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Normality Criteria for Families of Meromorphic Function Concerning Shared Values

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Abstract. Let k be a positive integer and let $\mathscr{F}$ be a family of meromorphic functions in the plane domain D all of whose zeros with multiplicity at least k. Let $P = a_p z^p + \cdots + a_2 z^2 + z$ be a polynomial, $a_p, a_2 \neq 0$ and $p = \deg(P) \geq k+2$. If, for each $f, g \in \mathscr{F}$, $P(f)G(f)$ and $P(g)G(g)$ share a non-zero constant b in D, where $G(f) = f^{(k)} + H(f)$ be a differential polynomial of f satisfying $\frac{w}{\deg}|_H \leq \frac{k}{l+1} + 1$ or $w(H) - \deg(H) < k$, then $\mathscr{F}$ is normal in D.

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1. Introduction and main results

Let D be a domain in $\mathbb{C}$ and $\mathscr{F}$ is a family of meromorphic in D. The family $\mathscr{F}$ is said to be normal in D, in the sense of Montel, if each sequence $\{f_n\} \subset \mathscr{F}$ had a subsequence $\{f_{n_j}\}$ which converges spherically locally uniformly in D, to a meromorphic function or ∞ (see [9, 21, 25]). Suppose that $f(z)$, $g(z)$ are meromorphic functions in D and $a \subset \mathbb{C} \cup \{\infty\}$. If $f(z) = a$ if and only if $g(z) = a$, we say that f and g share a IM (ignoring multiplicity) (see [24]).

Definition 1.1. *Let $D \subseteq \mathbb{C}$ be an arbitrary domain, $m, l_1, l_2, \cdots, l_m$ be non-negative integers and $(0 \leq l_i \leq k)$, if*

$$M(f, f', \cdots, f^{(k)}) = a(z) \prod_{i=1}^{m} f^{(l_i)},$$

where f is meromorphic and a is a holomorphic function in D ($a \not\equiv 0$), then $M(f, f', \cdots, f^{(k)})$ is called a differential monomial of degree $\deg(M) := m$ *and weight* $w(M) := \sum_{i=1}^{m}(1 + l_i)$. *The summation $H := M_1 + \cdots + M_n$ of differential monomials M_j is called the differential polynomial of degree of* $\deg(H) := \max\{\deg(M_1), \cdots, \deg(M_n)\}$ *and weight* $w(H) :=$

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$\max\{w(M_1),\cdots,w(M_n)\}$). *Furthermore, we set*

$$\frac{w}{\deg}|_H = \max\left\{\frac{w(M_1)}{\deg(M_1)}, \frac{w(M_2)}{\deg(M_2)}, \cdots, \frac{w(M_n)}{\deg(M_n)}\right\},$$

$$G(f) = f^{(k)} + H(f, f', \cdots, f^{(k)}).$$

In 1959, Hayman [10] proposed:

Conjecture 1.1. *If f is a transcendental meromorphic function, then $f^n f'$ assumes every finite non-zero complex number infinitely often for any positive integer n.*

Hayman [10, 11] himself confirmed it for $n \geq 3$ and for $n \geq 2$ in the case of entire f. Further, it was proved by Mues [16] when $n \geq 2$; Clunie [6] when $n \geq 1$ and f is entire; Bergweiler and Eremenko [2] verified the case when $n = 1$ and f is of finite order, and finally by Chen and Fang [5] for the case $n = 1$. Correspondingly, there is a conjecture of Hayman [11] related to above problem concerning the normality of $\mathscr{F}$ (see [1]).

Conjecture 1.2. *If each $f \in \mathscr{F}$ satisfies $f^n f' \neq a$ for a positive integer n and a finite non-zero complex number a, then $\mathscr{F}$ is normal.*

Concerning this conjecture, there are many significant results have been obtained by Yang and Zhang [26], Gu [8], Oshkin [17], Li and Xie [14], Pang [18] and Zalcman [27]. Chen and Fang [5] verified the Conjecture 1.2 completely. Schick [22] was the first author to draw a connection between value shared by functions in $\mathscr{F}$ and the normality of the family $\mathscr{F}$. Moreover, many scholars had studied normality criterions such as Meng [3], Lei and Fang [13], Li and Gu [15], Pang and Zalcman [19], Xia and Xu [23].

In 2004, Fang and Zalcman [7] proved:

Theorem 1.1. *Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, let $n \geq 1$ be a positive integer, and b be a finite non-zero complex number. If, for each $f, g \in \mathscr{F}$, f and g share 0 IM; $f^n f'$ and $g^n g'$ share b IM in D, then $\mathscr{F}$ is normal in D.*

Lately, Zhang [28] obtained:

Theorem 1.2. *Let $\mathscr{F}$ be a family of holomorphic functions in a domain D, let $n \geq 1$ be a positive integer, and b be a finite complex number. If, for each $f, g \in \mathscr{F}$, $f^n(f-1)f'$ and $g^n(g-1)g'$ share b CM, then $\mathscr{F}$ is normal in D.*

In 2008, Zhang [29] weakened the condition of Theorem 1.2.

Theorem 1.3. *Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, let $n \geq 2$ be a positive integer, and b be a finite non-zero complex number. If, for each $f, g \in \mathscr{F}$, $f^n f'$ and $g^n g'$ share b IM, then $\mathscr{F}$ is normal in D.*

There are examples showing that this result is not true if $n = 1$. Recently, Lei and Fang [12] extended Theorems 1.1–1.2. They have arrived at:

Theorem 1.4. *Let $\mathscr{F}$ be a family of meromorphic functions in the plane domain D, let P be a polynomial with either* $\deg(P) \geq 3$ *or* $\deg(P) = 2$ *and P having only one distinct zero. If, for each $f, g \in \mathscr{F}$, $P(f)f'$ and $P(g)g'$ share a nonzero constant b IM in D, then $\mathscr{F}$ is normal in D.*

Theorem 1.5. *Let $\mathscr{F}$ be a family of meromorphic functions in the domain D, all of whose poles are multiple, and let P be a polynomial with two distinct zeros. If, for each $f,g \in \mathscr{F}$, $P(f)f'$ and $P(g)g'$ share complex number b IM in D, then $\mathscr{F}$ is normal in D.*

In this paper, we obtain the following extensions of Theorem 1.4 and Theorem 1.5.

Theorem 1.6. *Let k be a positive integer and let $\mathscr{F}$ be a family of meromorphic functions in the plane domain D all of whose zeros with multiplicity at least k. Let $P = a_p z^p + \cdots + a_2 z^2 + z$ be a polynomial, $a_p, a_2 \neq 0$ and $p = \deg(P) \geq k+2$. If, for each $f,g \in \mathscr{F}$, $P(f)G(f)$ and $P(g)G(g)$ share a non-zero constant b IM in D, where $G(f) = f^{(k)} + H(f)$ be a differential polynomial of f satisfying $\frac{w}{\deg}|_H \leq \frac{k}{l+1} + 1$ or $w(H) - \deg(H) < k$, then $\mathscr{F}$ is normal in D.*

Remark 1.1. If the polynomial $P(z)$ has only one zero, Theorem 1.6 is established for $\deg(P) \geq k+1$.

Corollary 1.1. *Let k be a positive integer and $\mathscr{F}$ be a family of meromorphic functions in the plane domain D all of whose zeros with multiplicity at least k. Let P be a polynomial as in Theorem 1.6. If, for each $f,g \in \mathscr{F}$, $P(f)f^{(k)}$ and $P(g)g^{(k)}$ share a non-zero constant b IM in D, then $\mathscr{F}$ is normal in D.*

Theorem 1.7. *Let k be a positive integer, suppose that $\mathscr{F}$ be a family of meromorphic functions in the plane domain D all of whose zeros and poles with multiplicity at least k and 2 respectively. Let P be a polynomial with two distinct zeros at least. If, for each $f,g \in \mathscr{F}$, $P(f)G(f)$ and $P(g)G(g)$ share a constant b IM in D, where $G(f) = f^{(k)} + H(f)$ be a differential polynomial of f with $w(H) - \deg(H) < k$, then $\mathscr{F}$ is normal in D.*

Corollary 1.2. *Let k be a positive integer, suppose that $\mathscr{F}$ be a family of meromorphic functions in the plane domain D all of whose zeros and poles with multiplicity at least k and 2 respectively. Let P be a polynomial as in Theorem 1.7. If, for each $f,g \in \mathscr{F}$, $P(f)f^{(k)}$ and $P(g)g^{(k)}$ share b IM in D, then $\mathscr{F}$ is normal in D.*

2. Preliminary Lemmas

In order to prove our theorem, we need the following lemmas:

Lemma 2.1 (Zalcman's lemma)**.** [4, 20] *Let k be a positive integer, let $\mathscr{F}$ be a family of meromorphic functions in the unit disc $\triangle$ with the property that for each $f \in \mathscr{F}$, all zeros of multiplicity at least k. Suppose that there exists a number $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f = 0$. Suppose that $\mathscr{F}$ is not normal at z_0, then for $0 \leq \alpha \leq k$, there exist*

a) *points $z_n \in \triangle$, $z_n \to z_0$;*
b) *functions $f_n \in \mathscr{F}$; and*
c) *positive numbers $\rho_n \to 0^+$;*

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ locally uniformly with respect to the spherical metric, where $g(\xi)$ is a non-constant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $g^{\sharp}(\xi) \leq g^{\sharp}(0) = kA+1$. In particular, g has order at most 2.

Lemma 2.2. [30] *Let $n \geq 2$, k be a positive integer. If f is a transcendental meromorphic function, then $f^n f^{(k)}$ assume every finite non-zero complex value infinitely often; if f is a non-constant rational function, then $f^n f^{(k)}$ assume every finite non-zero complex number one time at least.*

Lemma 2.3. [12] *Let f be a non-constant rational function, let k be a positive integer, and let b be a non-zero finite complex number. Then, $f^k f' - b$ has two distinct zeros at least.*

Lemma 2.4. *Let n, k be two positive integers such that $n \geq k+1$, and let $b \neq 0$ be a finite complex number. If f be a non-constant rational function and f has only zeros of multiplicity at least k, then $f^n f^{(k)} - b$ has two distinct zeros at least.*

Proof. Assume, to the contrary, that $f^n f^{(k)} - b$ has one zero at most. Then $f^n f^{(k)} - b$ has exactly one zero because of Lemma 2.2. Suppose that f is a non-constant polynomial, we have

$$f^n f^{(k)}(z) = A(z-z_0)^l + b, \tag{2.1}$$

where A is a non-zero constant and $l \geq 2$ is a positive integer. The right hand side of (2.1) has only simple zeros, but the left has multiple zeros, a contradiction. Thus f is a non-polynomial rational function. Next we distinguish two cases:

Case 1. When the positive integer $k \geq 2$. Set

$$f(z) = A\frac{(z-\alpha_1)^{m_1}(z-\alpha_2)^{m_2}\cdots(z-\alpha_s)^{m_s}}{(z-\beta_1)^{n_1}(z-\beta_2)^{n_2}\cdots(z-\beta_t)^{n_t}}, \tag{2.2}$$

where A is a non-zero constant. By the zeros of f are at least k, we obtain $m_i \geq k$ $(i = 1,2,\cdots,s)$, $n_j \geq 1 (j = 1,2,\cdots,t)$. Hence

$$m_1 + m_2 + \cdots + m_s \geq ks, \tag{2.3}$$

$$n_1 + n_2 + \cdots + n_t \geq t. \tag{2.4}$$

From (2.2), we obtain

$$f^{(k)}(z) = \frac{(z-\alpha_1)^{m_1-k}(z-\alpha_2)^{m_2-k}\cdots(z-\alpha_s)^{m_s-k}g(z)}{(z-\beta_1)^{n_1+k}(z-\beta_2)^{n_2+k}\cdots(z-\beta_t)^{n_t+k}}. \tag{2.5}$$

Where g is a polynomial of degree at most $k(s+t-1)$.

From (2.2) and (2.5), we obtain

$$f^n f^{(k)}(z) = \frac{A^n(z-\alpha_1)^{M_1}(z-\alpha_2)^{M_2}\cdots(z-\alpha_s)^{M_s}g(z)}{(z-\beta_1)^{N_1}(z-\beta_2)^{N_2}\cdots(z-\beta_t)^{N_t}} = \frac{P}{Q}. \tag{2.6}$$

Where P and Q are polynomials of degree M and N respectively. Also P and Q have no common factor, where $M_i = (n+1)m_i - k$ and $N_j = (n+1)n_j + k$. By (2.3) and (2.4), we deduce $M_i = (n+1)m_i - k \geq k(n+1) - k = nk$, $N_j = (n+1)n_j + k \geq n+k+1$. Thus

$$\deg(P) = M = \sum_{i=1}^{s} M_i + \deg(g) \geq nks, \tag{2.7}$$

$$\deg(Q) = N = \sum_{j=1}^{t} N_j \geq (n+k+1)t. \tag{2.8}$$

Since, $f^n f^{(k)} - a = 0$ a zero z_0 exactly, from (2.6) we obtain

$$f^n f^{(k)}(z) = a + \frac{B(z-z_0)^l}{(z-\beta_1)^{N_1}(z-\beta_2)^{N_2}\cdots(z-\beta_t)^{N_t}} = \frac{P}{Q}. \tag{2.9}$$

Note that $a \neq 0$, we obtain $z_0 \neq \alpha_i$ $(i = 1,\cdots,s)$, where B is a non-zero constant.

From (2.6), we obtain

$$[f^n f^{(k)}(z)]' = \frac{(z-\alpha_1)^{M_1-1}(z-\alpha_2)^{M_2-1}\cdots(z-\alpha_s)^{M_s-1}g_1(\xi)}{(z-\beta_1)^{N_1+1}\cdots(z-\beta_t)^{N_t+1}}. \tag{2.10}$$

Where $g_1(\xi)$ is a polynomial of degree at most $(k+1)(s+t-1)$.

From (2.9), we obtain

$$[f^n f^{(k)}(z)]' = \frac{(z-z_0)^{l-1}g_2(z)}{(z-\beta_1)^{N_1+1}\cdots+(z-\beta_t)^{N_t+1}}. \tag{2.11}$$

Where $g_2(\xi) = B(l-N)z^t + B_1 z^{t-1}\cdots+B_t$ is a polynomial ($B_1,\cdots,B_t$ are constants).

Case 1.1. If $l \neq N$, by (2.9), then we obtain the $\deg(P) \geq \deg(Q)$. So $M \geq N$. By (2.10) and (2.11), we obtain $\sum_{i=1}^{s}(M_i-1) \leq \deg(g_2) = t$. So $M-s-\deg(g) \leq t$, and $M \leq s+t+\deg(g) \leq (k+1)(s+t)-k < (k+1)(s+t)$. By (2.6) and (2.7), we obtain

$$M < (k+1)(s+t) \leq (k+1)\left[\frac{M}{nk}+\frac{N}{n+k+1}\right] \leq (k+1)\left[\frac{1}{nk}+\frac{1}{n+k+1}\right]M.$$

Since $n \geq k+1$, we deduce that $M < M$. Which is impossible.

Case 1.2. If $l = N$, then we consider two subcases.

Case 1.2.1. If $M \geq N$, by (2.10) and (2.11), we obtain $\sum_{i=1}^{s}(M_i-1) \leq \deg(g_2) = t$. So $M-s-\deg(g) \leq t$, and $M \leq s+t+\deg(g) \leq (k+1)(s+t)-k < (k+1)(s+t)$, by the same reasoning mentioned in the case 1.1. This is impossible.

Case 1.2.2. If $M < N$, by (2.10) and (2.11), we obtain $l-1 \leq \deg g_1 \leq (s+t-1)(k+1)$, then

$$N = l \leq \deg(g_1)+1 \leq (k+1)(s+t)-k < (k+1)(s+t) \leq (k+1)\left[\frac{1}{nk+k}+\frac{1}{n+k+1}\right]N \leq N.$$

Since $n \geq k+1$, we deduce that $N < N$. Which is impossible.

Case 2. When $k = 1$, then Lemma 2.3 imply this result. ∎

Lemma 2.5. [24] *Let $f(z)$ be a non-constant rational function, then $f(z)$ has only one deficient value.*

3. Proof of theorems

Proof of Theorem 1.7. Without loss of generality, we assume that $P(z) = Q(z)z(z-1)$, where $Q(z) \not\equiv 0$ is a polynomial. Suppose that $\mathscr{F}$ is not normal in D. Then there exists at least one z_0 such that $\mathscr{F}$ is not normal at z_0, we assume that $z_0 = 0$. By Lemma 2.1, there exist points $z_j \to 0$; a sequence $\rho_j \to 0^+$ and functions $f_j \in \mathscr{F}$ such that

$$g_j(\xi) = f_j(z_j+\rho_j\xi) \to g(\xi), \tag{3.1}$$

locally uniformly with respect to the spherical metric, where g is a non-constant meromorphic function in $\mathbb{C}$, all of whose zeros and poles are of multiplicity at least k and 2 respectively. If $Q(g)g(g-1)g^{(k)} \equiv 0$, then g is a constant, a contradiction.

If $Q(g)g(g-1)g^{(k)}\neq 0$, because of the zeros of g have multiplicity at least k, we obtain $g\neq 0,1$. We can claim that g is not a transcendental meromorphic function. In fact, if it is not true, we have

$$\begin{aligned} T(r,g) &\leq \overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+S(r,g) \\ &\leq \frac{1}{2}N(r,g)+S(r,g) \\ &\leq \frac{1}{2}T(r,g)+S(r,g) \end{aligned}.$$

Thus $T(r,g)=S(r,g)$, a contradiction. So g is a rational function. Since $g\neq 0,1$, then g is a constant, a contradiction. Thus $Q(g)g(g-1)g^{(k)}$ is a non-constant meromorphic function and has one zero at least.

Next we will prove that $Q(g)g(g-1)g^{(k)}$ has just a single zero. In fact, let ξ_0 and ξ_0^* be two distinct solutions of $Q(g)g(g-1)g^{(k)}$. We choose a positive number δ small enough such that g and g_j are holomorphic in $D(\xi_0,\delta_1)D(\xi_0^*,\delta_1)$ and $D(\xi_0,\delta_1)\cap D(\xi_0^*,\delta_1)=\emptyset$. From (3.1), we have

$$\begin{aligned} &\rho_j^k[Q(f_j(z_j+\rho_j\xi)f_j(z_j+\rho_j\xi)(f_j(z_j+\rho_j\xi)-1)\cdot G(f_j(z_j+\rho_j\xi)-b] \\ &=\rho_j^k[Q(g_j(\xi)g_j(\xi)(g_j(\xi)-1)\cdot(\rho_j^{-k}g_j^{(k)}(\xi)+\sum_{j=1}^{n}a_j^*\rho_j^{\deg(M_j)-w(M_j)} \\ &\quad M_i(g,g',\cdots,g^{(k)}))-b]\to Q(g(\xi))g(\xi)(g(\xi)-1)g^{(k)}(\xi). \end{aligned} \tag{3.2}$$

By Hurwitz's theorem, there exist points $\xi_j\in D(\xi_0,\delta)$, $\xi_j^*\in D(\xi_0^*,\delta)$ such that for sufficiently large j

$$Q(f_j(z_j+\rho_j\xi_j))f_j(z_j+\rho_j\xi_j)(f_j(z_j+\rho_j\xi_j)-1)G(f_j(z_j+\rho_j\xi_j))=b,$$
$$Q(f_j(z_j+\rho_j\xi_j^*))f_j(z_j+\rho_j\xi_j^*)(f_j(z_j+\rho_j\xi_j^*)-1)G(f_j(z_j+\rho_j\xi_j^*))=b.$$

By the hypothesis that for each pair of functions f and g in $\mathscr{F}$, $P(f)G(f)$ and $P(g)G(g)$ share 0 in D, we know that for any positive integer m

$$Q(f_m(z_j+\rho_j\xi_j))f_m(z_j+\rho_j\xi_j)(f_m(z_j+\rho_j\xi_j)-1)G(f_m(z_j+\rho_j\xi_j))=b,$$
$$Q(f_m(z_j+\rho_j\xi_j^*))f_m(z_j+\rho_j\xi_j^*)(f_m(z_j+\rho_j\xi_j^*)-1)G(f_m(z_j+\rho_0^*\xi_j))=b.$$

Fix m, take $j\to\infty$, and note $z_j+\rho_j\xi_j\to 0$, $z_j+\rho_j\xi_j^*\to 0$, then

$$Q(f_m(0))f_m(0)(f_m(0)-1)G(f_m(0))=b.$$

Since the zeros of $P(f_m)G(f_m)-b$ has no accumulation point, so $z_j+\rho_j\xi_j=0$, $z_j+\rho_j\xi_j^*=0$. Hence

$$\xi_j=-\frac{z_j}{\rho_j},\qquad \xi_j^*=-\frac{z_j}{\rho_j}.$$

This contradicts with $\xi_j\in D(\xi_0,\delta)$, $\xi_j^*\in D(\xi_0^*,\delta)$ and $D(\xi_0,\delta)\cap D(\xi_0^*,\delta)=\emptyset$ So $Q(g)g(g-1)g^{(k)}$ has just a single zero, which can be denoted by ξ_0.

Suppose that g is a transcendental meromorphic function. Since $Q(g)g(g-1)g^{(k)}$ has only one zero, so $g=0$ and $g=1$ has only finite zeros. As the above argument, we obtain $T(r,g)=S(r,g)$, a contradiction. Thus g is a rational function which is not a polynomial.

Because $Q(g)g(g-1)g^{(k)}$ has only one zero, we have $g \neq 0$ or $g \neq 1$. If $g \neq 0$, then $g(\xi) = 1/H(\xi)$ where $H(\xi)$ is a non-constant polynomial. Since $g(\xi)-1 = (1-H(\xi))/H(\xi)$ has just a single zero, so

$$1-H(\xi) = A(\xi - B)^k, \tag{3.3}$$

where $A \neq 0, B$ are constant, $k \geq 2$ is a positive integer.

We claim $H(\xi)$ has only simple zeros. Suppose, on the contrary, that $H(z_0) = 0$ and z_0 is multiple. Form (3.3), we arrive at $0 = H'(z_0) = (1-H(z_0))' = Ak(z_0-B)^{(k-1)}$, a contradiction, since $z_0 \neq B$. Thus $H(\xi)$ has just simple zeros, this contradicts that g has no simple pole. If $g \neq 1$, we can argue it in the same way. So $\mathscr{F}$ is normal on D. ∎

Proof of Theorem 1.6. We may assume that $D = \{|z| < 1\}$. Suppose that $\mathscr{F}$ is not normal in D. Without loss of generality, we assume that $\mathscr{F}$ is not normal at $z_0 = 0$. Then, by Lemma 2.1, there exist a sequence z_j of complex numbers with $z_j \to 0$ $(j \to \infty)$; a sequence f_j of $\mathscr{F}$; and a sequence $\rho_j \to 0^+$ such that

$$g_j(\xi) = \rho_j^{-\frac{k}{l+1}} f_j(z_j + \rho_j \xi) \tag{3.4}$$

converges uniformly to a non-constant mermorphic functions $g(\xi)$ in C with respect to the spherical metric. Moreover, $l(\geq k+1)$ be a constant and $g(\xi)$ is of order at most 2. By Hurwitz's theorem, the zeros of $g(\xi)$ have at least multiplicity k. Next we will distinguish two cases:

Case 1. When $P(z)$ has two distinct zeros, then we can denote $P(f) = f^l(f+1)$ $(l \geq k+1)$.

If $g^l g^{(k)} \equiv b$, then g has no zeros. Of course, g also has no poles. Since g is a non-constant meromorphic function of order at morst 2, we obtain $g(\xi) = e^{d\xi^2 + h\xi + c}$ (where D, h, c are constants and $dh \neq 0$). At this moment $g^n(\xi)g^{(k)}(\xi) \not\equiv b$. Which is a contradiction.

If $g^l g^{(k)} \neq b$, then by Lemma 2.2, we obtain that g is a constant. This contradicts that g is a non-zero meromorphic function. Thus $g^l g^{(k)} - b$ is a non-constant meromorphic function and has one zero at least.

Next we will prove that $g^l g^{(k)} - b$ has just a single zero. In fact, let ξ_0 and ξ_0^* be two distinct solutions of $g^l g^{(k)} - b$. We choose a positive number δ_1 small enough such that g and g_j are holomorphic in $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$.

From (3.4), we have

$$\begin{aligned}
&[f_j^{l+1}(z_j+\rho_j\xi) + f_j^l(z_j+\rho_j\xi)] \cdot [f_j^{(k)}(z_j+\rho_j\xi) + H(f, f', \cdots, f^{(k)})] - b \\
&= \left[\rho_j^{-\frac{lk}{l+1}} g_j^{(k)}(\xi) + \sum_{j=1}^{n} a_j^* \rho_j^{(\frac{k}{l+1}+1)\deg(M_j) - w(M_j)} M_i(g, g', \cdots, g^{(k)})\right] \cdot \\
&\quad [\rho_j^k g_j^{l+1}(\xi) + \rho_j^{\frac{lk}{l+1}} g_j^l(\xi)] - b \to g^l(\xi) g^{(k)}(\xi) - b.
\end{aligned} \tag{3.5}$$

Choose δ_2 such that $D(\xi_0, \delta_2) \cap D(\xi_0^*, \delta_2) = \emptyset$ and such that $g^n g^{(k)} - b$ has no other zeros in $D(\xi_0, \delta) \cup D(\xi_0^*, \delta)$. By Hurwitz's theorem, there exist points $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$ such that for sufficiently large j

$$[f_j^{l+1}(z_j+\rho_j\xi_j) + f_j^l(z_j+\rho_j\xi_j)]G(f_j(z_j+\rho_j\xi_j)) - b = 0,$$

$$[f_j^{l+1}(z_j+\rho_j\xi_j^*) + f_j^l(z_j+\rho_j\xi_j^*)]G(f_j(z_j+\rho_j\xi_j^*)) - b = 0.$$

By the hypothesis that for each pair of functions f and g in $\mathscr{F}$, $P(f)G(f^{(k)})$ and $P(g)G(g^{(k)})$ share b in D, we know that for any positive integer m

$$[f_m^{l+1}(z_j+\rho_j\xi_j)+f_m^l(z_j+\rho_j\xi_j)]G(f_m(z_j+\rho_j\xi_j))-b=0,$$
$$[f_m^{l+1}(z_j+\rho_j\xi_j^*)+f_m^l(z_j+\rho_j\xi_j^*)]G(f_m(z_j+\rho_j\xi_j^*))-b=0.$$

Fix m, take $j\to\infty$, and note $z_j+\rho_j\xi_j\to 0$, $z_j+\rho_j\xi_j^*\to 0$, then

$$[f_m^{l+1}(0)+f_m^l(0)]G(f_m(0))-b=0.$$

Since the zeros of $P(f)G(f)-b$ has no accumulation point, so $z_j+\rho_j\xi_j=0$, $z_j+\rho_j\xi_j^*=0$. Hence

$$\xi_j=-\frac{z_j}{\rho_j},\qquad \xi_j^*=-\frac{z_j}{\rho_j}.$$

This contradicts with $\xi_j\in D(\xi_0,\delta)$, $\xi_j^*\in D(\xi_0^*,\delta)$ and $D(\xi_0,\delta)\cap D(\xi_0^*,\delta)=\emptyset$. So $g^l g^{(k)}-b$ has just a single zero, which can be denoted by ξ_0. From the above, we know $g^l g^{(k)}-b$ has just a unique zero. This contradicts Lemma 2.2 and Lemma 2.4.

Case 2. If $P(z)$ has more than three distinct zeros, we can denote $P(z)=Q(g)g(g-1)(g-a)$.

Without loss of generality, we assume that $\mathscr{F}$ is not normal at $z_0=0$. Then, by Lemma 2.1, there exist a sequence z_j of complex numbers with $z_j\to 0$ $(j\to\infty)$; a sequence f_j of $\mathscr{F}$; and a sequence $\rho_j\to 0^+$ such that

$$g_j(\xi)=f_j(z_j+\rho_j\xi) \tag{3.6}$$

converges uniformly to a non-constant mermorphic functions $g(\xi)$ in C with respect to the spherical metric.

Proceeding as in the proof of Theorem 1.7, we have $Q(g)g(g-1)(g-a)G(g)$ only one zero. Obviously, g is not a transcendental meromorphic function from Picard Theorem. Thus g is a non-constant rational function and g doesn't assume two complex number of $\{0,1,a\}$, a contradiction, because of Lemma 2.5. So $\mathscr{F}$ is normal in z_0. ∎

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References

[1] W. Bergweiler, Bloch's principle, *Comput. Methods Funct. Theory* **6** (2006), no. 1, 77–108.
[2] W. Bergweiler and A. Eremenko, Complex dynamics and value distribution, in: *International Conference of Complex Analysis*, Nanjing, 1994.
[3] C. Meng, Normal families and shared values of meromorphic functions, *Bull. Malays. Math. Sci. Soc. (2)* **31** (2008), no. 1, 85–90.
[4] H. H. Chen and Y. X. Gu, Improvement of Marty's criterion and its application, *Sci. China Ser. A* **36** (1993), no. 6, 674–681.
[5] H. H. Chen and M. L. Fang, The value distribution of $f^n f'$, *Sci. China Ser. A* **38** (1995), no. 7, 789–798.
[6] J. Clunie, On a result of Hayman, *J. London Math. Soc.* **42** (1967), 389–392.
[7] M. Fang and L. Zalcman, A note on normality and shared values, *J. Aust. Math. Soc.* **76** (2004), no. 1, 141–150.

[8] Y. X. Gu, On normal families of meromorphic functions, *Sci. China Ser. A* **36** (1978), 373-384.

[9] W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs Clarendon Press, Oxford, 1964.

[10] W. K. Hayman, Picard values of meromorphic functions and their derivatives, *Ann. of Math. (2)* **70** (1959), 9–42.

[11] W. K. Hayman, *Research Problems in Function Theory*, The Athlone Press University of London, London, 1967.

[12] C. Lei and M. Fang, Normality and shared values concerning differential polynomials, *Sci. China Math.* **53** (2010), no. 3, 749–754.

[13] C. Lei, M. Fang and D. Yang, Normal families and shared values of meromorphic functions, *Proc. Japan Acad. Ser. A Math. Sci.* **83** (2007), no. 3, 36–39.

[14] S.-Y. Li and H. C. Xie, On normal families of meromorphic functions, *Acta Math. Sinica* **29** (1986), no. 4, 468–476.

[15] Y. Li and Y. Gu, On normal families of meromorphic functions, *J. Math. Anal. Appl.* **354** (2009), no. 2, 421–425.

[16] E. Mues, Über ein Problem von Hayman, *Math. Z.* **164** (1979), no. 3, 239–259.

[17] I. B. Oshkin, On a condition for the normality of families of holomorphic functions, *Uspekhi Mat. Nauk* **37** (1982), no. 2(224), 221–222.

[18] X. C. Pang, Bloch's principle and normal criterion, *Sci. China Ser. A* **32** (1989), no. 7, 782–791.

[19] X. Pang and L. Zalcman, Normal families and shared values, *Acta Math.* **76** (2000), 171–182.

[20] X. Pang and L. Zalcman, Normal families and shared values, *Bull. London Math. Soc.* **32** (2000), no. 3, 325–331.

[21] J. L. Schiff, *Normal families*, Universitext, Springer, New York, 1993.

[22] W. Schwick, Sharing values and normality, *Arch. Math.* (Basel) **59** (1992), no. 1, 50–54.

[23] J. Xia and Y. Xu, Normality criterion concerning sharing functions II, *Bull. Malays. Math. Sci. Soc. (2)* **33** (2010), no. 3, 479–486.

[24] C.-C. Yang and H.-X. Yi, *Uniqueness theory of meromorphic functions*, Mathematics and its Applications, 557, Kluwer Acad. Publ., Dordrecht, 2003.

[25] Yang, Lo. Value distribution theory. Translated and revised from the 1982 Chinese original. Springer-Verlag, Berlin; Science Press Beijing, Beijing, 1993. xii+269 pp.

[26] L. Yang, G. H. Zhang, Recherches sur la normalité des familles de fonction analytiques à des valeurs multiples, I.Un nouveau critère et quelques applications. *Sci. China Ser. A* **14** (1965), 1258–1271; II. Géneralisations, *Ibid.* **15** (1966), 433–453.

[27] L. Zalcman, Normal families: new perspectives, *Bull. Amer. Math. Soc.* (N.S.) **35** (1998), no. 3, 215–230.

[28] Q. C. Zhang, Normality criteria for holomorphic functions, *Math. Practice Theory* **36** (2006), no. 6, 283–286.

[29] Q. Zhang, Some normality criteria of meromorphic functions, *Complex Var. Elliptic Equ.* **53** (2008), no. 8, 791–795.

[30] Z. L. Zhang and W. Li, Picard exceptional values for two classes of differential polynomials, *Acta Math. Sinica* **37** (1994), no. 6, 828–835.