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A Characterization of Cayley Graphs of Brandt Semigroups

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Abstract. In this paper, first we characterize Cayley graphs of finite Brandt semigroups, and we give a criterion to check whether a finite digraph is a Cayley graph of a finite Brandt semigroup. Also Kelarev and Praeger gave necessary and sufficient conditions for Cayley graphs of semigroups to be vertex-transitive. Then, some authors gave descriptions for all vertex-transitive Cayley graphs of some special classes of semigroups. In this note similar descriptions for all vertex-transitive Cayley graphs of Brandt semigroups are given.

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1. Introduction

Let S be a semigroup and C be a subset of S. Recall that the *Cayley graph* $\mathrm{Cay}(S,C)$ of S with the *connection set* C is defined as the digraph with vertex set S and arc set $E(\mathrm{Cay}(S,C)) = \{(s,cs) : s \in S, c \in C\}$.

Cayley graphs of groups have been extensively studied and some interesting results have been obtained (see for example, [1]). Also, the Cayley graphs of semigroups have been considered by some authors (see for example, [2], [3], [6]–[17]).

It is known that the Cayley graphs of groups are *vertex transitive*; i.e. for every two vertices g_1, g_2 there exists a graph automorphism ϕ such that $\phi(g_1) = g_2$. In [10], Kelarev and Praeger characterized vertex transitive Cayley graphs $\mathrm{Cay}(S,C)$ of semigroups S for which all principal left ideals of the subsemigroup generated by the connection set C are finite. Using this result, in [3, 14, 15, 17], descriptions of vertex transitive Cayley graphs of some special classes of semigroups are given. In this paper we give similar descriptions for all vertex-transitive Cayley graphs of Brandt semigroups which form one of the most popular classes of semigroups. Sabidussi in [18] presented a criterion to check whether a digraph is a Cayley graph of a group. In [16] by presenting a characterization of the Cayley graphs

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of Clifford semigroups, a similar criterion for these Cayley graphs is obtained. Similarly in [15], a characterization of the Cayley graphs of rectangular groups is obtained. Also in this note, we present a characterization of Cayley graphs of finite Brandt semigroups and we give a criterion to check whether a finite digraph is a Cayley graph of a finite Brandt semigroup.

2. Preliminaries

A *digraph* (*directed graph*) Γ is a nonempty set $V = V(\Gamma)$ of *vertices*, together with a binary relation $E = E(\Gamma)$ on V. We denote the digraph Γ by $\Gamma = (V,E)$. A digraph is *symmetric* if the relation E is symmetric. Symmetric digraphs are more conveniently viewed as (undirected) graphs. The elements $a = (u,v)$ of E are called the *arcs* of Γ, u is said the *tail* of a and v is its *head*. An *empty digraph* is one with no arcs. Given a digraph Γ, the *underlying graph* of Γ which is denoted by $\bar{\Gamma}$, is the graph with the same vertices of Γ and $(u,v),(v,u) \in E(\bar{\Gamma})$ if (u,v) or (v,u) belongs to $E(\Gamma)$. A digraph Γ is said to be *connected* if its underlying graph is connected. If for each pair of vertices u,v of Γ, there exists a directed path from u to v, then Γ is said to be *strongly connected*. By a *connected component* of a digraph Γ we mean any component of the underlying graph of Γ. The *in-degree* $d^-_\Gamma(v)$ of a vertex v in a digraph Γ is the number of arcs with head v; the *out-degree* $d^+_\Gamma(v)$ of v is the number of arcs with tail v.

Let $\Gamma = (V,E)$ be a digraph. Suppose that V' is a nonempty subset of V. The subgraph of Γ whose vertex set is V' and whose arc set is the set of those arcs of Γ that have both ends in V' is called the *subgraph of* Γ *induced by* V' and is denoted by $\Gamma[V']$. The *union* of digraphs Γ_1 and Γ_2, written $\Gamma_1 \cup \Gamma_2$, is the digraph with vertex set $V(\Gamma_1) \cup V(\Gamma_2)$ and arc set $E(\Gamma_1) \cup E(\Gamma_2)$. If Γ_1 and Γ_2 are disjoint, we denote their union by $\Gamma_1 + \Gamma_2$. In this paper, the i-th projection map is denoted by π_i.

Let S be a semigroup, and C be a nonempty subset of S. The *Cayley digraph* $\mathrm{Cay}(S,C)$ *of* S *relative to* C (which is simply called Cayley graph) is defined as the digraph with vertex set S and arc set $E(C)$ consisting of those ordered pairs (s,t) such that $cs = t$, for some $c \in C$. The set C is called the *connection set* of $\mathrm{Cay}(S,C)$ (see [7]). Obviously, if C is an empty set, then $\mathrm{Cay}(S,C)$ is an empty digraph.

Let $\Gamma_1 = (V_1,E_1)$ and $\Gamma_2 = (V_2,E_2)$ be digraphs. A *graph (digraph) homomorphism* $\phi : \Gamma_1 \to \Gamma_2$ is a mapping $\phi : V_1 \to V_2$ such that $(u,v) \in E_1$ implies $(\phi(u),\phi(v)) \in E_2$, and is called a *graph (digraph) isomorphism* if it is bijective and both ϕ and ϕ^{-1} are graph homomorphisms. A graph homomorphism $\phi : \Gamma \to \Gamma$ is called an *endomorphism*, and a graph isomorphism $\phi : \Gamma \to \Gamma$ is said to be an *automorphism*. We denote the set of all endomorphisms on a digraph Γ by $\mathrm{End}(\Gamma)$, and the set of all automorphisms on Γ by $\mathrm{Aut}(\Gamma)$.

For a Cayley graph $\mathrm{Cay}(S,C)$, we denote End $(\mathrm{Cay}(S,C))$ by $\mathrm{End}_C(S)$, and Aut $(\mathrm{Cay}(S,C))$ by $\mathrm{Aut}_C(S)$. An element $f \in \mathrm{End}_C(S)$ is called a *color-preserving endomorphism* if $cx = y$ implies $cf(x) = f(y)$ for every $x,y \in S$ and $c \in C$. The set of all color-preserving endomorphisms of $\mathrm{Cay}(S,C)$ is denoted by $\mathrm{Col\,End}_C(S)$, and the set of all color-preserving automorphisms of $\mathrm{Cay}(S,C)$ by $\mathrm{Col\,Aut}_C(S)$. Obviously $\mathrm{Col\,End}_C(S) \subseteq \mathrm{End}_C(S)$ and $\mathrm{Col\,Aut}_C(S) \subseteq \mathrm{Aut}_C(S)$.

The following proposition, known as Sabidussi's Theorem, gives a criterion to check whether a digraph is a Cayley graph of a group (see also [16, Theorem 2.5]).

Proposition 2.1. [18] *A finite digraph $\Gamma = (V, E)$ is a Cayley graph of a group G if and only if the automorphism group of Γ contains a subgroup Δ isomorphic to G such that for every two vertices $u, v \in V$ there exists a unique $\sigma \in \Delta$ such that $\sigma(u) = v$.*

The Cayley graph $\mathrm{Cay}(S,C)$ is said to be *automorphism-vertex transitive* or simply $\mathrm{Aut}_C(S)$-*vertex-transitive* if, for every two vertices $x, y \in S$, there exists $f \in \mathrm{Aut}_C(S)$ such that $f(x) = y$. The notions of $\mathrm{Col\,Aut}_C(S)$-*vertex-transitive*, $\mathrm{Col\,End}_C(S)$-*vertex-transitive*, and $\mathrm{End}_C(S)$-*vertex-transitive* for Cayley graphs are defined similarly.

A *right zero semigroup* (*left zero semigroup*) is a semigroup S satisfying the identity $xy = y$ ($xy = x$). Also, recall that a semigroup is said to be *left simple* (*right simple*) if it has no proper left (right) ideals. A semigroup is called a *left group* (*right group*) if it is left (right) simple and right (left) cancellative. It is known that a semigroup is a right (left) group if and only if it is isomorphic to the direct product of a group and a right (left) zero semigroup (see [5]). The following proposition describes all semigroups S and all subsets C of S, satisfying a certain finiteness condition, such that the Cayley graph $\mathrm{Cay}(S,C)$ is $\mathrm{Col\,Aut}_C(S)$-vertex-transitive.

Proposition 2.2. [10, Theorem 2.1] *Let S be a semigroup, and C be a subset of S which generates a subsemigroup $\langle C\rangle$ such that all principal left ideals of $\langle C\rangle$ are finite. Then, the Cayley graph $\mathrm{Cay}(S,C)$ is $\mathrm{Col\,Aut}_C(S)$-vertex-transitive if and only if the following conditions hold:*

(i) *$cS = S$, for all $c \in C$;*
(ii) *$\langle C\rangle$ is isomorphic to a right group;*
(iii) *$|\langle C\rangle s|$ is independent of the choice of $s \in S$.*

A semigroup is *completely simple* if it has no proper ideals and has an idempotent element which is minimal with respect to the partial order on idempotents $e \le f \Leftrightarrow e = ef = fe$.

Proposition 2.3. [10, Theorem 2.2] *Let S be a semigroup, and C be a subset of S such that all principal left ideals of the subsemigroup $\langle C\rangle$ are finite. Then, the Cayley graph $\mathrm{Cay}(S,C)$ is $\mathrm{Aut}_C(S)$-vertex-transitive if and only if the following conditions hold:*

(i) *$CS = S$;*
(ii) *$\langle C\rangle$ is a completely simple semigroup;*
(iii) *the Cayley graph $\mathrm{Cay}(\langle C\rangle, C)$ is $\mathrm{Aut}_C(\langle C\rangle)$-vertex-transitive;*
(iv) *$|\langle C\rangle s|$ is independent of the choice of $s \in S$.*

Let G be a group and I_λ be a set of cardinality $\lambda > 0$. Now we define a semigroup operation on $S = (I_\lambda \times G \times I_\lambda) \cup \{0\}$ as follows:

$$(i,g,j)(l,h,k) = \begin{cases} (i,gh,k), & if \quad j = l, \\ 0, & if \quad j \neq l; \end{cases}$$

and $(i,g,j)0 = 0(i,g,j) = 00 = 0$, for all $i, j, l, k \in I_\lambda$ and $g, h \in G$. Then the semigroup S is called a *Brandt semigroup* and is denoted by $B(G,\lambda)$.

Lemma 2.1. [10, Lemma 6.1] *Let S be a semigroup, and C be a subset of S.*

(i) *If $\mathrm{Cay}(S,C)$ is $\mathrm{End}_C(S)$-vertex-transitive, then $CS = S$.*
(ii) *If $\mathrm{Cay}(S,C)$ is $\mathrm{Col\,End}_C(S)$-vertex-transitive, then $cS = S$ for each $c \in C$.*

Lemma 2.2. [10, Lemma 5.2, Corollary 5.3] *Let S be a semigroup with a subset C such that $\langle C\rangle$ is completely simple, and $CS = S$. Then, every connected component of the Cayley graph*

$\mathrm{Cay}(S,C)$ is strongly connected, and for every $v \in S$, the connected component containing v is equal to $\langle C\rangle v$. Also, if $\langle C\rangle$ is isomorphic to a right group, then the right $\langle C\rangle$-cosets are the connected components of $\mathrm{Cay}(S,C)$.

For more information on graphs, we refer to [4], and for semigroups see [5].

3. Characterization of Cayley graphs of Brandt semigroups

In this section, we suppose that every digraph is finite. To provide a criterion for Cayley graphs of finite Brandt semigroups, we present a characterization of Cayley graphs of finite Brandt semigroups. Let S be a finite Brandt semigroup and $C \subseteq S$. Then it is obvious that if $0 \in C$, then each vertex of $\mathrm{Cay}(S,C)$ is joined to 0. Also if $C = \emptyset$, then $\mathrm{Cay}(S,C)$ is an empty digraph. Therefore in the sequel of this section we suppose that C is a nonempty set and $0 \notin C$.

Theorem 3.1. *A finite digraph D is a Cayley graph of a finite Brandt semigroup if and only if D consists of a vertex v_0, with a loop on it, and λ mutually disjoint subgraphs $\{D_\alpha\}_{\alpha=1}^{\lambda}$ such that $v_0 \notin V(D_\alpha)$, for each α. Also the arc set of D satisfies the following conditions: there exists no arc between $V(D_\alpha)$ and $V(D_{\alpha'})$, for $1 \le \alpha, \alpha' \le \lambda$ and $\alpha \neq \alpha'$, and every D_α is isomorphic to a digraph denoting by $\Gamma = (V,E)$ such that*

1. *$V = \bigcup_{i=1}^{\lambda} V_i$, where V_i's are pairwise disjoint and have the same cardinality,*
2. *there exists a group G such that for every $1 \le i \le \lambda$, if $\Gamma_i = \Gamma[V_i]$, then $\Gamma_i \cong \mathrm{Cay}(G,C_i)$, for some $C_i \subseteq G$,*
3. *there exists a family of graph isomorphisms $\{f_i\}_{i=1}^{\lambda}$, $f_i : \mathrm{Cay}(G,C_i) \to \Gamma_i$, for $1 \le i \le \lambda$ such that if, for $x \in G$ and e the identity of G, $f_i(e)$ is joined to $f_j(x)$, then $f_i(g)$ is joined to $f_j(xg)$ for every $g \in G$. Also there is not any other arc from Γ_i to Γ_j. Let C_{ij} be the elements of G, say x, such that $f_i(e)$ is joined to $f_j(x)$.*

Moreover let $\eta_\alpha : \Gamma \to D_\alpha$, where $1 \le \alpha \le \lambda$, be the isomorphism between Γ and D_α. For every $1 \le \alpha \le \lambda$, if $C_i \neq \emptyset$, for some $1 \le i \le \lambda$ or $C_{ij} \neq \emptyset$, for some $1 \le i, j \le \lambda$ and $i \neq j$, then all vertices in $\eta_\alpha(V \setminus V_i)$ are joined to v_0 in D.

Proof. ($\Rightarrow$) Let $D = \mathrm{Cay}(S,C)$, where $S = (I_\lambda \times G \times I_\lambda) \cup \{0\}$ is a finite Brandt semigroup and $C \subseteq S$. By the definition of Brandt semigroup we know that I_λ is a set of cardinality λ, G is a group, and 0 is the zero of S. Without loss of generality we can assume that $I_\lambda = \{1,2,\ldots,\lambda\}$. Let $v_0 = 0$. Also since for every $c \in C$, $c0 = 0$, there exists a loop on 0. We know that $S = (\bigcup_{1\le i,j\le\lambda}\{(i,g,j)|g \in G\}) \cup \{0\}$. For every $1 \le i,j \le \lambda$, let $D_{ij} = D[\{(i,g,j)|g \in G\}]$ and $A_{ij} = \{(i,g,j) \in C|g \in G\}$. We claim that $D_{ij} \cong \mathrm{Cay}(G,C_i)$, where $C_i = \{g \in G|(i,g,i) \in C\}$. To prove it, we define $\psi_{ij} : D_{ij} \to \mathrm{Cay}(G,C_i)$, by $(i,g,j) \mapsto g$. Obviously ψ_{ij} is one-to-one and onto. So it is enough to check that ψ_{ij} preserves adjacency and non-adjacency. To prove ψ_{ij} preserves adjacency, let $v_1 = (i,g_1,j)$, $v_2 = (i,g_2,j) \in V(D_{ij})$ and $(v_1,v_2) \in E(D_{ij})$. So there exists $c \in C$ such that $v_2 = cv_1$. So $(i,g_2,j) = c(i,g_1,j)$. Thus $g_2 = \pi_2(c)g_1$, $\pi_1(c) = i$, and also since $(i,g_2,j) \neq 0$, $\pi_3(c) = i$. Hence $\pi_2(c) \in C_i$. Therefore $(g_1,g_2) \in E(\mathrm{Cay}(G,C_i))$. So $(\psi_{ij}(v_1),\psi_{ij}(v_2)) \in E(\mathrm{Cay}(G,C_i))$. To prove ψ_{ij} preserves non-adjacency, let $(\psi_{ij}(v_1),\psi_{ij}(v_2)) = (g_1,g_2) \in E(\mathrm{Cay}(G,C_i))$. Then, there exists $h \in C_i$, such that $g_2 = hg_1$. Since $h \in C_i$, $(i,h,i) \in A_{ii}$. Also since $v_1, v_2 \in V(D_{ij})$ and $(i,g_2,j) = (i,h,i)(i,g_1,j)$, we conclude that $((i,g_1,j),(i,g_2,j)) = (v_1,v_2) \in E(D_{ij})$. Therefore

$$D_{ij} \cong \mathrm{Cay}(G,C_i), \tag{3.1}$$

for each $1 \leq i, j \leq \lambda$.

Now we show that there exists no arc between $V(D_{ij})$ and $V(D_{i'j'})$, for $1 \leq i, i' \leq \lambda$, $1 \leq j, j' \leq \lambda$ and $j \neq j'$. On the contrary if there exists some arcs between $V(D_{ij})$ and $V(D_{i'j'})$ in D, there exist $(i,g,j) \in V(D_{ij})$ and $(i',g',j') \in V(D_{i'j'})$ such that $((i,g,j),(i',g',j')) \in E(D)$. Since $D = \mathrm{Cay}(S,C)$, there exists $(l,h,k) \in C$ such that $(i',g',j') = (l,h,k)(i,g,j)$. Since $(i',g',j') \neq 0$, we get that $k = i$. Thus $(i',g',j') = (l,hg,j)$. Hence $j = j'$, which is a contradiction. Now we prove that D has λ subgraphs $\{D_\alpha\}_{\alpha=1}^{\lambda}$ such that D_α's are pairwise disjoint and isomorphic to each other. Let $D_\alpha = D[\bigcup_{i=1}^{\lambda} V(D_{i\alpha})]$, for $1 \leq \alpha \leq \lambda$. Then the D_α's are pairwise disjoint and there exists no arc between D_α and $D_{\alpha'}$ if $\alpha \neq \alpha'$. Obviously, $V(D) = \bigcup_{\alpha=1}^{\lambda} V(D_\alpha) \cup \{0\}$. Now we prove that D_α's are isomorphic to each other. To prove it, for every arbitrary $1 \leq \alpha, \alpha' \leq \lambda$, we define $\psi : D_\alpha \to D_{\alpha'}$, by $\psi(i,g,\alpha) = (i,g,\alpha')$, for every $(i,g,\alpha) \in V(D_\alpha)$. Since $(i_1,g_1,\alpha) = (i_2,g_2,\alpha)$ if and only if $(i_1,g_1,\alpha') = (i_2,g_2,\alpha')$, we get that ψ is well-defined and one-to-one. Also it is obvious that ψ is onto. So it is enough to prove that ψ preserves adjacency and non-adjacency. To prove ψ preserves adjacency, let $(u,v) \in E(D_\alpha)$, $u = (i_1,g_1,\alpha)$ and $v = (i_2,g_2,\alpha)$. Hence there exists $c = (l,h,k) \in C$ such that $(i_2,g_2,\alpha) = (l,h,k)(i_1,g_1,\alpha)$. So $l = i_2$, $g_2 = hg_1$ and $k = i_1$. Thus $c = (i_2,h,i_1)$ and $(i_2,g_2,\alpha') = (i_2,h,i_1)(i_1,g_1,\alpha')$. Therefore $(\psi(u),\psi(v)) \in E(D_{\alpha'})$. Similarly if $(\psi(u),\psi(v)) = ((i_1,g_1,\alpha'),(i_2,g_2,\alpha')) \in E(D_{\alpha'})$, then $((i_1,g_1,\alpha),(i_2,g_2,\alpha)) \in E(D_\alpha)$, which proves that ψ preserves non-adjacency. Without loss of generality we can assume that $\Gamma = (V,E)$ is equal to D_1. Let $\eta_\alpha : D_1 \to D_\alpha$ by

$$\eta_\alpha(i,g,1) = (i,g,\alpha), \tag{3.2}$$

where $(i,g,\alpha) \in V(D_\alpha)$ and $1 \leq \alpha \leq \lambda$.

Now we prove that conditions (1) and (2) are satisfied. Let $V_i = V(D_{i1})$ and $\Gamma_i = \Gamma[V_i]$, $1 \leq i \leq \lambda$. Therefore $\Gamma_i = D_{i1}$ and, by (3.1), we have $\Gamma_i = D_{i1} \cong \mathrm{Cay}(G,C_i)$. Also we note that $V(D_1) = \bigcup_{i=1}^{\lambda} V(D_{i1})$ and so $V = \bigcup_{i=1}^{\lambda} V_i$. Since by (3.1), $D_{i1} \cong \mathrm{Cay}(G,C_i)$, we get that $|V(D_{i1})| = |G|$. So V_i's have the same cardinality. Hence conditions (1) and (2) are satisfied.

To prove condition (3), for every $1 \leq i \leq \lambda$, we define $f_i : \mathrm{Cay}(G,C_i) \to \Gamma_i$, for $1 \leq i \leq \lambda$, by $f_i(g) = (i,g,1)$. It is easy to check that the f_i's are well-defined, one-to-one and onto. So it is enough to prove that f_i preserves adjacency and non-adjacency. To prove that f_i preserves adjacency for every arc $(g_1,g_2) \in E(\mathrm{Cay}(G,C_i))$, we know that there exists $d \in C_i$ such that $g_2 = dg_1$. So $(i,d,i) \in A_{ii}$ and $f_i(g_2) = (i,g_2,1) = (i,d,i)(i,g_1,1) = (i,d,i)f_i(g_1)$. Hence $(f_i(g_1),f_i(g_2)) \in E(\Gamma_i)$. Therefore f_i preserves adjacency. To prove f_i preserves non-adjacency, let $(f_i(g_1),f_i(g_2)) \in E(\Gamma_i)$. There exists $c \in C$ such that $f_i(g_2) = cf_i(g_1)$, since $D = \mathrm{Cay}(S,C)$. Let $c = (l,d,k)$. Similarly to the above, we conclude that $\pi_1(c) = i, \pi_3(c) = i$. Thus, $c = (i,d,i)$, $d \in C_i$ and $g_2 = dg_1$. Therefore $(g_1,g_2) \in E(\mathrm{Cay}(G,C_i))$. Hence f_i preserves adjacency and non-adjacency. Therefore f_i is a graph isomorphism. Since $(i,e,1)$ is joined to $(j,x,1)$, where $x \in C_{ij}$, it follows that $(j,x,i) \in C$ and so $\{j\} \times C_{ij} \times \{i\} \subseteq C$. Thus, for every $g \in G$, $f_i(g)$ is joined to each vertex of $\{(j,d,i)(i,g,1) | d \in C_{ij}\} = f_j(C_{ij}g)$. Now we prove that all arcs from Γ_i to Γ_j are arcs mentioned above. Let there exists an arc from a vertex $f_i(g) \in V_i = V(\Gamma_i)$, for some $g \in G$, to a vertex $f_j(g') \in V_j = V(\Gamma_j)$, where $g' \in G$. Since $D = \mathrm{Cay}(S,C)$, there exists $(l,h,k) \in C$ such that $(j,g',1) = (l,h,k)(i,g,1)$. So $l = j$, $k = i$ and $g' = hg$. Since $(j,h,i)(i,e,1) = (j,h,1)$, it follows that $f_i(e)$ is joined to $f_j(h)$. Thus $h \in C_{ij}$, and so $g' \in C_{ij}g$. Therefore $f_j(g') \in f_j(C_{ij}g)$ and condition (3) is satisfied.

Now we prove that if $C_i \neq \emptyset$ or $C_{ij} \neq \emptyset$, then each vertex of $\eta_\alpha(V \setminus V_i)$ are joined to v_0 in D, where $1 \leq \alpha \leq \lambda$. If $C_i \neq \emptyset$, then there exists $d \in C_i$ such that $(i,d,i) \in C$. Thus, for every vertex $(i',g,1) \in V \setminus V_i$, we have $i \neq i'$ and since $(i,d,i)(i',g,1) = 0$, we conclude that $(i',g,1)$ is joined to 0. Also since, for every $1 \leq \alpha \leq \lambda$, $(i,d,i)(i',g,\alpha) = 0$, we get that $\eta_\alpha(i',g,1) = (i',g,\alpha)$ is joined to 0 in D.

If $C_{ij} \neq \emptyset$, then as we mentioned above $(j,h,i) \in C$, for $h \in C_{ij}$. For every vertex $(i',g,1) \in V \setminus V_i$, we have $i \neq i'$ and since $(j,d,i)(i',g,1) = 0$, we conclude that $(i',g,1)$ is joined to 0. Also since, for every $1 \leq \alpha \leq \lambda$, $(j,h,i)(i',g,\alpha) = 0$, we get that $\eta_\alpha(i',g,1)$ is joined to 0 in D.

$(\Leftarrow)$ Take a digraph $\Gamma = (V,E)$ with properties (1)-(3) and take a digraph D with the given properties. Then D consists of a vertex v_0 with a loop on it and λ mutually disjoint subgraphs $\{D_\alpha\}_{\alpha=1}^{\lambda}$ such that each D_α is isomorphic to $\Gamma = (V,E)$. We define a Brandt semigroup S as $S = (I_\lambda \times G \times I_\lambda) \cup \{0\}$, where G is the group given in part (2) and $I_\lambda = \{1,2,\ldots,\lambda\}$. Let

$$C = \Big(\bigcup_{i=1}^{\lambda} \{i\} \times C_i \times \{i\}\Big) \cup \Big(\bigcup_{\substack{1 \leq i,j \leq \lambda \\ i \neq j}} \{j\} \times C_{ij} \times \{i\}\Big), \tag{3.3}$$

where C_i and C_{ij} are given in parts (2) and (3), respectively. Let $D' = \mathrm{Cay}(S,C)$ and $D'_\alpha = D'[\{(i,g,\alpha) | g \in G, 1 \leq i \leq \lambda\}]$, for $1 \leq \alpha \leq \lambda$. Using the $(\Rightarrow)$ part of the theorem, we conclude that $D' = \mathrm{Cay}(S,C)$ consists of the vertex 0 with a loop on it and λ pairwise disjoint subgraphs D'_α which are isomorphic to a graph satisfying conditions (1)-(3) and there exists no arc between these subgraphs. We claim that D is isomorphic to $D' = \mathrm{Cay}(S,C)$.

To prove D is isomorphic to D', first we prove that $\Gamma \cong D'_1$. Using (2), we know that $\Gamma_i = \Gamma[V_i] \cong \mathrm{Cay}(G,C_i)$, for $1 \leq i \leq \lambda$, and by (3) there exists a graph isomorphism $f_i : \mathrm{Cay}(G,C_i) \to \Gamma_i$. For every $v \in V = V(\Gamma)$, using (1) we get that there exists a unique $1 \leq i \leq \lambda$ such that $v \in V_i = V(\Gamma_i)$. To prove $\Gamma \cong D'_1$, we define $\psi : \Gamma \to D'_1$, by $\psi(v) = (i, f_i^{-1}(v), 1)$, where $v \in V_i = V(\Gamma_i)$. Now we prove that ψ is a graph isomorphism. Since f_i^{-1} is a graph isomorphism, we get that ψ is one-to-one and onto. So it is enough to show that ψ preserves adjacency and non-adjacency. Let $(u,v) \in E(\Gamma)$. There exists $1 \leq i,j \leq \lambda$ such that $u \in V_i = V(\Gamma_i)$ and $v \in V_j = V(\Gamma_j)$. Now we consider two cases. If $i = j$, then using (2) we get that there exists $d \in C_i$ such that $f_i^{-1}(v) = d f_i^{-1}(u)$. So by the definition of C in (3.3), we conclude that $(i,d,i) \in C$. Now since $(i, f_i^{-1}(v), 1) = (i,d,i)(i, f_i^{-1}(u), 1)$, we conclude that $(\psi(u), \psi(v)) \in E(D'_1)$. If $i \neq j$, then there exist $g, g' \in G$ such that $f_i(g) = u$, $f_j(g') = v$. Using (3), we get that $f_i(g)$ is joined in Γ_j only to $f_j(C_{ij}g)$. Hence $g'g^{-1} \in C_{ij}$. By the definition of C in (3.3), we get that $(j, g'g^{-1}, i) \in C$. Hence $(j, f_j^{-1}(v), 1) = (j, g'g^{-1}, i)(i,g,1) = (j, g'g^{-1}, i)(i, f_i^{-1}(u), 1)$. Thus, $(\psi(u), \psi(v)) \in E(D'_1)$. Therefore ψ preserves adjacency. To prove ψ preserves non-adjacency, let $(\psi(u), \psi(v)) \in E(D'_1)$. Also let $\psi(u) = (i,g,1)$ and $\psi(v) = (i',g',1)$. Therefore $g = f_i^{-1}(u)$ and $g' = f_{i'}^{-1}(v)$. By definition of Cayley graph, there exists $(i_c, g_c, j_c) \in C$ such that $(i',g',1) = (i_c,g_c,j_c)(i,g,1)$. So $i_c = i'$, $j_c = i$, and $g' = g_c g$. If $i = i'$, then by the definition of C in (3.3), we get that $g_c \in C_i$. Since $i = i'$, we have $g = f_i^{-1}(u)$ and $g' = f_i^{-1}(v)$. Since f_i is a graph isomorphism and $(g,g') \in E(\mathrm{Cay}(G,C_i))$, $(f_i(g), f_i(g')) = (u,v) \in E(\Gamma_i) \subseteq E(\Gamma)$. If $i \neq i'$, then $(i', g_c, i) \in C$ and so $g_c \in C_{ii'}$. Using (3), each vertex $f_i(g'')$, $g'' \in G$, is joined to $f_{i'}(g_c g'')$. Thus $f_i(g)$ is joined to $f_{i'}(g_c g) = f_{i'}(g')$. Hence u is joined to v. So $(u,v) \in E(\Gamma)$. Therefore ψ preserves non-adjacency. Hence $\Gamma \cong D'_1$.

Now we prove that $D \cong D' = \mathrm{Cay}(S,C)$. By assumption, $D' = \mathrm{Cay}(S,C)$ is a Cayley graph of a Brandt semigroup. Therefore as we mentioned in the necessary part of the proof, for each $1 \leq \alpha \leq \lambda$, there exists a graph isomorphism $\eta'_\alpha : D'_1 \to D'_\alpha$, where $\eta'_\alpha(i,g,1) = (i,g,\alpha)$ (see 3.2). To prove $D \cong D' = \mathrm{Cay}(S,C)$, we define $\mu : D \to D'$ by $\mu(v_0) = 0$ and $\mu(v) = \eta'_\alpha \psi \eta_\alpha^{-1}(v)$ if $v \in V(D_\alpha)$, for some $1 \leq \alpha \leq \lambda$. It is easy to check that μ is bijection since η'_α, ψ and η_α^{-1} are bijection and v_0 does not belong to any $V(D_\alpha)$, for $1 \leq \alpha \leq \lambda$. Hence to prove μ is a graph isomorphism, it is enough to prove that μ preserves adjacency and non-adjacency. For this purpose let $v_1, v_2 \in V(D)$ and $(v_1,v_2) \in E(D)$. Since in the graph D there does not exist any arc from v_0 to any other vertex of D, we have three following cases.

Case (1). Let $v_1 = v_2 = v_0$. Since we know that there is a loop on v_0 in D, and there is a loop on $\mu(v_0) = 0$ in D', we conclude that $(\mu(v_1),\mu(v_2)) = (0,0) \in E(D')$.

Case (2). Let $v_1 \neq v_0$ and $v_2 \neq v_0$. Since there does not exist any arc between D_α and $D_{\alpha'}$, for $1 \leq \alpha, \alpha' \leq \lambda$ and $\alpha \neq \alpha'$, we conclude that there exists some $1 \leq \alpha \leq \lambda$ such that $v_1, v_2 \in V(D_\alpha)$. Since η'_α, ψ and η_α^{-1} are graph isomorphisms, we get that $(\mu(v_1),\mu(v_2)) = (\eta'_\alpha \psi \eta_\alpha^{-1}(v_1), \eta'_\alpha \psi \eta_\alpha^{-1}(v_2)) \in E(D'_\alpha) \subseteq E(D')$.

Case (3). Let $v_1 \neq v_0$ and $v_2 = v_0$. Then $v_1 \in V(D_\alpha)$, for some $1 \leq \alpha \leq \lambda$. By the hypothesis, v_1 is joined to v_0. Therefore $C_i \neq \emptyset$, for some $1 \leq i \leq \lambda$, or $C_{ij} \neq \emptyset$, for some $1 \leq i,j \leq \lambda$, $i \neq j$ and $\eta_\alpha^{-1}(v_1) \in V \setminus V_i$. Let $\eta_\alpha^{-1}(v_1) \in V_{i'} = V(\Gamma_{i'})$, for some $1 \leq i' \leq \lambda$, where $i' \neq i$. By the definition of ψ, we know that $\psi(\eta_\alpha^{-1}(v_1)) = (i', f_{i'}^{-1}(\eta_\alpha^{-1}(v_1)), 1)$. Therefore $\mu(v_1) = \eta'_\alpha(\psi(\eta_\alpha^{-1}(v_1))) = (i', f_{i'}^{-1}(\eta_\alpha^{-1}(v_1)), \alpha) \in V(D'_\alpha)$. If $C_i \neq \emptyset$, then there exists $d \in C_i$ and so $(i,d,i) \in C$. Then $(i,d,i)(i', f_{i'}^{-1}(\eta_\alpha^{-1}(v_1)), \alpha) = 0$ shows that $\mu(v_1)$ is joined to $\mu(v_0) = 0$. Similarly if $C_{ij} \neq \emptyset$ and $d \in C_{ij}$, then by the definition of C, $(j,d,i) \in C$. Similarly to the above, we conclude that $\mu(v_1) = \eta'_\alpha \psi \eta_\alpha^{-1}(v_1) = (i', f_{i'}^{-1}(\eta_\alpha^{-1}(v_1)), \alpha)$ is joined to $\mu(v_2) = 0$ in D'.

Thus $\mu(v_1)$ is joined to $\mu(v_2)$ in D'. Therefore μ preserves adjacency. Similarly we can conclude that μ preserves non-adjacency. Hence μ is a graph isomorphism. Thus $D \cong D' = \mathrm{Cay}(S,C)$. Therefore D is isomorphic to a Cayley graph of a finite Brandt semigroup. ∎

In the next example we show that the following digraph is not a Cayley graph of a Brandt semigroup, because condition (3) of the above theorem is not satisfied.

Example 3.1. Let D be the following digraph. By Theorem 3.1, we show that D is not a Cayley graph of a Brandt semigroup. Throughout of the proof, we use the notations of Theorem 3.1. On the contrary suppose that D is a Cayley graph of a Brandt semigroup. Let $S = (I_\lambda \times G \times I_\lambda) \cup \{0\}$ be a Brandt semigroup and $C \subseteq S$ such that $D \cong \mathrm{Cay}(S,C)$.

Since $|S| = \lambda^2|G| + 1 = 17$, we get that $\lambda \in \{1,2,4\}$. In any case $v_0 = 0$. If $\lambda = 1$, then $S \cong G^0$. So, by conditions (1) and (2) of Theorem 3.1 we conclude that $D[V \setminus \{0\}]$ must be isomorphic to a Cayley graph of a group. By Proposition 2.1, we know that every Cayley graph of a group is vertex-transitive. Also we know that in a finite vertex-transitive graph the in-degree is the same for each vertex, and is equal to its out-degree. Now we note that D is not vertex-transitive because $d^-_{D[V\setminus\{0\}]}(v_3) = 1$ and $d^-_{D[V\setminus\{0\}]}(v_6) = 2$. Since $D[V \setminus \{0\}]$

is not vertex-transitive, we get that $D[V \setminus \{0\}]$ can not be isomorphic to a Cayley graph of a group, which is a contradiction. Hence $\lambda > 1$. Then there exist λ mutually disjoint subgraphs, $\{D_i\}_{i=1}^{\lambda}$ such that there exists no arc between them. Let $v_1 \in V(D_1)$. Since there does not exist any arc between D_i's, we get that $v_2, v_4, v_8 \in V(D_1)$. Since $v_2, v_4, v_8 \in V(D_1)$, similarly to the above we conclude that $v_3, v_5, v_6, v_7 \in V(D_1)$, too. Similarly we conclude that there exists D_i, where $2 \leq i \leq \lambda$, such that $v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, v'_8$ belong to $V(D_i)$. This implies that $\lambda = 2$. Without loss of generality, we can assume that $I_\lambda = \{1,2\}$. We choose $D_1 = D[\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}]$ and $D_2 = D[\{v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, v'_8\}]$. It is obvious that D_1 and D_2 are isomorphic to each other and up to isomorphism the choices of D_1 and D_2 are unique. Without loss of generality, we can assume that $\Gamma = D_1$. By condition (1), we get that $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\} = \bigcup_{i=1}^{2} V_i$ such that $|V_1| = |V_2| = 4$ and $\Gamma[V_i]$ is isomorphic to a Cayley graph of a group, for $i = 1, 2$. Without loss of generality let $v_1 \in V_1$. Now we consider the following four cases.

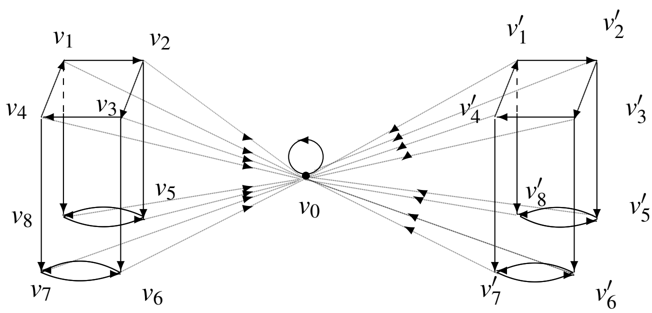

Figure 1: Digraph D.

Case (1). Let $v_2 \in V_1$ and $v_8 \in V_1$. We claim that this case can not occur. Since $v_2, v_8 \in V_1$, $d^+_{\Gamma_1}(v_1) = 2$. But $d^-_{\Gamma_1}(v_1) \leq d^-_{\Gamma}(v_1) = 1$, which is a contradiction because Γ_1 is vertex-transitive.

Case (2). Let $v_2 \notin V_1$ and $v_8 \in V_1$. Since Γ_1 is vertex-transitive, we get that $d^-_{\Gamma_1}(v_1) = d^+_{\Gamma_1}(v_1) = 1$. So $v_4 \in V_1$ and $d^-_{\Gamma_1}(v_4) = d^+_{\Gamma_1}(v_4) = 1$. Therefore $v_3 \in V_1$ and $d^-_{\Gamma_1}(v_3) = d^+_{\Gamma_1}(v_3) = 1$ which implies that $v_2 \in V_1$, and this is a contradiction.

Case (3). Let $v_2 \in V_1$ and $v_8 \notin V_1$. Then $d^-_{\Gamma_1}(v_1) = d^+_{\Gamma_1}(v_1) = 1$ and so $v_4 \in V_1$. Now similar to the above cases, we conclude that $v_3 \in V_1$. Therefore

$$V_1 = \{v_1, v_2, v_3, v_4\}, \quad V_2 = \{v_5, v_6, v_7, v_8\}.$$

So $\Gamma[V_1] \cong \mathrm{Cay}(\mathbb{Z}_4, \{c\})$, where $c = \bar{1}$ or $c = \bar{3}$, and $\Gamma[V_2] \cong \mathrm{Cay}(\mathbb{Z}_4, \{\bar{2}\})$ (we note that since $\Gamma[V_1]$ is a square, then c must be an element of order 4 and so G can be only $\mathbb{Z}_4$). Hence $S = (I_2 \times \mathbb{Z}_4 \times I_2) \cup \{0\}$. Let $f_1 : \mathrm{Cay}(\mathbb{Z}_4, \{c\}) \to \Gamma[V_1]$, where $c \in \{\bar{1}, \bar{3}\}$ and $f_2 : \mathrm{Cay}(\mathbb{Z}_4, \{\bar{2}\}) \to \Gamma[V_2]$. Now we claim that condition (3) of Theorem 3.1 can not be satisfied. To prove it we note that $v_1 = f_1(g_1)$ is joined to $v_2 = f_1(g_2) \in V_1$ and $v_8 = f_2(g') \in V_2$, for some $g_1, g_2, g' \in \mathbb{Z}_4$. Since f_1 is a graph isomorphism, $(g_1, g_2) \in E(\mathrm{Cay}(\mathbb{Z}_4, \{c\}))$ and so $g_2 = g_1 + c$. We note that $v_1 = f_1(g_1)$ is joined to $v_8 = f_2(g')$. Hence $f_1(e)$ is joined to $f_2(g' - g_1)$. By condition (3) of Theorem 3.1, since $v_2 = f_1(g_2) = f_1(g_1 + c)$ is joined to v_5, we get that $v_5 = f_2(g' - g_1 + g_1 + c)$. Therefore $v_5 = f_2(g' + c)$. Since f_2 is a graph isomorphism and $(v_5, v_8) \in E(\Gamma_2)$, we get that $(f_2^{-1}(v_5), f_2^{-1}(v_8)) \in E(\mathrm{Cay}(\mathbb{Z}_4, \{\bar{2}\}))$ and so $f_2^{-1}(v_8) = f_2^{-1}(v_5) + \bar{2}$. Thus $g' = g' + c + \bar{2}$. Hence $c = \bar{2}$, which is a contradiction because $c \in \{\bar{1}, \bar{3}\}$. Therefore in this case the graph D can not be a Cayley graph of a Brandt semigroup.

Case (4). Let $v_2 \notin V_1$ and $v_8 \notin V_1$. Then $d^-_{\Gamma_1}(v_1) = d^+_{\Gamma_1}(v_1) = 0$. So $v_4 \in V_2$. Also $d^-_{\Gamma_2}(v_2) = d^+_{\Gamma_2}(v_2) = 0$ implies that $v_3, v_5 \in V_1$. Finally $d^-_{\Gamma_2}(v_4) = 0$ and so $v_7 \in V_1$. Therefore

$$V_1 = \{v_1, v_3, v_5, v_7\}, \quad V_2 = \{v_2, v_4, v_6, v_8\}.$$

Also we note that by condition (3) of Theorem 3.1, each vertex of Γ_1 is joined to exactly $|C_{12}|$ vertices of Γ_2. Now v_1 is joined to v_2 and v_8 in $V_2 = V(\Gamma_2)$ but v_7 is joined only to v_6 in $V_2 = V(\Gamma_2)$, which is a contradiction. Therefore in this case the graph D can not be a Cayley graph of a Brandt semigroup.

So D is not a Cayley graph of a finite Brandt semigroup.

4. Vertex-transitive Cayley graphs of Brandt semigroups

In this section, we describe Cayley graphs of Brandt semigroups which are vertex transitive. Throughout this section, we assume that S is a Brandt semigroup and C is a nonempty subset of S.

Theorem 4.1. *Let $S = (I_\lambda \times G \times I_\lambda) \cup \{0\}$ be a Brandt semigroup. Let C be a subset of S which generates a subsemigroup $\langle C \rangle$ such that all principal left ideals of $\langle C \rangle$ are finite. Then the following statements are equivalent:*

(i) $\mathrm{Cay}(S, C)$ *is* $\mathrm{Col\,Aut}_C(S)$*-vertex-transitive;*
(ii) $\mathrm{Cay}(S, C)$ *is* $\mathrm{Aut}_C(S)$*-vertex-transitive;*
(iii) $\mathrm{Cay}(S, C)$ *is* $\mathrm{Col\,End}_C(S)$*-vertex-transitive;*
(iv) $|I_\lambda| = 1$*,* $S \cong G^0$ *and* $C = \{(i, e_G, i)\}$*, where* $I_\lambda = \{i\}$*;*
(v) $\mathrm{Cay}(S, C) \cong |S|\vec{K}_1$.

Proof. (i)⇒(iv) By Proposition 2.2, we get that $cS = S$, for every $c \in C$. Let $c = (i_0, g_0, j_0) \in C$. For every $s = (i, g, j) \in S$, since $cS = S$, there exists $s' = (j_0, g', j) \in S$ such that $(i, g, j) = (i_0, g_0, j_0)(j_0, g', j)$. Since s is arbitrary, for every $i \in I_\lambda$, $i = i_0$. Therefore $|I_\lambda| = 1$. Let $I_\lambda = \{i\}$. Now we define $\psi : (\{i\} \times G \times \{i\}) \cup \{0\} \to G^0$, by $(i, g, i) \mapsto g$ and $0 \mapsto 0$. Obviously, ψ is a semigroup isomorphism. Hence $S \cong G^0$. Since for every $c \in C$, $cS = S$, we get that $0 \notin C$. So $C \subseteq \{i\} \times G \times \{i\}$.

By Proposition 2.2, we conclude that $\langle C \rangle$ is isomorphic to a right group. By Lemma 2.2, we conclude that for every $v \in S$ the connected component containing v is equal to $\langle C \rangle v$. Since $|\langle C \rangle 0| = |\{0\}| = 1$, by Proposition 2.2, we conclude that for every $v \in S$, $|\langle C \rangle v| = 1$. So the cardinality of all connected components of $\mathrm{Cay}(S, C)$ are 1. Since C is not empty, all connected components of $\mathrm{Cay}(S, C)$ are isomorphic to $\vec{K}_1$. Since $C \subseteq \{i\} \times G \times \{i\}$ and all connected components of $\mathrm{Cay}(S, C)$ are isomorphic to $\vec{K}_1$, $C = \{(i, e_G, i)\}$.

(iv)⇒ (v) Since $C = \{(i, e_G, i)\}$ and for every (i, g, i) in S, $(i, e_G, i)(i, g, i) = (i, g, i)$, it follows that each vertex is joined only to itself. Therefore every connected component of $\mathrm{Cay}(S, C)$

is isomorphic to $\vec{K}_1$. Hence $\mathrm{Cay}(S,C) \cong |S|\vec{K}_1$.

(v)$\Rightarrow$ (i) It is routine to verify that the digraph $|S|\vec{K}_1$ is $\mathrm{Col\,Aut}_C(S)$-vertex-transitive.

(ii)$\Leftrightarrow$ (v) It is routine to verify that the digraph $|S|\vec{K}_1$ is $\mathrm{Aut}_C(S)$-vertex-transitive. Conversely let $\mathrm{Cay}(S,C)$ be an $\mathrm{Aut}_C(S)$-vertex-transitive Cayley graph. First we claim that $0 \notin C$. On the contrary let $0 \in C$. So all vertices of $\mathrm{Cay}(S,C)$ are joined to 0. Also we know that 0 is not adjacent to any other vertex of $\mathrm{Cay}(S,C)$. Since $\mathrm{Cay}(S,C)$ is $\mathrm{Aut}_C(S)$-vertex-transitive, for a non-zero vertex v, we conclude that there exists $f \in \mathrm{Aut}_C(S)$ such that $f(v) = 0$. Since $(v,0) \in E(\mathrm{Cay}(S,C))$, we get that $(f(v), f(0)) = (0, f(0)) \in E(\mathrm{Cay}(S,C))$. Since 0 is not adjacent to any other vertex of $\mathrm{Cay}(S,C)$, we conclude that $f(0) = 0$ which is a contradiction since $f(0) = 0 = f(v)$, $f \in \mathrm{Aut}_C(S)$ and $v \neq 0$. Therefore $0 \notin C$. On the other hand, by Proposition 2.3 we know that $|\langle C\rangle s|$ is independent of $s \in S$. Since $|\langle C\rangle 0| = |\{0\}| = 1$, and $C \neq \emptyset$, by Lemma 2.2 we conclude that all connected components of $\mathrm{Cay}(S,C)$ are isomorphic to $\vec{K}_1$. Therefore $\mathrm{Cay}(S,C) \cong |S|\vec{K}_1$.

(iii)$\Leftrightarrow$ (v) It is routine to verify that the digraph $|S|\vec{K}_1$ is $\mathrm{Col\,End}_C(S)$-vertex-transitive. Conversely let $\mathrm{Cay}(S,C)$ be a $\mathrm{Col\,End}_C(S)$-vertex-transitive Cayley graph. By Lemma 2.1, we get that $cS = S$, for every $c \in C$. Now similar to the proof of (i)$\Rightarrow$ (iv) we get that $|I_\lambda| = 1$, $0 \notin C$, and $S \cong G^0$. Let $I_\lambda = \{i\}$. Since $\mathrm{Cay}(S,C)$ is $\mathrm{Col\,End}_C(S)$-vertex-transitive and there exists a loop on the vertex 0, there exists a loop on each vertex of $\mathrm{Cay}(S,C)$. Hence $(i, e_G, i) \in C$, since $C \subseteq \{i\} \times G \times \{i\}$. Since $\mathrm{Cay}(S,C)$ is $\mathrm{Col\,End}_C(S)$-vertex-transitive, for every vertex $v \neq 0$, there exists a $\psi \in \mathrm{Col\,End}_C(S)$ such that $\psi(0) = v$. Since for every $c \in C$, $c0 = 0$, we get that $v = \psi(0) = \psi(c0) = c\psi(0) = cv$. So $(i, \pi_2(v), i) = (i, \pi_2(c), i)(i, \pi_2(v), i)$. Since $\pi_2(v) = \pi_2(c)\pi_2(v)$ and c is an arbitrary element of C, we conclude that $C = \{(i, e_G, i)\}$. So we get (iv) and we proved that (iv) and (v) are equivalent. ∎

Remark 4.1. Let $S = (I_\lambda \times G \times I_\lambda) \cup \{0\}$ be a Brandt semigroup, and let C be a subset of S. By the proof of Theorem 4.1 we conclude that the following statements are equivalent:

(i) $\mathrm{Cay}(S,C)$ is $\mathrm{Col\,End}_C(S)$-vertex-transitive;
(ii) $|I_\lambda| = 1$, $S \cong G^0$ and $C = \{(i, e_G, i)\}$ where $I_\lambda = \{i\}$;
(iii) $\mathrm{Cay}(S,C) \cong |S|\vec{K}_1$.

Now we present a necessary and sufficient condition for Cayley graphs of Brandt semigroups to be endomorphism-vertex-transitive.

Theorem 4.2. *Let $S = (I_\lambda \times G \times I_\lambda) \cup \{0\}$ be a Brandt semigroup, and let C be a subset of S such that all principal left ideals of the subsemigroup $\langle C\rangle$ are finite. Then the following statements are equivalent:*

(i) $\mathrm{Cay}(S,C)$ *is* $\mathrm{End}_C(S)$*-vertex-transitive;*
(ii) *there exists a loop on each vertex;*
(iii) $(i, e_G, i) \in C$*, for every* $i \in I_\lambda$.

Proof. (i)$\Rightarrow$(ii) Since $C \neq \emptyset$, there exists a loop on vertex 0. Also since $\mathrm{Cay}(S,C)$ is $\mathrm{End}_C(S)$-vertex-transitive, there exists a loop on each vertex of $\mathrm{Cay}(S,C)$.

(ii)$\Rightarrow$ (i) For every $s \in S$, we consider the map $\psi_s(v) = s$, which maps every vertex of $\mathrm{Cay}(S,C)$ to s. Since there exists a loop on each vertex of $\mathrm{Cay}(S,C)$, every ψ_s is a digraph endomorphism, for $s \in S$. Hence for every vertices $s,t \in S$, $\psi_s(t) = s$ and so $\mathrm{Cay}(S,C)$ is $\mathrm{End}_C(S)$-vertex-transitive.

(ii)$\Rightarrow$(iii) For every $(i,g,j) \in S\setminus\{0\}$, there exists $(i_c,g_c,j_c) \in C$ such that $(i,g,j) = (i_c,g_c,j_c)(i,g,j)$. So $j_c = i$, $i_c = i$ and $g_c g = g$. Hence $g_c = e_G$. Therefore for every $i \in I_\lambda$, $(i,e_G,i) \in C$.

(iii)$\Rightarrow$ (ii) It is obvious. ∎

Theorem 4.3. *Let* $S = (I_\lambda \times G \times I_\lambda) \cup \{0\}$ *be a Brandt semigroup and* $C \subseteq S$. *Then* $\Gamma = \mathrm{Cay}(S,C)$ *is symmetric if and only if*

(i) $|I_\lambda| = 1$;
(ii) $\pi_2(C) = (\pi_2(C))^{-1}$;
(iii) $0 \notin C$.

Proof. ($\Rightarrow$) We claim that $|I_\lambda| = 1$. On the contrary suppose that $|I_\lambda| > 1$. Since C is not empty, there exists $(i_c,g_c,j_c) \in C$. Since $|I_\lambda| > 1$, there exists $i \in I_\lambda$ such that $i \neq j_c$. So every vertex $(i,g,j) \in S$ is joined to 0, which is a contradiction since there does not exist any arc from 0 to (i,g,j) and we know that $\mathrm{Cay}(S,C)$ is symmetric. So $|I_\lambda| = 1$. Let $I_\lambda = \{i\}$. If $0 \in C$, then every vertex of Γ is joined to 0 and similarly we get a contradiction. Let $c \in C$. Since $I_\lambda = \{i\}$, we get that $c = (i,t,i)$, where $t \in G$. Therefore $(i,t,i)(i,g,i) = (i,tg,i)$ implies that $\mathrm{Cay}(S,C) \cong \mathrm{Cay}(G,\pi_2(C)) + \vec{K}_1$. To prove $\pi_2(C) = (\pi_2(C))^{-1}$, let $c \in C$. Then $c = (i,t,i)$, for some $t \in G$. For every $(i,g,i) \in S$, since $((i,g,i),(i,t,i)(i,g,i)) \in E(\mathrm{Cay}(S,C))$, then $((i,t,i)(i,g,i),(i,g,i)) \in E(\mathrm{Cay}(S,C))$. So there exists $(i,g',i) \in C$ such that $(i,g,i) = (i,g',i)(i,t,i)(i,g,i)$. Hence $t^{-1} = g' \in \pi_2(C)$. Therefore $\pi_2(C) = \pi_2(C)^{-1}$.

($\Leftarrow$) Since $|I_\lambda| = 1$, $S \cong G^0$. Also since $0 \notin C$, then as we mentioned above it follows that $\mathrm{Cay}(S,C) \cong \mathrm{Cay}(G,\pi_2(C)) + \vec{K}_1$. On the other hand we know that if $\pi_2(C) = (\pi_2(C))^{-1}$, then $\mathrm{Cay}(G,\pi_2(C))$ is symmetric. Therefore $\mathrm{Cay}(S,C)$ is symmetric. ∎

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