Capability of a Pair of Groups

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Abstract. A group G is called capable if it is the group of inner automorphisms of some group E. Capable pairs are defined in terms of a relative central extension. In this paper, we introduce the precise center for a pair of groups and prove that this subgroup makes a criterion for characterizing the capability of the pair. We also show that our result sharpens the obtained result in this area. A complete classification of finitely generated abelian capable pairs will also be given.

1. Introduction and motivation

Following Hall Jr. and Senior [5], a group G is called *capable* if it is the group of inner automorphisms of some group or equivalently if there exists a group E with $E/Z(E) \cong G$. The study of capable groups was started by Baer [1], who determined all capable abelian groups. Hall remarked in [6] that characterization of capable groups are important in classifying groups of prime power order. In 1979, Beyl, Felgner and Schmid [2] studied capable groups by focusing on a characteristic subgroup $Z^*(G)$ which is called the precise center of G. They actually proved that the triviality of the precise center is a criterion for capability of the group itself.

The theory of capability of groups may be extended to the theory of pairs of groups. In fact capable pairs are defined in terms of Loday's notion [7] of a relative central extension. By a pair of groups we mean a group G and a normal subgroup N and this is denoted by (G, N). Let M be another group on which an action of G is given. The *G-commutator subgroup* of M is defined by the subgroup $[M, G]$ of M generated by all the G-commutators

$$[m, g] = m^g m^{-1},$$

in which $g \in G$, $m \in M$ and m^g is the action of g on m. Also we define the *G-center* of M to be the subgroup

$$Z(M,G) = \{m \in M | m^g = m, \forall g \in G\}.$$

For introducing the capable pair, we need to define a relative central extension as follows:

Definition 1.1. *Let* (G, N) *be a pair of groups. A relative central extension of the pair* (G, N) *consists of a group homomorphism* $\sigma : M \to G$, *together with an action of* G *on* M *such that*

    (i) $\sigma(M) = N$,
    (ii) $\sigma(m^g) = g^{-1}\sigma(m)g$, *for all* $g \in G$, $m \in M$,
    (iii) $m'^{\sigma(m)} = m^{-1}m'm$, *for all* $m, m' \in M$,
    (iv) $\ker(\sigma) \subseteq Z(M,G)$.

We shall say that the pair (G, N) is *capable* if it admits such a relative central extension with $\ker \sigma = Z(M,G)$.

Note that $Z(M,G)$ is a central subgroup of M and therefore, if N is equal to G, then the relative central extension $\sigma : M \to G$ gives the following central extension of G

$$1 \to \ker \sigma \to M \to G \to 1.$$

Now, it is obvious that a group G is capable precisely when the pair (G, G) is capable.

One of the interesting results of Beyl *et al.*'s type [2] for the capability of pairs of groups was proved by Ellis [4] in 1996. He actually introduced the exterior G-center of N for a pair of groups (G, N), and proved that the pair (G, N) is capable if and only if the exterior G-center of N is the trivial group. Using this, he could generalize Baer's [1] characterization of finitely generated capable abelian groups to capable pairs (G, N) of finitely generated abelian groups. His method is also based on the tensor product and exterior product of groups.

This paper is organized as follows. In the next section we give some more properties of capable pairs of groups of Beyl *et al.*'s type [2]. A description of the exterior G-center of N, in terms of a free presentation of G, is given in Section 3. In Section 4, we introduce a central subgroup $Z^*(G, N)$ of G, for a pair of groups (G, N), and we shall call it the precise center of the pair (G, N) throughout the article. We go on to show that the precise center of a pair (G, N) provides a criterion for recognizing the capability of the pair and also it is a subgroup contained in the exterior G-center of N. This shows that the precise center is a smaller and more suitable subgroup of G, with respect to the to the exterior G-center of N, for characterizing the capability of the pair (G, N). Therefore the attained result sharpens the one obtained by Ellis (see [4, Theorem 3]) and the important point is the easier technique applied to attain the conclusion. Finally in Section 5, we turn our attention to determining the capability of a pair of finitely generated abelian groups, and give a complete classification of finitely generated abelian capable pairs.

2. Some results of Beyl *et al.*'s type

In what follows, we present some properties of a capable pair of groups.

Theorem 2.1. *Let (G, N) be a pair of groups and $\{K_i\}_{i \in I}$ be a family of normal subgroups of N. If the pair of groups $(G/K_i, N/K_i)$ is capable for all $i \in I$, then*

$$\left(\frac{G}{\bigcap_{i \in I} K_i}, \frac{N}{\bigcap_{i \in I} K_i} \right)$$

is capable.

Proof. Suppose that for each $i \in I$, $\delta_i : E_i \to G/K_i$, together with an action of G/K_i on E_i, is a relative central extension of the pair $(G/K_i, N/K_i)$ such that $\delta_i(E_i) = N/K_i$ and $\ker \delta_i = Z(E_i, G/K_i)$. Put

$$H = \left\{ \{e_i\}_{i \in I} \in \prod_{i \in I} E_i | \exists g \in G; \delta_i(e_i) = gK_i \right\}$$

and $K = \bigcap_{i \in I} K_i$, where $\prod_{i \in I} E_i$ denotes the Cartesian product of the groups E_i. It is readily verified that $\delta : H \to G/K$ defined by $\delta(\{e_i\}_{i \in I}) = gK$ is a relative central extension of the pair $\left(G/\bigcap_{i \in I} K_i, N/\bigcap_{i \in I} K_i \right)$ and that $\ker \delta = Z(H, G/K)$. So the result follows. ∎

Theorem 2.2. *If (G, N) is capable and G/G' is of finite exponent, then the exponent of $Z(G) \cap N$ divides that of G/G'.*

Proof. By the assumption, there is a relative central extension $\varphi : M \to G$ with $\ker \varphi = Z(M, G)$. Thus for every $x \in N$ there exist $t_x \in M$ such that $\varphi(t_x) = x$. Now consider the map $\gamma : Z(G) \cap N \times G \longrightarrow Z(M, G)$ with $\gamma((x, g)) = [t_x, g]$, for $x \in Z(G) \cap N$ and $g \in G$. One can easily verify that the map γ is well defined and since $Z(M, G) \leq Z(M)$ and $\ker \varphi = Z(M, G)$, then γ is left linear. On the other hand, for all $g_1, g_2 \in G$ and $x \in Z(G) \cap N$ we have

$$[t_x, g_1 g_2] = t_x^{-1} t_x^{g_1 g_2} = t_x^{-1} t_x^{g_2} [t_x, g_1]^{g_2} = [t_x, g_1][t_x, g_2].$$

This proves that γ is right linear and therefore γ is a bilinear map. Also the equality $[t_x, g] = 1$, for all $g \in G$, implies that $t_x \in Z(M, G)$ and this shows that the left kernel of γ is trivial. The right kernel of γ also must contain $G' = [G, G]$. So if G/G' is of exponent n, then $\gamma(x^n, g) = \gamma(x, g^n) = 1$, for all $x \in Z(G) \cap N$ and $g \in G$. It follows that $x^n = 1$, for all $x \in Z(G) \cap N$ and this completes the proof. ∎

Using Theorem 2.2, we obtain the following corollary which states a necessary condition for the capability of the pair (G, N), when G is a perfect group.

Corollary 2.1. *If G is a perfect group, then the capability of the pair (G, N) implies that $Z(G) \cap N = 1$.*

3. The exterior G-center subgroup and free presentation

In order to study the capability of a pair of groups (G, N), Ellis [4] introduced a subgroup $Z_G^\wedge(N)$ with the property that the pair is capable if and only if $Z_G^\wedge(N) = 1$. But to define this subgroup, we need to recall the definition of exterior product from [3] as follows.

Definition 3.1. *Let N and P be arbitrary normal subgroups of G. The exterior product $P \wedge N$ is the group generated by symbols $p \wedge n$ for $p \in P$, $n \in N$ subject to the relations*

$$pp' \wedge n = (p'^p \wedge n^p)(p \wedge n),$$
$$p \wedge nn' = (p \wedge n)(p^n \wedge n'^n),$$
$$x \wedge x = 1,$$

for $x \in P \cap N$, $n, n' \in N$, $p, p' \in P$.

Now for a group G and normal subgroups N and P, the *exterior P-center* of N is denoted by $Z_P^\wedge(N)$, and is defined to be

$$\{n \in N | 1 = p \wedge n \in P \wedge N, \quad \text{for all} \quad p \in P\},$$

(see [4]). Clearly $Z_P^\wedge(N)$ is a central subgroup of N, if P contains N. As Ellis proved in [4], the pair (G, N) is capable if and only if $Z_G^\wedge(N) = 1$. Therefore determining the structure of exterior G-center of N is useful for studying the capability of a pair (G, N). In what follows we intend to describe the exterior G-center of N, $Z_G^\wedge(N)$, in terms of a free presentation for G. The description has some interesting applications which are stated in the last section of the paper. To prove the main purpose of the section, we need the following lemma which gives a considerable isomorphism. But first note that for a group G with a free presentation $G \cong F/R$ and a normal subgroup $N \trianglelefteq G$ with $N \cong S/R$, we consider the action of G on $S/[R, F]$, defined by $(s[R, F])^g := s^f[R, F]$, such that $g = \pi(f)$, for $f \in F$, $s \in S$ and $g \in G$, where π is the natural epimorphism from F to G.

Lemma 3.1. *Let F/R be a free presentation of G and $N \trianglelefteq G$ with $N = S/R$. Then*

$$N \wedge G \cong \left[\frac{S}{[R, F]}, G \right].$$

Proof. Put $\bar{F} = F/[R, F]$ and $\bar{S} = S/[R, F]$. It is easy to see that $[\bar{S}, G] = [\bar{S}, \bar{F}] = [S, F]/[R, F]$. On the other hand, by a theorem of Lue [8] there exists the epimorphism

$$\varphi : N \wedge G \to [S, F]/[R, F]$$
$$sR \wedge fR \mapsto [s, f][R, F].$$

It remains to prove that φ is an isomorphism. Using the universal property of free groups and tensor products, we obtain an isomorphism $\theta : [F, F] \to F \wedge F$ with $\theta([x, y]) = x \wedge y$, for $x, y \in F$. Then the restriction of θ to $[S, F]$ is the homomorphism $\theta|_{[S,F]} : [S, F] \to S \wedge F$. Now considering the natural epimorphism $S \wedge F \to N \wedge G$, we obtain the homomorphism

$$\psi : [S, F] \to N \wedge G,$$
$$[s, f] \mapsto sR \wedge fR$$

whose kernel contains $[R, F]$. Hence $\bar{\psi} : [S, F]/[R, F] \to N \wedge G$ is a homomorphism such that $\bar{\psi} \circ \varphi = 1$ and $\varphi \circ \bar{\psi} = 1$, and the proof is completed. ∎

Let G, N, F, R and S be the above groups and the action of G on $S/[R,F]$ be considered as in Lemma 3.1. Define the group homomorphism

$$\sigma : \frac{S}{[R,F]} \to G,$$
$$s[R,F] \mapsto \pi(s)$$

where π is the natural epimorphism from F to G. It is straightforward to check that σ is a relative central extension. Now using this, a description of the exterior G-center of N, in terms of the given free presentation of G, is presented.

Theorem 3.1. *Considering the above notation and assumption, we have*

$$Z_G^\wedge(N) = \sigma \left(Z \left(\frac{S}{[R,F]}, G \right) \right).$$

Proof. Lemma 3.1 implies that $[\bar{s}, \bar{f}] = 1$ if and only if $\pi(s) \wedge g = 1$, for all $s \in S$, $f \in F$ and $g \in G$ with $\pi(f) = g$. (Note that for $x \in F$, $\bar{x}$ denotes the image of x in $F/[R,F]$). Hence

$$\sigma \left(Z \left(\frac{S}{[R,F]}, G \right) \right) = \{ \sigma(\bar{s}) |\ [\bar{s}, g] = 1,\ \text{for all } g \in G \}$$
$$= \{ \sigma(\bar{s}) |\ [\bar{s}, \bar{f}] = 1,\ \text{for all } f \in F \}$$
$$= \{ \pi(s) |\ \pi(s) \wedge g = 1,\ \text{for all } g \in G \}$$
$$= Z_G^\wedge(N). \qquad \blacksquare$$

4. The precise center of a pair of groups

In this section, we introduce a central subgroup $Z^*(G, N)$ of G for a pair (G, N), which is as useful and important as $Z_G^\wedge(N)$. It is also shown that the subgroup $Z^*(G, N)$ is actually a subgroup of $Z_G^\wedge(N)$ with the property that (G, N) is capable if and only if $Z^*(G, N) = 1$.

Definition 4.1. *Let G be a group and $N \trianglelefteq G$. Then the precise center of the pair (G, N) is denoted by $Z^*(G, N)$ and is defined to be*

$$\cap \{ \psi(Z(M, G)) \,|\, \psi : M \to G \quad \text{is a relative central extension} \}.$$

One may see that if $N = G$, then $Z^*(G, G)$ is exactly $Z^*(G)$ defined in [2].

Theorem 4.1. *If G is a group with a normal subgroup N, then*

$$\left(\frac{G}{Z^*(G, N)}, \frac{N}{Z^*(G, N)} \right)$$

is capable.

Proof. Let $\varphi : M \to G$ be a relative central extension of the pair (G, N) and $\pi : G \to G/\varphi(Z(M, G))$ be the natural epimorphism. Put $\psi = \pi o \varphi$. It is straightforward to check that ψ is a relative central extension of the pair

$$\left(\frac{G}{\varphi(Z(M, G))}, \frac{N}{\varphi(Z(M, G))} \right)$$

such that

$$\ker \psi = Z\left(M, \frac{G}{\varphi(Z(M,G))}\right).$$

It follows that the pair

$$\left(\frac{G}{\varphi(Z(M,G))}, \frac{N}{\varphi(Z(M,G))}\right)$$

is capable. So the result follows from Theorem 2.1. ∎

Now the criterion for capability of a pair of groups is an immediate consequence of Theorem 4.1 as follows.

Corollary 4.1. *A pair (G, N) of groups is capable if and only if $Z^*(G, N) = 1$.*

Another useful and interesting property of the precise center is given in the next theorem.

Theorem 4.2. *Let (G, N) be a pair of groups. Then $Z^*(G, N)$ is the smallest normal subgroup K of G such that $(G/K, N/K)$ is capable.*

Proof. Let K be a normal subgroup of G such that the pair $(G/K, N/K)$ is capable. Then there exists a relative central extension $\varphi : M \to G/K$ of $(G/K, N/K)$ with $\ker \varphi = Z(M, G/K)$. Define $H = \{(m, x) \in M \times N | \varphi(m) = xK\}$ with an action of G on H defined by $(m, x)^g = (m^{gK}, x^g)$, for all $g \in G$, $x \in N$ and $m \in M$. Note that m^{gK} is the action of gK on m. Now considering $\psi : H \to G$ by $\psi(m, x) = x$, one can easily see that ψ is a relative central extension of (G, N). So $Z^*(G, N) \subseteq \psi(Z(H, G))$. On the other hand, $(m, x) \in Z(H, G)$ implies that $x \in K$. Therefore $Z^*(G, N) \subseteq K$. ∎

Invoking Theorem 3.1, one can observe that $Z^*(G, N) \subseteq Z_G^{\wedge}(N)$. This point together with Corollary 4.1, show that the precise center of a pair (G, N) is a smaller and more suitable subgroup for characterizing the capability of the pair with respect to the exterior G-center of N. This means that Corollary 4.1 sharpens the criterion obtained by Ellis [4, Theorem 3], while in fact the method applied in this article is easier and also much shorter than [4].

We now intend to determine a sufficient condition under which the precise center of a pair (G, N) coincides with the exterior G-center of N. For this aim, we should recall from Ellis [4] that all relative central extensions of a pair (G, N) form a category and this category is denoted by $\mathcal{RCE}(G, N)$. Let $\delta : M \to G$ and $\delta' : M' \to G$ be two relative central extensions of a pair (G, N). A morphism between these relative central extensions is a group homomorphism $\psi : M \to M'$ satisfying $\delta' \circ \psi = \delta$ and $\psi(m^g) = \psi(m)^g$, for all $g \in G$ and $m \in M$. A universal object in this category is naturally called a *universal relative central extension*. Now the above mentioned sufficient condition is stated.

Theorem 4.3. *Let (G, N) be a pair of groups and F/R be a free presentation of G with $N = S/R$. If $\sigma : S/[R, F] \to G$ (defined in the Section 3) is a universal relative central extension, then*

$$Z^*(G, N) = Z_G^{\wedge}(N).$$

Proof. Let $\varphi : M \to G$ be an arbitrary relative central extension of (G, N). Since σ is universal, then there exists a group homomorphism $\psi : S/[R, F] \to M$, such that $\varphi \circ \psi = \sigma$ and $\psi(\bar{s}^g) = \psi(\bar{s})^g$, for all $g \in G$ and $\bar{s} \in S/[R, F]$. Thus if $\bar{s} \in Z(S/[R, F], G)$, then $\psi(\bar{s}) \in Z(M, G)$. So $\sigma(\bar{s}) = \varphi \circ \psi(\bar{s}) \in \varphi(Z(M, G))$ and therefore $\sigma(Z(S/[R, F], G)) \leq Z^*(G, N)$. This implies that $\sigma(Z(S/[R, F], G)) = Z^*(G, N)$. Now the result is an immediate consequence of Theorem 3.1. ∎

The precise center of a pair of products of groups has the following property.

Theorem 4.4. *Let I be an ordered set and G_i be a group with $N_i \trianglelefteq G_i$, for all $i \in I$. Then*

$$Z^* \left(\prod_{i \in I} G_i, \prod_{i \in I} N_i \right) \subseteq \prod_{i \in I} Z^*(G_i, N_i).$$

Proof. Let $\psi_i : M_i \to G_i$ be an arbitrary relative central extension of (G_i, N_i), for all $i \in I$. Put $G = \prod_{i \in I} G_i$, $N = \prod_{i \in I} N_i$ and $M = \prod_{i \in I} M_i$. Define

$$\Psi : M \to G.$$

$$\{m_i\}_{i \in I} \mapsto \{\psi_i(m_i)\}_{i \in I}$$

It is easy to check that Ψ is a relative central extension of (G, N) and $\Psi(Z(M, G)) = \prod_{i \in I} \psi_i(Z(M_i, G_i))$. Therefore $Z^*(G, N) \leq \prod_{i \in I} \psi_i(Z(M_i, G_i))$. Since ψ_i's are arbitrary, the result follows. ∎

The above theorem has an immediate consequence for the capability of a pair of products of groups.

Corollary 4.2. *Let (G_i, N_i) be a capable pair of groups, for all $i \in I$. Then $\left(\prod_{i \in I} G_i, \prod_{i \in I} N_i \right)$ is capable.*

5. Finitely generated Abelian group

In this section we present some interesting applications of Theorem 3.1 for finitely generated abelian groups. We give a necessary and sufficient condition under which the pair (G, N) of finitely generated abelian groups is capable. Another description of the exterior G-center subgroup $Z_G^{\wedge}(N)$, in terms of the precise center $Z^*(G)$, is also provided.

Now, first recall that if a group G is presented as a quotient of a free group F by a normal subgroup R, then the *Schur multiplier* of G is defined to be

$$M(G) = \frac{R \cap F'}{[R, F]}.$$

To attain the mentioned results, we need the following lemma from [2].

Lemma 5.1. *Let N be a central subgroup of G. Then $N \subseteq Z^*(G)$ if and only if the natural map $M(G) \to M(G/N)$ is monomorphism.*

Theorem 5.1. *Let (G, N) be a pair of groups and $K \leq N$. Let F/R be a free presentation of G with $N = S/R$ and $K = T/R$. Then $K \leq Z_G^{\wedge}(N)$ if and only if $[T, F]/[R, F] = 1$.*

Proof. Using Theorem 3.1, we have

$$K \leq Z_G^\wedge(N) \Longleftrightarrow \sigma\left(\frac{T}{[R,F]}\right) \leq \sigma\left(Z\left(\frac{S}{[R,F]}, G\right)\right)$$

$$\Longleftrightarrow \frac{T}{[R,F]} \leq Z\left(\frac{S}{[R,F]}, G\right)$$

$$\Longleftrightarrow \left[\frac{T}{[R,F]}, G\right] = 1$$

$$\Longleftrightarrow \left[\frac{T}{[R,F]}, \frac{F}{[R,F]}\right] = 1$$

$$\Longleftrightarrow \frac{[T,F]}{[R,F]} = 1. \qquad \blacksquare$$

Now Theorem 5.1 gives a relationship between the precise center $Z^*(G)$ and the exterior G-center subgroup $Z_G^\wedge(N)$ for an abelian group G as follows.

Corollary 5.1. *Let G be an abelian group and $N \leq G$. Then*

$$Z_G^\wedge(N) = N \cap Z^*(G).$$

Proof. Let F/R be a free presentation of G and $K \leq N$ with $K = T/R$. Then $K \leq Z^*(G) \cap N$ if and only if $M(G) \to M(G/K)$ is injective and this is equivalent to the equality $[T,F]/[R,F] = 1$. Now the required assertion follows from Theorem 5.1. $\blacksquare$

As an application of Corollary 5.1, we can establish a complete classification of finitely generated abelian capable pairs.

Theorem 5.2. *Let G be a finitely generated abelian group as follows:*

$$G = \langle x_1 \rangle \oplus ... \oplus \langle x_m \rangle \oplus \langle y_1 \rangle \oplus ... \oplus \langle y_r \rangle,$$

where $\langle x_i \rangle \cong \mathbf{Z}$, for $1 \leq i \leq rm$ and $|y_i| = d_i$ for $1 \leq i \leq r$, such that $d_{i+1} \mid d_i$. If $N \leq G$ such that $N = \langle x_1^{\alpha_1} \rangle \oplus ... \oplus \langle x_m^{\alpha_m} \rangle \oplus \langle y_1^{\beta_1} \rangle \oplus ... \oplus \langle y_r^{\beta_r} \rangle$, then (G, N) is capable if and only if

(i) $m \geq 2$, *or*
(ii) $m = 0$, $r \geq 2$ *and* $d_1 \mid [d_2, \beta_1]$,

in which $[d_2, \beta_1]$ means the least common multiple of d_2 and β_1.

Proof. It follows from [2, Proposition 7.3] that

$$Z^*(G) = \begin{cases} 1; & m \geq 2, \\ \langle x_1^{d_1} \rangle; & m = 1, \\ \langle y_1^{d_2} \rangle; & m = 0. \end{cases}$$

Then, by Corollary 5.1, we have

$$Z_G^\wedge(N) = Z^*(G) \cap N = \begin{cases} 1; & m \geq 2, \\ \langle x_1^{[d_1, \alpha_1]} \rangle; & m = 1, \\ \langle y_1^{[d_2, \beta_1]} \rangle; & m = 0. \end{cases}$$

The result now follows easily. $\blacksquare$

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