4) also hold in **A**. From Corollary 2.2 it follows that **A** is a bounded commutative pseudo-BCK algebra. ∎

By the proof of Theorem 3.1 we get:

Corollary 3.1. *If* $\mathbf{A} = (A; *, \circ, 0, 1)$ *is a bounded commutative pseudo-*BCK *algebra, then* $\mathbf{A}^d = (A; \circ, *, 0, 1)$ *is also a bounded commutative pseudo-*BCK *algebra.*

Corollary 3.2. *The class of bounded commutative pseudo-*BCK *algebras is a variety defined by the identities* (B1)–(B5) *or* (B1')–(B5')*.*

Theorem 3.2. *The axioms* (B1)–(B5) *are independent, that is, none of them can be deduced from the others.*

Proof. We are going to give some examples of algebras in which only four of the axioms hold. Let $A = \{0, a, 1\}$. Define binary operations $*$ and $\circ$ on A as follows:

$*$	0	a	1
0	0	0	0
a	a	0	0
1	1	1	0

$\circ$	0	a	1
0	0	0	0
a	a	0	a
1	1	1	0

Then $(A; *, *, 0, 1)$ fulfils the axioms (B2)–(B5), but not (B1), since $1 * (1 * a) = 0 \neq a = a * (a * 1)$. However, $(A; \circ, \circ, 0, 1)$ satisfies (B1) and (B3)–(B5), but not (B2). Now we define the binary operations $*$ and $\circ$ on the set A by the following tables:

$*$	0	a	1
0	0	a	0
a	a	a	0
1	1	a	0

$\circ$	0	a	1
0	a	a	a
a	a	a	a
1	a	a	a

The identities (B1), (B2), (B4), and (B5) are valid in $(A; *, \circ, 0, 1)$, but (B3) does not hold because $(0 * 1) \circ a = a$, while $(0 \circ a) * 1 = 0$.

Now, consider the set $A = \{0, 1\}$ with the operations $*$ and $\circ$ given as follows:

$*$	0	1
0	0	0
1	1	0

$\circ$	0	1
0	0	0
1	0	0

Then the algebra $(A; *, \circ, 0, 1)$ satisfies (B1)–(B3) and (B5), but not (B4). Indeed, $1 \circ (1 * 1) = 0 \neq 1 = 1 * (1 \circ 0)$. Finally, it is evident that the axioms (B1)–(B4) hold in $(A; \circ, \circ, 0, 1)$, while (B5) does not. ∎

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