BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **34**(2) (2011), 295–305

Real Hypersurfaces in Complex Two-Plane Grassmannians with Recurrent Shape Operator

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Abstract. We introduce the notion of recurrent hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ and give a non-existence theorem for a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with recurrent shape operator.

2010 Mathematics Subject Classification: Primary: 53C40; Secondary: 53C15

Keywords and phrases: Real hypersurfaces, complex two-plane Grassmannians, Hopf hypersurface, recurrent shape operator, recurrent hypersurfaces.

1. Introduction

The notion of recurrent tensor field of type (r, s) on a differentiable manifold M with a linear connection was well introduced in [7] and [15]. A non-zero tensor field K of type (r, s) on M which is said to be *recurrent* if there exists a 1-form ω such that

$$\nabla K = K \otimes \omega\,.$$

Moreover, they gave some geometric interpretation of such a manifold M with recurrent curvature tensor K in terms of holonomy group, see also [7] and [15].

Now let us denote by A the shape operator of real hypersurfaces in non flat complex space form $M_n(c)$. Recently, Hamada [5, 6] applied such a notion of recurrent tensor to a shape operator or a Ricci tensor for real hypersurfaces M in complex projective space $\mathbb{C}P^n$ in such a way that

$$\nabla A = \omega \otimes A$$

or

$$\nabla S = \omega \otimes S$$

for a certain 1-form ω defined on M, and proved the following:

Communicated by Young Jin Suh.
Received: July 29, 2009; *Revised:* November 13, 2009.

Theorem 1.1. *The complex projective space $\mathbb{C}P^n$ does not admit any real hypersurfaces with recurrent shape operator or recurrent Ricci tensor.*

On the other hand, Suh [9] have explained the geometrical meaning of recurrent shape operator A as follows:

$$[\nabla_X A, A] = \omega(X)[A, A] = 0$$

for any tangent vector field X defined on M. That is, *the eigenspaces of the shape operator A of M are parallel along any curve γ in M*. Here, the eigenspaces of the shape operator A are said to be *parallel* along γ if they are invariant with respect to any parallel translation along γ.

Now let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometrical structure. It is the unique compact irreducible Riemannian manifold being equipped with both a Kaehler structure J and a quaternionic Kaehler structure $\mathfrak{J}$ not containing J.

In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kaehler, quaternionic Kaehler manifold which is not a hyper-Kaehler manifold. So, in $G_2(\mathbb{C}^{m+2})$ we have the two natural geometrical conditions for real hypersurfaces M that $[\,\xi\,] = \text{Span}\{\,\xi\,\}$ or $\mathfrak{D}^\perp = \text{Span}\{\,\xi_1,\ \xi_2,\ \xi_3\,\}$ are invariant under the shape operator A of M. The almost contact structure vector field ξ mentioned above is defined by $\xi = -JN$ is said to be a *Reeb* vector field, where N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. If the Reeb vector field ξ of M in $G_2(\mathbb{C}^{m+2})$ is invariant by the shape operator, ξ is said to be a *Hopf*. The *almost contact 3-structure* vector fields $\{\,\xi_1,\ \xi_2,\ \xi_3\,\}$ for the 3-dimensional distribution $\mathfrak{D}^\perp$ of M in $G_2(\mathbb{C}^{m+2})$ are defined by $\xi_\nu = -J_\nu N$, $\nu = 1,\ 2,\ 3$, where J_ν denotes a canonical local basis of a quaternionic Kaehler structure $\mathfrak{J}$, such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$.

When the Reeb vector field ξ and the distribution $\mathfrak{D}^\perp$ is invariant by the shape operator A of real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$, Berndt and Suh [2] have proved the following:

Theorem 1.2. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\,\xi\,]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

Now we introduce the notion of recurrent shape operator tensor defined in such a way that

$$(\nabla_X A)Y = \omega(X)AY \tag{1.1}$$

for a 1-form ω and any vector fields X and Y on M in $G_2(\mathbb{C}^{m+2})$. When the shape operator A of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ satisfies the formula (1.1), a hypersurface M is said to be a *recurrent hypersurface* in $G_2(\mathbb{C}^{m+2})$.

Related to such a notion, Suh [14] has proved the non-existence for recurrent hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-invariant shape operator as follows:

Theorem 1.3. *There do not exist any recurrent real hypersurfaces in* $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ *with* $\mathfrak{D}$ *(resp.* $\mathfrak{D}^{\perp}$*)-invariant shape operator.*

On the other hand, the 1-dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurfaces* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. By the formulas in Section 3 it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf. Such a notion of Hopf hypersurface in complex projective space $\mathbb{C}P^n$ is mainly discussed by Cecil and Ryan [4] and the invariancy of the distribution $\mathfrak{D}^{\perp}$ for hypersurface in quaternionic space forms was investigated in Berndt [1].

In this paper, we have considered the notion of Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ and give another non-existence theorem for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with recurrent shape operator as follows:

Theorem 1.4 (Main Theorem)**.** *There do not exist any Hopf recurrent hypersurfaces in complex two-plane Grassmannian,* $G_2(\mathbb{C}^{m+2})$, $m \geq 3$.

2. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [2] and [3]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight. Since $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight we will assume $m \geq 2$ from now on. Note that the isomorphism $Spin(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^6$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kaehler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kaehler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_1 is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $\mathrm{tr}(JJ_1) = 0$.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of

$(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{2.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\bar{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3}\{g(J\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y \\ & - 2g(J_\nu X,Y)J_\nu Z\} + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX \\ & - g(J_\nu JX,Z)J_\nu JY\},\end{aligned} \tag{2.2}$$

where $\{J_1, J_2, J_3\}$ is any canonical local basis of $\mathfrak{J}$.

3. Some fundamental formulas in $G_2(\mathbb{C}^{m+2})$

In this section we derive some basic formulae and the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (see [2, 3, 8, 10–14]).

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal field of M and A the shape operator of M with respect to N. The Kaehler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ, ξ, η, g). Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M. Using the above expression (1.2) for the curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$, the Codazzi equation becomes

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X = {} & \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\ & + \sum_{\nu=1}^{3}\big\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\big\} \\ & + \sum_{\nu=1}^{3}\big\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\big\} \\ & + \sum_{\nu=1}^{3}\big\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\big\}\xi_\nu .\end{aligned}$$

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

$$\begin{aligned}& \phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \\ & \phi\xi_\nu = \phi_\nu\xi, \quad \eta_\nu(\phi X) = \eta(\phi_\nu X), \\ & \phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu,\end{aligned} \tag{3.1}$$

$$\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}.$$

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N$$

for any tangent vector X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector of M in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas (2.1) and (3.1) we have the following

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{3.2}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{3.3}$$

$$\begin{aligned}(\nabla_X\phi_\nu)Y = &- q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX \\ &- g(AX,Y)\xi_\nu.\end{aligned} \tag{3.4}$$

Summing up these formulas, we find the following

$$\begin{aligned}\nabla_X(\phi_\nu\xi) &= \nabla_X(\phi\xi_\nu) \\ &= (\nabla_X\phi)\xi_\nu + \phi(\nabla_X\xi_\nu) \\ &= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu\phi AX \\ &\quad - g(AX,\xi)\xi_\nu + \eta(\xi_\nu)AX.\end{aligned} \tag{3.5}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1,2,3$, it follows that

$$\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu. \tag{3.6}$$

4. A key lemma

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with recurrent shape operator. Then it satisfied the condition that

$$(\nabla_X A)Y = \omega(X)AY$$

for a 1-form ω and any vector fields X, Y defined on M. By the equation of Codazzi in Section 3 we have that

$$\begin{aligned}(\nabla_\xi A)Y - (\nabla_Y A)\xi &= \omega(\xi)AY - \omega(Y)A\xi \\ &= \phi Y + \sum_{v=1}^{3}\{\,\eta_\nu(\xi)\phi_\nu Y - \eta_\nu(Y)\phi_\nu\xi - 2g(\phi_\nu\xi,Y)\xi_\nu\,\} \\ &\quad + \sum_{\nu=1}^{3}\eta_\nu(\phi Y)\xi_\nu\,.\end{aligned} \tag{4.1}$$

Since we assumed that M is Hopf, (4.1) gives

$$\omega(\xi)AY = \alpha\omega(Y)\xi + \phi Y + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu Y - \eta_\nu(Y)\phi_\nu\xi + 3\eta_\nu(\phi Y)\xi_\nu\,\}. \tag{4.2}$$

Now we assert the key lemma as following:

Lemma 4.1. *Let M be a recurrent hypersurface in $G_2(\mathbb{C}^{m+2})$. If the Reeb vector ξ is principal, then ξ belongs to either the distribution $\mathfrak{D}$ or to the distribution $\mathfrak{D}^\perp$.*

Proof. To prove this lemma we put $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for some unit vector $X_0 \in \mathfrak{D}$. Here we notice that $\eta(X_0)$ and $\eta(\xi_1)$ are not zero. In (4.2), by putting $Y = \xi_1$ we have

$$\begin{aligned} \omega(\xi)A\xi_1 &= \alpha\omega(\xi_1)\xi + \phi_1\xi + \sum_{\nu=1}^{3}\{\, 3\eta_\nu(\phi_1\xi)\xi_\nu - \eta_\nu(\xi_1)\phi_\nu\xi \,\} \\ &= \alpha\omega(\xi_1)\xi + \phi_1\xi - 3\eta_3(\xi)\xi_2 + 3\eta_2(\xi)\xi_3 - \phi_1\xi \\ &= \alpha\omega(\xi_1)\xi\,. \end{aligned} \tag{4.3}$$

We get also the following equations by putting $Y = \xi_2$ and $Y = \xi_3$ in (4.2), similarly.

$$\begin{aligned} \omega(\xi)A\xi_2 &= \alpha\omega(\xi_2)\xi - 2\eta_1(\xi)\xi_3\,, \\ \omega(\xi)A\xi_3 &= \alpha\omega(\xi_3)\xi + 2\eta_1(\xi)\xi_2\,. \end{aligned} \tag{4.4}$$

From these equations, taking an inner product with ξ, it follows that

$$\begin{aligned} \alpha\{\omega(\xi_1) - \omega(\xi)\eta_1(\xi)\} &= 0\,, \\ \alpha\omega(\xi_2) &= 0\,, \\ \alpha\omega(\xi_3) &= 0\,. \end{aligned} \tag{4.5}$$

Thus we can consider two cases that the first is $\alpha = 0$ and the second is not. For the first case $\alpha = 0$, by the lemma due to Pérez and Suh [8] we know that ξ belongs to either the distribution $\mathfrak{D}$ or to the distribution $\mathfrak{D}^\perp$.

Now let us consider the remaining case, $\alpha \neq 0$. From (4.5), we have

$$\begin{aligned} \omega(\xi_1) &= \omega(\xi)\eta_1(\xi)\,, \\ \omega(\xi_2) = 0\,, &\quad \omega(\xi_3) = 0\,. \end{aligned} \tag{4.6}$$

Substituting these equations into (4.3) and (4.4) gives

$$\begin{aligned} \omega(\xi)A\xi_1 &= \alpha\omega(\xi)\eta_1(\xi)\xi, \\ \omega(\xi)A\xi_2 &= -2\eta_1(\xi)\xi_3, \\ \omega(\xi)A\xi_3 &= 2\eta_1(\xi)\xi_2. \end{aligned} \tag{4.7}$$

From this, we consider the following two subcases:

Subcase II(1) . $\omega(\xi) = 0$.

Then (4.7) gives $\eta_1(\xi) = 0$. This implies $\xi \in \mathfrak{D}$.

Subcase II(2) . $\omega(\xi) \neq 0$.

Then (4.7) give the following

$$\begin{aligned} A\xi_1 &= \alpha\eta_1(\xi)\xi\,, \\ A\xi_2 &= -\frac{2\eta_1(\xi)}{\omega(\xi)}\xi_3\,, \\ A\xi_3 &= \frac{2\eta_1(\xi)}{\omega(\xi)}\xi_2\,. \end{aligned} \tag{4.8}$$

From this, taking an inner product with ξ_3 to the second formula of (4.8), it follows that

$$g(A\xi_2, \xi_3) = g(-\frac{2\eta_1(\xi)}{\omega(\xi)}\xi_3, \xi_3) = -\frac{2\eta_1(\xi)}{\omega(\xi)}. \tag{4.9}$$

On the other hand, from the third formula of (4.8) we have

$$g(A\xi_2, \xi_3) = g(A\xi_3, \xi_2) = g(\frac{2\eta_1(\xi)}{\omega(\xi)}\xi_2, \xi_2) = \frac{2\eta_1(\xi)}{\omega(\xi)}.$$

From this, together with (4.9), it follows $\frac{\eta_1(\xi)}{\omega(\xi)} = 0$. That is, $\eta_1(\xi) = 0$, which gives $\xi \in \mathfrak{D}$. This complete the proof of our Lemma 4.1. ∎

5. Recurrent hypersurfaces for $\xi \in \mathfrak{D}^\perp$

In this section by Lemma 4.1 we consider the case that $\xi \in \mathfrak{D}^\perp$. That is, we consider a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ with recurrent shape operator and $\xi \in \mathfrak{D}^\perp$. Accordingly, we may put $\xi = \xi_1$. Then (4.2) implies the following

Lemma 5.1. *Let M be a Hopf recurrent hypersurface in $G_2(\mathbb{C}^{m+2})$. If $\xi \in \mathfrak{D}^\perp$, then $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

Proof. Since we have assumed $\xi \in \mathfrak{D}^\perp$, we may put $\xi = \xi_1$. Then from (4.3) and (4.4) we know that

$$\begin{aligned} \omega(\xi_1)A\xi_1 &= \alpha\omega(\xi_1)\xi_1, \\ \omega(\xi_1)A\xi_2 &= \alpha\omega(\xi_2)\xi_1 - 2\xi_3, \\ \omega(\xi_1)A\xi_3 &= \alpha\omega(\xi_3)\xi_1 + 2\xi_2. \end{aligned} \tag{5.1}$$

From this, if we take an inner product with $X \in \mathfrak{D}$, then we have

$$\omega(\xi_1)g(A\xi_\nu, X) = 0, \quad \nu = 1, 2, 3. \tag{5.2}$$

So, for the case where $\omega(\xi_1) \neq 0$ in (5.2) we have our assertion. Now let us consider the case that $\omega(\xi_1) = 0$. Then (5.1) gives the following

$$\alpha\omega(\xi_2)\xi_1 = 2\xi_3 \quad \text{and} \quad \alpha\omega(\xi_3)\xi_1 = -2\xi_2, \tag{5.3}$$

which makes a contradiction. So, we complete the proof of Lemma 5.1. ∎

Now in order to complete the proof of our main theorem we recall a proposition due to Berndt and Suh [2] as follows:

Proposition 5.1. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r)\,, \ \beta = \sqrt{2}\cot(\sqrt{2}r)\,, \ \gamma = -\sqrt{2}\tan(\sqrt{2}r)\,, \ \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1\ ,\ m(\beta) = 2\ ,\ m(\gamma) = 2m-2 = m(\mu),$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1,$$

$$T_\beta = \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3,$$
$$T_\gamma = \{X | X \perp \mathbb{H}\xi, JX = J_1X\},$$
$$T_\mu = \{X | X \perp \mathbb{H}\xi,\ JX = -J_1X\},$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ and $\mathbb{H}\xi$ respectively denotes real, complex and quaternionic span of the structure vector ξ and $\mathbb{C}^\perp\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

Without loss of generality we may put $\xi = \xi_1$. Now let us put $Y = \xi_2$ in T_β in (4.2). Then by using (4.5), we have

$$\begin{aligned}\omega(\xi_1)A\xi_2 &= \alpha\omega(\xi_2)\xi_1 + \phi\xi_2 + \phi_1\xi_2 + 3\sum_{\nu=1}^{3}\eta_\nu(\phi Y)\xi_\nu - \phi_2\xi \\ &= \alpha\omega(\xi_2)\xi_1 - \xi_3 + \xi_3 - 3\xi_3 + \xi_3 \\ &= \sqrt{8}\cot(\sqrt{8}r)\omega(\xi_2)\xi_1 - 2\xi_3.\end{aligned}$$

On the other hand, by Proposition 5.1, we know that

$$A\xi_2 = \beta\xi_2 = \sqrt{2}\cot(\sqrt{2}r)\xi_2.$$

Then summing up these two formulas, we have

$$\sqrt{2}\cot(\sqrt{2}r)\omega(\xi_1)\xi_2 = \sqrt{8}\cot(\sqrt{8}r)\omega(\xi_2)\xi_1 - 2\xi_3\,. \tag{5.4}$$

If we take the scalar product of (5.4) and ξ_3 then we derive a contradiction. So we assert the following:

Theorem 5.1. *There do not exist any Hopf recurrent hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying $\xi \in \mathfrak{D}^\perp$.*

6. Recurrent hypersurfaces for $\xi \in \mathfrak{D}$

Now by Lemma 4.1, we consider the case that the Reeb vector belongs to $\mathfrak{D}$. In this section, we give a complete classification of Hopf recurrent hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\xi\in\mathfrak{D}$. Thus we assert the following:

Lemma 6.1. *Let M be a Hopf recurrent hypersurface in $G_2(\mathbb{C}^{m+2})$. If the Reeb vector ξ belongs to the distribution $\mathfrak{D}$, then $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

Proof. By using $\xi \in \mathfrak{D}$ in (4.3) and (4.4) we have the following for $\nu = 1, 2, 3$,

$$\omega(\xi)A\xi_\nu = \alpha\omega(\xi_\nu)\xi, \quad \nu = 1, 2, 3.$$

From this, by taking an inner product with ξ, it follows that

$$0 = \alpha\omega(\xi)\eta_\nu(\xi) = \alpha\omega(\xi_\nu).$$

That is, $\omega(\xi)A\xi_\nu = 0$. Thus we consider the following two cases:

Case I. $A\xi_\nu = 0$.
Then naturally we have $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.

Case II. $\omega(\xi) = 0$.

Let us take an inner product of the equation of Codazzi with ξ and using the differentiation of $A\xi = \alpha\xi$. Then we get

$$\begin{aligned} -2g(\phi X, Y) &+ 2\sum_{\nu=1}^{3}\{\eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi)\} \\ &= g((\nabla_X A)Y, \xi) - g((\nabla_Y A)X, \xi) \\ &= g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X) \\ &= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y)\,. \end{aligned} \tag{6.1}$$

From this, if we put $X = \xi$, then

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y)\,. \tag{6.2}$$

In this case the recurrence of Hopf hypersurfaces and $\omega(\xi) = 0$ give

$$(\nabla_\xi A)Y = \omega(\xi)AY = 0$$

for any tangent vector field Y on M. This gives

$$\nabla_\xi(AY) = A(\nabla_\xi Y).$$

Then by putting $Y = \xi$ and using M is Hopf, we have

$$0 = \nabla_\xi(A\xi) = \nabla_\xi(\alpha\xi) = (\xi\alpha)\xi.$$

So, we see that $\xi\alpha = 0$. From this and using (6.2) and $\xi\in\mathfrak{D}$, it follows that

$$Y\alpha = 0 \tag{6.3}$$

for any tangent vector field Y on M. This means that the function α is constant on M. On the other hand, by differentiating $A\xi = \alpha\xi$ and using (6.3) we have the following

$$\alpha\omega(X)\xi + A\phi AX = \alpha\phi AX.$$

So, it follows that for any tangent vector field X on M

$$\alpha\omega(X)\xi = \alpha\phi AX - A\phi AX. \tag{6.4}$$

Now we consider a subdistribution $\mathfrak{D}_1$ of the distribution $\mathfrak{D}$ defined in such a way that

$$\mathfrak{D}_1 = \{X \in \mathfrak{D} \mid X\perp\xi,\ X\perp\phi_i\xi,\ i = 1, 2, 3\}.$$

Then from (4.2) and using $\xi \in \mathfrak{D}$ in Case II we have

$$0 = \alpha\omega(X)\xi + \phi X \tag{6.5}$$

for any $X\in\mathfrak{D}_1$. Then (6.4) and (6.5) give the following

$$\alpha\phi AX - A\phi AX + \phi X = 0 \tag{6.6}$$

for any $X\in\mathfrak{D}_1$.

On the other hand, (6.1) and (6.3) give the following

$$-2\phi X = \alpha(A\phi + \phi A)X - 2A\phi AX$$

for any $X \in \mathfrak{D}_1$ where we have used the fact that $\xi \in \mathfrak{D}$. From this and together with (6.6) we get

$$2\alpha\phi AX = \alpha(A\phi + \phi A)X,$$

which gives

$$\alpha\phi AX = \alpha A\phi X$$

for any $X\in\mathfrak{D}_1$. Then we have the following two subcases.

Subcase II(1). $\alpha = 0$.

From (6.4) and (6.6) we have $\phi X = 0$ for any $X\in\mathfrak{D}_1$. Then by applying ϕ we have $X = 0$ for any $X\in\mathfrak{D}_1$. But this case can not appear.

Subcase II(2). $\alpha \neq 0$.

By putting $Y = \xi$ in the Codazzi equation in Section 3, we have

$$(\nabla_X A)\xi - (\nabla_\xi A)X = -\phi X + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu\xi - 2g(\phi_\nu X, \xi)\xi_\nu - \eta_\nu(\phi X)\xi_\nu\}.$$

From this, together with the recurrence and $\omega(\xi) = 0$, it follows that

$$\alpha\omega(X)\xi = -\phi X \tag{6.7}$$

for any $X\in\mathfrak{D}_1$. Taking an inner product with ξ we have $\alpha\omega(X) = 0$ for any $X\in\mathfrak{D}_1$. This gives $\omega(X) = 0$ for any $X\in\mathfrak{D}_1$. Then (6.7) gives $\phi X = 0$ for any $X\in\mathfrak{D}_1$, which also makes a contradiction. So, Case II can not appear. ∎

Then by virtue of Theorem 1.1 in the introduction, a Hopf recurrent hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\xi \in \mathfrak{D}$ is congruent to of type B, that is, a tube over a totally real quaternionic projective space $\mathbb{H}P^n$, $m = 2n$. Now for this type of hypersurface we introduce the following (see [2]).

Proposition 6.1. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r)\,, \quad \beta = 2\cot(2r)\,, \quad \gamma = 0\,, \quad \lambda = \cot(r)\,, \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1\,, \quad m(\beta) = 3 = m(\gamma)\,, \quad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi\,, \quad T_\beta = \mathfrak{J}J\xi\,, \quad T_\gamma = \mathfrak{J}\xi\,, \quad T_\lambda\,, \quad T_\mu\,,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp\,, \quad \mathfrak{J}T_\lambda = T_\lambda\,, \quad \mathfrak{J}T_\mu = T_\mu\,, \quad JT_\lambda = T_\mu\,.$$

Then by putting $Y = \phi_1\xi$ in T_γ in Proposition 6.1, we have

$$\begin{aligned}0 = \gamma\omega(\xi)\phi_1\xi = \omega(\xi)A\phi_1\xi &= \alpha\omega(\phi_1\xi)\xi + \phi^2\xi_1 + 3\sum_{\nu=1}^{3}\eta_\nu(\phi^2\xi_1)\xi_\nu \\ &= \alpha\omega(\phi_1\xi)\xi - \xi_1 - 3\xi_1 \\ &= \alpha\omega(\phi_1\xi)\xi - 4\xi_1,\end{aligned}$$

which gives a contradiction for $\xi\in\mathfrak{D}$. So we assert the following:

Theorem 6.1. *There do not exist any Hopf recurrent hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\xi \in \mathfrak{D}$.*

Then summing up Theorems 5.1 and 6.1 we complete the proof of our Main Theorem in the introduction.

Acknowledgement. The authors would like to express their hearty thanks to Professor Young Jin Suh for his valuable suggestions and continuous encouragement during the preparation of this work. This work was supported by grant Proj. No. R17-2008-001-01001-0 from Korea Science and Engineering Foundation.

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