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Hypersurfaces with Constant k-th Mean Curvature in a Unit Sphere and Euclidean Space

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Abstract. Let M^n be an $n(n \geq 3)$-dimensional complete connected and oriented hypersurface in a unit sphere $S^{n+1}(1)$ or Euclidean space $\mathbf{R}^{n+1}$ with constant k-th mean curvature $H_k > 0 (k < n)$ and with two distinct principal curvatures λ and μ such that the multiplicity of λ is $n-1$. We show that (1) in the case of $S^{n+1}(1)$, if $k \geq 3$ and $|h|^2 \leq (n-1)t_2^{2/k} + t_2^{-2/k}$, then M^n is isometric to the Riemannian product $S^1(\sqrt{1-a^2}) \times S^{n-1}(a)$, where t_2 is the positive real root of the function $P_{H_k}(t) = kt^{\frac{k-2}{k}} - (n-k)t + nH_k$; (2) in the case of $\mathbf{R}^{n+1}$, if $|h|^2 \leq (n-1)(nH_k/(n-k))^{\frac{2}{k}}$, then M^n is isometric to the Riemannian product $S^{n-1}(a) \times \mathbf{R}$. We extend some recent results to the case $k \geq 3$.

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1. Introduction

Let $M^{n+1}(c)$ be an $(n+1)$-dimensional connected Riemannian manifold with constant sectional curvature c. According to $c > 0$ or $c = 0$, it is called sphere space or Euclidean space, respectively, and it is denoted by $S^{n+1}(c), \mathbf{R}^{n+1}$. Let M^n be an n-dimensional hypersurface in a unit sphere $S^{n+1}(1)$. As it is well known there are many rigidity results for hypersurfaces with constant mean curvature or constant scalar curvature in $S^{n+1}(1)$, for example, see [1–8]. In [7], Wei proved:

Theorem 1.1. *Let M^n be an $n(n \geq 3)$-dimensional complete connected and oriented hypersurface with constant mean curvature H and with two distinct principal curvatures such that the multiplicity of one of the principal curvatures is $n-1$ in $S^{n+1}(1)$. If*

$$|h|^2 \leq n + \frac{n^3H^2}{2(n-1)} - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2},$$

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then M^n is isometric to the Riemannin product $S^1(a)\times S^{n-1}(\sqrt{1-a^2})$, where

$$a^2=(1/2n(1+H^2))[2+nH^2+\sqrt{n^2H^4+4(n-1)H^2}]$$

On the other hand, if M^n has constant scalar curvature $n(n-1)r$ and two distinct principal curvatures, one of which is simple, Wei [8] also proved the following theorems:

Theorem 1.2. *Let M^n be an n-dimensional complete connected and oriented hypersurface in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)\ r(r\neq\frac{n-2}{n-1})$ and with two distinct principal curvatures, one of which is simple. If*

$$|h|^2\leq(n-1)\frac{n(r-1)+2}{n-2}+\frac{n-2}{n(r-1)+2},$$

then M^n is isometric to the Riemannian product $S^1(\sqrt{1-a^2})\times S^{n-1}(a)$, where $a^2=(n-2)/nr$.

Let H_k be the normalized k-th symmetric function of principal curvatures of the hypersurface M^n defined by

$$C_n^kH_k=\sum_{1\leq i_1<i_2<\cdots<i_k\leq n}\lambda_{i_1}\cdots\lambda_{i_k}, \tag{1.1}$$

where $C_n^k=n!/k!(n-k)!$. We call H_k the k-th mean curvature of M^n.

We should note that for $c=1$, if $k=1$, H_1 is the mean curvature of M^n and if $k=2$, from (1.1) and (2.11), we have $H_2=r-1$, where r is the normalized scalar curvature of M^n.

Denote by $P_{H_1}(t)$ and $P_{H_2}(t)$ the following function:

$$P_{H_1}(t)=\frac{1}{t}-(n-1)t+nH_1, \tag{1.2}$$

$$P_{H_2}(t)=-(n-2)t+nH_2+2, \tag{1.3}$$

we can rewritten Theorem 1.1 and Theorem 1.2 as follows:

Theorem 1.3. *Let M^n be an $n(n\geq 3)$-dimensional complete connected and oriented hypersurface with constant mean curvature H_1 and with two distinct principal curvatures such that the multiplicity of one of the principal curvatures is $n-1$ in $S^{n+1}(1)$. If*

$$|h|^2\leq(n-1)t_2^2+t_2^{-2},$$

then M^n is isometric to the Riemannin product

$$S^1(a)\times S^{n-1}(\sqrt{1-a^2}), a^2=\frac{1}{2n(1+H^2)}[2+nH^2+\sqrt{n^2H^4+4(n-1)H^2}],$$

where t_2 is the positive root of (1.2).

Theorem 1.4. *Let M^n be an n-dimensional complete connected and oriented hypersurface in $S^{n+1}(1)$ with constant 2-th mean curvature $H_2(H_2\neq-1/n-1)$ and with two distinct principal curvatures, one of which is simple. If*

$$|h|^2\leq(n-1)t_2+t_2^{-1},$$

then M^n is isometric to the Riemannian product $S^1(\sqrt{1-a^2})\times S^{n-1}(a)$, $a^2=(n-2)/n(H_2+1)$, where t_2 is the root of (1.3).

Now it is natural for us to consider the problem: May we extend recent results Theorem 1.3 and Theorem 1.4 of G. Wei [7, 8] to the case that M^n has constant $k(k \geq 3)$-th mean curvature $H_k(k < n)$ and with two distinct principal curvatures. Denote by $P_{H_k}(t)$ the following function:

$$P_{H_k}(t) = kt^{\frac{k-2}{k}} - (n-k)t + nH_k, \tag{1.4}$$

We may solve the problem and obtain the following:

Theorem 1.5. *Let M^n be an $n(n \geq 3)$-dimensional complete connected and oriented hypersurface with constant $k(k \geq 3)$-th mean curvature $H_k > 0(k < n)$ and with two distinct principal curvatures λ and μ such that the multiplicity of λ is $n-1$ in $S^{n+1}(1)$. If*

$$|h|^2 \leq (n-1)t_2^{2/k} + t_2^{-2/k},$$

then M^n is isometric to the Riemannin product $S^1(\sqrt{1-a^2}) \times S^{n-1}(a)$, where t_2 is the positive root of (1.4).

Let M^n be an n-dimensional hypersurface in a Euclidean space $\mathbf{R}^{n+1}$ with constant k-th mean curvature $H_k(k < n)$. Recently, Wei [9] also obtained the following result:

Theorem 1.6. *Let M^n be an $n(n \geq 3)$-dimensional complete connected and oriented hypersurface in $\mathbf{R}^{n+1}$ with constant k-th mean curvature $H_k > 0(k < n)$ and with two distinct principal curvatures, one of which is simple. If*

$$|h|^2 \geq (n-1)\left(\frac{nH_k}{n-k}\right)^{\frac{2}{k}},$$

then M^n is isometric to Riemannin product $S^{n-1}(a) \times \mathbf{R}$.

It is natural for us again to consider the problem: May we extend recent results Theorem 1.6 of Wei [9] to the case that M^n has constant $k(k \geq 3)$-th mean curvature $H_k(k < n)$ and with two distinct principal curvatures and satisfies $|h|^2 \leq (n-1)(nH_k/(n-k))^{2/k}$.

We solve the problem and obtain the following result:

Theorem 1.7. *Let M^n be an $n(n \geq 3)$-dimensional complete connected and oriented hypersurface in $\mathbf{R}^{n+1}$ with constant k-th mean curvature $H_k > 0(k < n)$ and with two distinct principal curvatures, one of which is simple. If*

$$|h|^2 \leq (n-1)\left(\frac{nH_k}{n-k}\right)^{\frac{2}{k}},$$

then M^n is isometric to Riemannin product $S^{n-1}(a) \times \mathbf{R}$.

For $c = 0$, if $k = 1$, H_1 is the mean curvature of M^n and if $k = 2$, from (1.1) and (2.11), we have $H_2 = r$, where r is the normalized scalar curvature of M^n. From Theorem 1.7, we have the following important corollaries:

Corollary 1.1. *Let M^n be an $n(n \geq 3)$-dimensional complete connected and oriented hypersurface in $\mathbf{R}^{n+1}$ with constant mean curvature H and with two distinct principal curvatures, one of which is simple. If $|h|^2 \leq n^2H^2/(n-1)$, then M^n is isometric to the Riemannian product $S^{n-1}(a) \times \mathbf{R}$.*

Corollary 1.2. *Let M^n be an $n(n \geq 3)$-dimensional complete connected and oriented hypersurface in $\mathbf{R}^{n+1}$ with constant scalar curvature $n(n-1)r$ and with two distinct principal curvatures, one of which is simple. If $|h|^2 \leq n(n-1)r/(n-2)$, then M^n is isometric to the Riemannian product $S^{n-1}(a) \times \mathbf{R}$.*

2. Preliminaries

Let $M^{n+1}(c)$ be an $(n+1)$-dimensional connected Riemannian manifold with constant sectional curvature $c(\geq 0)$. Let M^n be an n-dimensional complete connected and oriented hypersurface in $M^{n+1}(c)$. We choose a local orthonormal frame $e_1, \cdots, e_{n+1}$ in $M^{n+1}(c)$ such that $e_1, \cdots, e_n$ are tangent to M^n. Let $\omega_1, \cdots, \omega_{n+1}$ be the dual coframe. We use the following convention on the range of indices:

$$1 \leq A, B, C, \cdots \leq n+1; \quad 1 \leq i, j, k, \cdots \leq n.$$

The structure equations of $M^{n+1}(c)$ are given by

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} + \Omega_{AB}, \tag{2.2}$$

where

$$\Omega_{AB} = -\frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D, \tag{2.3}$$

$$K_{ABCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.4}$$

Restricting to M^n,

$$\omega_{n+1} = 0. \tag{2.5}$$

$$\omega_{n+1i} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji}. \tag{2.6}$$

The structure equations of M^n are

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.7}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \tag{2.8}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.9}$$

$$R_{ij} = (n-1)c\delta_{ij} + nHh_{ij} - \sum_k h_{ik}h_{kj}, \tag{2.10}$$

$$n(n-1)(r-c) = n^2H^2 - |h|^2, \tag{2.11}$$

where $n(n-1)r$ is the scalar curvature, H is the mean curvature and $|h|^2$ is the squared norm of the second fundamental form of M^n.

The Codazzi equation and the Ricci identity are

$$h_{ijk} = h_{ikj}, \tag{2.12}$$

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj} R_{mikl} + \sum_m h_{im} R_{mjkl}, \tag{2.13}$$

where h_{ijk} and h_{ijkl} denote the first and the second covariant derivatives of h_{ij}.

We choose $e_1, \cdots, e_n$ such that $h_{ij} = \lambda_i \delta_{ij}$. From (2.6), we have

$$\omega_{n+1i} = \lambda_i \omega_i, \quad i = 1, 2, \cdots, n. \tag{2.14}$$

Hence, we have from the structure equations of M^n

$$\begin{aligned} d\omega_{n+1i} &= d\lambda_i \wedge \omega_i + \lambda_i d\omega_i \\ &= d\lambda_i \wedge \omega_i + \lambda_i \sum_j \omega_{ij} \wedge \omega_j. \end{aligned} \tag{2.15}$$

On the other hand, we have on the curvature forms of $M^{n+1}(c)$,

$$\begin{aligned} \Omega_{n+1i} &= -\frac{1}{2} \sum_{C,D} K_{n+1iCD} \omega_C \wedge \omega_D \\ &= -\frac{1}{2} \sum_{C,D} c(\delta_{n+1C} \delta_{iD} - \delta_{n+1D} \delta_{iC}) \omega_C \wedge \omega_D \\ &= -c\omega_{n+1} \wedge \omega_i = 0. \end{aligned} \tag{2.16}$$

Therefore, from the structure equations of $M^{n+1}(c)$, we have

$$\begin{aligned} d\omega_{n+1i} &= \sum_j \omega_{n+1j} \wedge \omega_{ji} + \omega_{n+1n+1} \wedge \omega_{n+1i} + \Omega_{n+1i} \\ &= \sum_j \lambda_j \omega_{ij} \wedge \omega_j. \end{aligned} \tag{2.17}$$

From (2.15) and (2.17), we obtain

$$d\lambda_i \wedge \omega_i + \sum_j (\lambda_i - \lambda_j) \omega_{ij} \wedge \omega_j = 0. \tag{2.18}$$

Putting

$$\psi_{ij} = (\lambda_i - \lambda_j) \omega_{ij}. \tag{2.19}$$

Then $\psi_{ij} = \psi_{ji}$,(2.18) can be written as

$$\sum_j (\psi_{ij} + \delta_{ij} d\lambda_j) \wedge \omega_j = 0. \tag{2.20}$$

By Cartan's lemma, we get

$$\psi_{ij} + \delta_{ij} d\lambda_j = \sum_k Q_{ijk} \omega_k, \tag{2.21}$$

where Q_{ijk} are uniquely determined functions such that

$$Q_{ijk} = Q_{ikj}. \tag{2.22}$$

3. Proof of main theorems

We firstly have the following Proposition 3.1 original due to Otsuki[6].

Proposition 3.1. *Let M^n be a hypersurface in $M^{n+1}(c)$ $(c \geq 0)$ such that the multiplicities of the principal curvatures are constant. Then the distribution of the space of the principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of the principal vectors.*

Let M^n be an $n(n \geq 3)$-dimensional complete connected and oriented hypersurface with constant k-th mean curvature $H_k > 0$ and with two distinct principal curvatures one of which is simple, that is, without loss of generality, we may assume

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \lambda, \quad \lambda_n = \mu, \tag{3.1}$$

where λ_i for $i = 1, 2, \cdots, n$ are the principal curvatures of M^n. Therefore, we obtain

$$C_n^k H_k = C_{n-1}^k \lambda^k + C_{n-1}^{k-1} \lambda^{k-1} \mu,$$

this implies that

$$\lambda^{k-1}[(n-k)\lambda + k\mu] = nH_k. \tag{3.2}$$

By changing the orientation for M^n and renumbering $e_1, \cdots, e_n$ if necessary, we may assume that $\lambda > 0$. From (3.2), we have

$$\mu = \frac{n}{k} H_k \lambda^{1-k} - \frac{n-k}{k} \lambda. \tag{3.3}$$

Since

$$\lambda - \mu = n \frac{\lambda^k - H_k}{k\lambda^{k-1}} \neq 0,$$

we know that $\lambda^k - H_k \neq 0$.

Let $\varpi = |\lambda^k - H_k|^{-\frac{1}{n}}$. We denote the integral submanifold through $x \in M^n$ corresponding to λ by $M_1^{n-1}(x)$. Putting

$$d\lambda = \sum_{k=1}^{n} \lambda_{,k} \omega_k, \quad d\mu = \sum_{k=1}^{n} \mu_{,k} \omega_k. \tag{3.4}$$

From Proposition 3.1, we have

$$\lambda_{,1} = \lambda_{,2} = \cdots = \lambda_{,n-1} = 0 \text{ on } M_1^{n-1}(x). \tag{3.5}$$

From (3.3), we have

$$d\mu = \left[\frac{n(1-k)}{k} H_k \lambda^{-k} - \frac{n-k}{k} \right] d\lambda. \tag{3.6}$$

Hence, we also have

$$\mu_{,1} = \mu_{,2} = \cdots = \mu_{,n-1} = 0 \quad \text{on } M_1^{n-1}(x). \tag{3.7}$$

In this case, we may consider locally λ is a function of the arc length s of the integral curve of the principal vector field e_n corresponding to the principal curvature μ. From (2.21), (3.4) and (3.5), we have for $1 \le j \le n-1$,

$$\begin{aligned} d\lambda = d\lambda_j &= \sum_{k=1}^{n} Q_{jjk}\omega_k \\ &= \sum_{k=1}^{n-1} Q_{jjk}\omega_k + Q_{jjn}\omega_n \\ &= \lambda_{,n}\,\omega_n. \end{aligned} \tag{3.8}$$

Therefore, we have

$$Q_{jjk} = 0, \;\; 1 \le k \le n-1, \quad \text{and} \quad Q_{jjn} = \lambda_{,n}\,. \tag{3.9}$$

By (2.21), (3.4) and (3.7), we have

$$\begin{aligned} d\mu = d\lambda_n &= \sum_{k=1}^{n} Q_{nnk}\omega_k \\ &= \sum_{k=1}^{n-1} Q_{nnk}\omega_k + Q_{nnn}\omega_n \\ &= \sum_{i=1}^{n} \mu_{,i}\,\omega_i \\ &= \mu_{,n}\,\omega_n. \end{aligned} \tag{3.10}$$

Hence, we obtain

$$Q_{nnk} = 0, \quad 1 \le k \le n-1, \;\; \text{and} \;\; Q_{nnn} = \mu_{,n}\,. \tag{3.11}$$

From (3.6), we get

$$\begin{aligned} Q_{nnn} &= \mu_{,n} \\ &= \left[\frac{n(1-k)}{k} H_k \lambda^{-k} - \frac{n-k}{k}\right]\lambda_{,n}\,. \end{aligned} \tag{3.12}$$

From the definition of ψ_{ij}, if $i \ne j$, we have $\psi_{ij} = 0$ for $1 \le i \le n-1$ and $1 \le j \le n-1$. Therefore, from (2.21), if $i \ne j$ and $1 \le i \le n-1$ and $1 \le j \le n-1$ we have

$$Q_{ijk} = 0, \;\; \text{for any } \; k. \tag{3.13}$$

By (2.21), (3.9), (3.11), (3.12) and (3.13), we get

$$\begin{aligned} \psi_{jn} &= \sum_{k=1}^{n} Q_{jnk}\omega_k \\ &= Q_{jjn}\omega_j + Q_{jnn}\omega_n \\ &= \lambda_{,n}\,\omega_j. \end{aligned} \tag{3.14}$$

From (2.19), (3.3) and (3.14), we have

$$\begin{aligned} \omega_{jn} &= \frac{\psi_{jn}}{\lambda-\mu} \\ &= \frac{\lambda_{,n}}{\lambda-\mu}\omega_j \end{aligned}$$

$$= \frac{k\lambda^{k-1}\lambda_{,n}}{n(\lambda^k - H_k)}\omega_j. \tag{3.15}$$

Therefore, from the structure equations of M^n we have

$$d\omega_n = \sum_{k=1}^{n-1} \omega_k \wedge \omega_{kn} + \omega_{nn} \wedge \omega_n = 0.$$

Therefore, we may put $\omega_n = ds$. By (3.8) and (3.10), we get

$$d\lambda = \lambda_{,n} ds, \quad \lambda_{,n} = \frac{d\lambda}{ds},$$

and

$$d\mu = \mu_{,n} ds, \quad \mu_{,n} = \frac{d\mu}{ds}.$$

Then we have

$$\begin{aligned}
\omega_{jn} &= \frac{k\lambda^{k-1}\lambda_{,n}}{n(\lambda^k - H_k)}\omega_j \\
&= \frac{k\lambda^{k-1}\frac{d\lambda}{ds}}{n(\lambda^k - H_k)}\omega_j \\
&= \frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds}\omega_j. \qquad (3.16)
\end{aligned}$$

From (3.16) and the structure equations of $M^{n+1}(c)$, we have

$$\begin{aligned}
d\omega_{jn} &= \sum_{k=1}^{n-1} \omega_{jk} \wedge \omega_{kn} + \omega_{jn} \wedge \omega_{nn} + \omega_{jn+1} \wedge \omega_{n+1n} + \Omega_{jn} \\
&= \sum_{k=1}^{n-1} \omega_{jk} \wedge \omega_{kn} + \omega_{jn+1} \wedge \omega_{n+1n} - c\omega_j \wedge \omega_n \\
&= \frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds} \sum_{k=1}^{n-1} \omega_{jk} \wedge \omega_k - (\lambda\mu + c)\omega_j \wedge ds.
\end{aligned}$$

From (3.16), we have

$$\begin{aligned}
d\omega_{jn} &= \frac{d^2\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds^2} ds \wedge \omega_j + \frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds} d\omega_j \\
&= \frac{d^2\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds^2} ds \wedge \omega_j + \frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds} \sum_{k=1}^{n} \omega_{jk} \wedge \omega_k \\
&= \left\{ -\frac{d^2\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds^2} + \left[\frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds}\right]^2 \right\} \omega_j \wedge ds \\
&\quad + \frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds} \sum_{k=1}^{n-1} \omega_{jk} \wedge \omega_k.
\end{aligned}$$

From the above two equalities, we have

$$\frac{d^2\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds^2} - \left\{\frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds}\right\}^2 - (\lambda\mu + c) = 0. \tag{3.17}$$

From (3.3) we get

$$\frac{d^2\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds^2} - \left\{\frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds}\right\}^2 - \frac{n}{k}H_k\lambda^{2-k} + \frac{n-k}{k}\lambda^2 - c = 0.$$

Since we define $\varpi = |\lambda^k - H_k|^{-\frac{1}{n}}$, we obtain from the above equation

$$\frac{d^2\varpi}{ds^2} + \varpi\frac{nH_k - (n-k)\lambda^k + ck\lambda^{k-2}}{k\lambda^{k-2}} = 0. \tag{3.18}$$

We can prove the following Lemmas:

Lemma 3.1. *Let*

$$P_{H_k}(t) = kt^{\frac{k-2}{k}} - (n-k)t + nH_k,$$

where $t > 0$, $k \geq 3$ and $H_k = const. > 0$. Then $P_{H_k}(t)$ obtains its maximum at $t_0 = [(k-2)/(n-k)]^{k/2}$ and has a positive real root, denoted by t_2. In addition,

(1) *if $H_k \geq t_0$, then*
(i) *if $t \geq H_k$, then $t \leq t_2$ holds if and only if $P_{H_k}(t) \geq 0$;*
(ii) *if $t \leq H_k$, then $P_{H_k}(t) > 0$ for $t_0 < t < H_k$, $P_{H_k}(t) > 0$ for $0 < t \leq t_0$.*
(2) *if $H_k \leq t_0$, then*
(i) *if $t \geq H_k$, then $t_0 \leq t \leq t_2$ holds if and only if $P_{H_k}(t) \geq 0$ and $P_{H_k}(t) > 0$ for $H_k \leq t < t_0$;*
(ii) *if $t \leq H_k$, then $P_{H_k}(t) > 0$ for $0 < t \leq H_k$.*

Proof. We have

$$\frac{dP_{H_k}(t)}{dt} = -(n-k) + (k-2)t^{-2/k},$$

it follows that the solution of $dP_{H_k}(t)/dt = 0$ is $t_0 = [(k-2)/(n-k)]^{k/2} > 0$. Therefore, we know that if $t \leq t_0$ if and only if $P_{H_k}(t)$ is an increasing function, $t \geq t_0$ if and only if $P_{H_k}(t)$ is a decreasing function and $P_{H_k}(t)$ obtain its maximum at $t = t_0$.

Since $P_{H_k}(t)$ is continuous and $P_{H_k}(0) = nH_k > 0$, we infer that $P_{H_k}(t)$ has a positive real root, denoted by t_2. Since $P_{H_k}(H_k) = kH_k^{\frac{k-2}{k}} + kH_k > 0 = P_{H_k}(t_2)$, we have $t_2 > H_k$.

Now we prove the next part of Lemma 3.1.

(1) If $H_k \geq t_0$, we consider two cases $t \geq H_k$ and $t \leq H_k$.If $t \geq H_k$, since $t \geq t_0$ if and only if $P_{H_k}(t)$ is a decreasing function, we infer that if $t \geq H_k$, then $t \leq t_2$ if and only if $P_{H_k}(t) \geq P_{H_k}(t_2) = 0$. If $t \leq H_k$, we have $t \in (0,t_0] \cup (t_0,H_k]$, from the increasing and decreasing property of $P_{H_k}(t)$, we easily have $P_{H_k}(t) \geq P_{H_k}(H_k) = kH_k^{\frac{k-2}{k}} + kH_k > 0$ for $t_0 < t \leq H_k$, $P_{H_k}(t) > P_{H_k}(0) = nH_k > 0$ for $0 < t \leq t_0$.
(2) If $H_k \leq t_0$, we also consider two cases $t \geq H_k$ and $t \leq H_k$. If $t \geq H_k$, from the increasing and decreasing property of $P_{H_k}(t)$, we easily know that $t_0 \leq t \leq t_2$ holds if and only if $P_{H_k}(t) \geq P_{H_k}(t_2) = 0$ and $P_{H_k}(t) \geq P_{H_k}(H_k) = kH_k^{\frac{k-2}{k}} + kH_k > 0$ for $H_k \leq t < t_0$; If $t \leq H_k$, we have $t \in (0,H_k]$, from the increasing property of $P_{H_k}(t)$,

we easily have $P_{H_k}(t) > P_{H_k}(0) > 0$ for $0 < t \le H_k$. We complete the proof of Lemma 3.1. ∎

Lemma 3.2. *Let*

$$|h|^2(t) = \frac{1}{k^2 t^{(2k-2)/k}}\{(n-1)k^2t^2 + [nH_k - (n-k)t]^2\},$$

$t > 0$, $H_k = const. > 0$ *and* $k \ge 3$. *Then, if* $t \ge H_k$, $t \le t_2$ *holds if and only if* $|h|^2(t) \le (n-1)t_2^{2/k} + t_2^{-2/k}$, *where* t_2 *is the positive real root of (1.4)*

Proof. We have

$$\frac{d|h|^2(t)}{dt} = \frac{2t^{(2-3k)/k}}{k^3}\left((n^2-2nk+nk^2)t^2 + n(k-2)(n-k)H_kt + (1-k)n^2H_k^2\right),$$

it follows that the solution of $d|h|^2(t)/dt = 0$ is $t = H_k$. Therefore, we know that if $t \le H_k$ if and only if $|h|^2(t)$ is a decreasing function, $t \ge H_k$ if and only if $|h|^2(t)$ is an increasing function and $|h|^2(t)$ obtain its minimum at $t = H_k$.

From the proof of Lemma 3.1, we know that $t_2 > H_k$. Since $t \ge H_k$ if and only if $|h|^2(t)$ is an increasing function, we infer that if $t \ge H_k$, then $t \le t_2$ holds if and only if

$$\begin{aligned}
|h|^2(t) &\le |h|^2(t_2)\\
&= \frac{1}{k^2 t_2^{(2k-2)/k}}\{(n-1)k^2t_2^2 + [nH_k - (n-k)t_2]^2\}\\
&= \frac{1}{k^2 t_2^{(2k-2)/k}}\left((n-1)k^2t_2^2 + \left(\left[nH_k - (n-k)t_2 + kt_2^{\frac{k-2}{k}}\right] - kt_2^{\frac{k-2}{k}}\right)^2\right)\\
&= \frac{1}{k^2 t_2^{(2k-2)/k}}\left((n-1)k^2t_2^2 + \left(-kt_2^{\frac{k-2}{k}}\right)^2\right)\\
&= (n-1)t_2^{2/k} + t_2^{-2/k}.
\end{aligned}$$

This completes the proof of Lemma 3.2. ∎

Proof of Theorem 1.5. Firstly, we may prove that the positive function $\varpi = |\lambda^k - H_k|^{-1/n}$ is bounded from above. In fact, from the definition of ϖ and (3.18), we have

$$\frac{d^2\varpi}{ds^2} + \varpi\left(\frac{n}{k}H_k(H_k+\varpi^{-n})^{\frac{2}{k}-1} - \frac{n-k}{k}(H_k+\varpi^{-n})^{\frac{2}{k}} + c\right) = 0, \tag{3.19}$$

for $\lambda^k - H_k > 0$, or

$$\frac{d^2\varpi}{ds^2} + \varpi\left(\frac{n}{k}H_k(H_k-\varpi^{-n})^{\frac{2}{k}-1} - \frac{n-k}{k}(H_k-\varpi^{-n})^{\frac{2}{k}} + c\right) = 0, \text{ for } \lambda^k - H_k < 0. \tag{3.20}$$

Multiplying (3.19) or (3.20) by $2\frac{d\varpi}{ds}$ and integrating, we can get

$$\left(\frac{d\varpi}{ds}\right)^2 + c\varpi^2 + \varpi^2(H_k+\varpi^{-n})^{\frac{2}{k}} = C, \quad \text{for} \quad \lambda^k - H_k > 0,$$

or

$$\left(\frac{d\varpi}{ds}\right)^2 + c\varpi^2 + \varpi^2(H_k-\varpi^{-n})^{\frac{2}{k}} = C, \quad \text{for} \quad \lambda^k - H_k < 0,$$

where C is a constant. Therefore, we have

$$c\varpi^2 + \varpi^2 (H_k + \varpi^{-n})^{\frac{2}{k}} \le C, \quad \text{for} \quad \lambda^k - H_k > 0, \tag{3.21}$$

or

$$c\varpi^2 + \varpi^2 (H_k - \varpi^{-n})^{\frac{2}{k}} \le C, \quad \text{for} \quad \lambda^k - H_k < 0. \tag{3.22}$$

Since $c \ge 0$ and $H_k > 0$, from (3.21) and (3.22), we infer that the positive function ϖ is bounded from above.

Denote $t = \lambda^k (> 0)$, we have $|h|^2 = |h|^2(t)$. Putting $c = 1$ in (3.18), from (1.4), we have

$$\frac{d^2\varpi}{ds^2} + \varpi \frac{1}{kt^{(k-2)/k}} P_{H_k}(t) = 0. \tag{3.23}$$

Since

$$\lambda - \mu = n\frac{\lambda^k - H_k}{k\lambda^{k-1}} \ne 0,$$

we know that $\lambda^k - H_k \ne 0$, that is, $t \ne H_k$. Therefore, we may consider two cases $t > H_k$ and $t < H_k$.

Case (i). If $t > H_k$, we may also consider two subcases $H_k \ge t_0$ and $H_k \le t_0$, where t_0 is defined in Lemma 3.1.

Subcase (i). If $H_k \ge t_0$, since $t > H_k$, from Lemma 3.1, Lemma 3.2 and (3.23), we have $|h|^2(t) \le (n-1)t_2^{2/k} + t_2^{-2/k}$ if and only if $t \le t_2$ if and only if $P_{H_k}(t) \ge 0$ and if and only if $d^2\varpi/ds^2 \le 0$. Thus $d\varpi/ds$ is a monotonic function of $s \in (-\infty, +\infty)$. Therefore, by the similar assertion in Wei [7], we have $\varpi(s)$ must be monotonic when s tends to infinity. Since $\varpi(s)$ is bounded and monotonic when s tends to infinity, we know that both $\lim_{s\to-\infty} \varpi(s)$ and $\lim_{s\to+\infty} \varpi(s)$ exist and then we get

$$\lim_{s\to-\infty} \frac{d\varpi(s)}{ds} = \lim_{s\to+\infty} \frac{d\varpi(s)}{ds} = 0. \tag{3.24}$$

From the monotonicity of $d\varpi(s)/ds$, we have $d\varpi(s)/ds \equiv 0$ and $\varpi(s) = constant$. From $\varpi = |\lambda^k - H_k|^{-1/n}$ and (3.2), we have λ and μ are constant. Therefore, we know that M^n is isometric to the Riemannian product $S^1(\sqrt{1-a^2}) \times S^{n-1}(a)$.

Subcase (ii). If $H_k \le t_0$, since $t > H_k$, from Lemma 3.2, we have if $|h|^2(t) \le (n-1)t_2^{2/k} + t_2^{-2/k}$ then $t \le t_2$. Thus, we have $t \in (H_k, t_0] \cup (t_0, t_2]$.

If $t \in (H_k, t_0]$, from Lemma 3.1, we have $P_{H_k}(t) > 0$. From (3.23), we have $d^2\varpi/ds^2 < 0$. This implies that $d\varpi(s)/ds$ is a strictly monotone decreasing function of s and thus it has at most one zero point for $s \in (-\infty, +\infty)$. If $d\varpi(s)/ds$ has no zero point in $(-\infty, +\infty)$, then $\varpi(s)$ is a monotone function of s in $(-\infty, +\infty)$. If $d\varpi(s)/ds$ has exactly one zero point s_0 in $(-\infty, +\infty)$, then $\varpi(s)$ is a monotone function of s in both $(-\infty, s_0]$ and $[s_0, +\infty)$.

On the other hand, since $\varpi(s)$ is bounded and monotonic when s tends to infinity, we know that both $\lim_{s\to-\infty} \varpi(s)$ and $\lim_{s\to+\infty} \varpi(s)$ exist and (3.24) holds. This is impossible because $d\varpi(s)/ds$ is a strictly monotone decreasing function of s. Therefore, we know that the case $t \in (H_k, t_0]$ does not occur and we conclude that $t \in (t_0, t_2]$.

If $t \in (t_0, t_2]$, from Lemma 3.1, Lemma 3.2 and (3.23), we have $|h|^2(t) \leq (n-1)t_2^{2/k} + t_2^{-2/k}$ if and only if $t \leq t_2$ if and only if $P_{H_k}(t) \geq 0$ and if and only if $d^2\varpi/ds^2 \leq 0$. Thus $d\varpi/ds$ is a monotonic function of $s \in (-\infty, +\infty)$. By the same assertion in the proof of subcase (i), we know that M^n is isometric to the Riemannian product $S^1(\sqrt{1-a^2}) \times S^{n-1}(a)$.

Case (ii). If $t < H_k$, we may also consider two subcases $H_k \geq t_0$ and $H_k \leq t_0$.

Subcase (i). If $H_k \geq t_0$, since $t < H_k$, we have $t \in (0, t_0] \cup (t_0, H_k)$.

If $t \in (0, t_0]$, from Lemma 3.1, we have $P_{H_k}(t) > 0$. From (3.23), we have $d^2\varpi/ds^2 < 0$. This implies that $d\varpi(s)/ds$ is a strictly monotone decreasing function of s and thus it has at most one zero point for $s \in (-\infty, +\infty)$. By the same assertion in the proof of subcase (ii) in case (i), we know that the case $t \in (0, t_0]$ does not occur and we conclude that $t \in (t_0, H_k)$. If $t \in (t_0, H_k)$, from Lemma 3.1, we also have $P_{H_k}(t) > 0$. By the same assertion above, we also know that $t \in (t_0, H_k)$ does not occur.

Subcase (ii). If $H_k \leq t_0$, since $t < H_k$, from Lemma 3.1 and (3.23), we have $P_{H_k}(t) > 0$ and $d^2\varpi/ds^2 < 0$. By the same assertion above, we may know that this subcase does not occur. This completes the proof of Theorem 1.5. ∎

Proof of Theorem 1.7. Putting $c = 0$ and $t = \lambda^k$ in (3.18), we have

$$\frac{d^2\varpi}{ds^2} + \varpi \frac{nH_k - (n-k)t}{kt^{(k-2)/k}} = 0. \tag{3.25}$$

Since

$$\lambda - \mu = n\frac{\lambda^k - H_k}{k\lambda^{k-1}} \neq 0,$$

we know that $\lambda^k - H_k \neq 0$. If $\lambda^k - H_k < 0$, we have $\lambda^2 - \lambda\mu < 0$, then $\lambda^2 < \lambda\mu$, from the Gauss equation (2.9), we know that the sectional curvature of M^n is not less than $\lambda^2 > 0$. From the result of Hartman [5], we have M^n is isometric to a totally umbilical hypersurface. This is impossible because M^n has two distint principal curvatures. Thus, $\lambda^k - H_k > 0$, that is, $t = \lambda^k > H_k$

Since $|h|^2 = |h|^2(t)$, from the assumption of Theorem 1.7, we have

$$\frac{1}{k^2 t^{(2k-2)/k}}\{(n-1)k^2t^2 + [nH_k - (n-k)t]^2\} \leq (n-1)\left(\frac{nH_k}{n-k}\right)^{\frac{2}{k}},$$

that is

$$\frac{1}{k^2 t^{(2k-2)/k}}[nH_k - (n-k)t]^2 \leq \frac{n-1}{(n-k)^{\frac{2}{k}}}\left[(nH_k)^{\frac{2}{k}} - (n-k)^{\frac{2}{k}} t^{\frac{2}{k}}\right].$$

Therefore, we have $(nH_k)^{2/k} - (n-k)^{2/k}t^{2/k} \geq 0$, that is, $nH_k \geq (n-k)t$ because of $t > 0$. From (3.25), we have $d^2\varpi/ds^2 \leq 0$. By the same assertion in the proof of Theorem 1.5, we have λ and μ are constant. Therefore, we know that M^n is isometric to the Riemannian product $S^{n-1}(a) \times \mathbf{R}$. This completes the proof of Theorem 1.7. ∎

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