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On Meromorphic Starlike Functions of Reciprocal Order α

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Abstract. In the present paper, we introduce the concept of meromorphic starlike functions of reciprocal order α. Some sufficient conditions for functions belonging to this class are derived.

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1. Introduction

Let Σ denote the class of functions f of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \tag{1.1}$$

which are *analytic* in the *punctured* open unit disk

$$\mathbb{U}^* := \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\} =: \mathbb{U} \setminus \{0\}.$$

A function $f \in \Sigma$ is said to be in the class $\mathscr{MS}^*(\alpha)$ of *meromorphic starlike functions of order* α if it satisfies the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < -\alpha \quad (0 \leq \alpha < 1;\ z \in \mathbb{U}). \tag{1.2}$$

As usual, let $\mathscr{MS}^*(0) \equiv \mathscr{MS}^*$. Furthermore, a function $f \in \mathscr{MS}^*$ is said to be in the class $\mathscr{NS}^*(\alpha)$ of *meromorphic starlike of reciprocal order* α if and only if

$$\operatorname{Re}\left(\frac{f(z)}{zf'(z)}\right) < -\alpha \quad (0 \leq \alpha < 1;\ z \in \mathbb{U}). \tag{1.3}$$

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In the following, we give several examples of functions belonging to the class of meromorphic starlike of reciprocal order.

Example 1.1. In view of the fact that

$$\operatorname{Re}\,(p(z)) < 0 \Rightarrow \operatorname{Re}\,\left(\frac{1}{p(z)}\right) = \operatorname{Re}\,\left(\frac{p(z)}{|p(z)|^2}\right) < 0,$$

it follows that a meromorphic starlike function of reciprocal order 0 is same as a meromorphic starlike function. When $0 < \alpha < 1$, the function $f \in \Sigma$ is meromorphic starlike of reciprocal order α if and only if

$$\left|\frac{zf'(z)}{f(z)} + \frac{1}{2\alpha}\right| < \frac{1}{2\alpha} \quad (z \in \mathbb{U}). \tag{1.4}$$

Example 1.2. Let $f \in \Sigma$ satisfy the inequality

$$\left|\frac{zf'(z)}{f(z)} + 1\right| < 1 - \alpha \quad (0 \leq \alpha < 1;\ z \in \mathbb{U}). \tag{1.5}$$

Then

$$\left|\frac{zf'(z)}{f(z)} + \frac{2-\alpha}{2}\right| \leq \left|\frac{zf'(z)}{f(z)} + 1\right| + \frac{\alpha}{2} < 1 - \alpha + \frac{\alpha}{2} = \frac{2-\alpha}{2}$$

and therefore such functions are meromorphic starlike of reciprocal order $1/(2-\alpha)$.

Example 1.3. Let us define the function $f(z)$ by

$$f(z) = \frac{e^{(1-\alpha)z}}{z} \quad (0 < \alpha < 1;\ z \in \mathbb{U}).$$

This gives us that

$$\operatorname{Re}\,\left(\frac{zf'(z)}{f(z)}\right) = \operatorname{Re}\,(-1 + (1-\alpha)z) < -\alpha \quad (0 < \alpha < 1;\ z \in \mathbb{U}).$$

Therefore, we see that $f \in \mathscr{MS}^*(\alpha)$.

Moreover, we have

$$\frac{f(z)}{zf'(z)} = \frac{1}{-1 + (1-\alpha)z}.$$

It follows that

$$\frac{f(z)}{zf'(z)} = -1 \quad (z = 0)$$

and

$$\operatorname{Re}\,\left(\frac{f(z)}{zf'(z)}\right) = \operatorname{Re}\,\left(\frac{1}{-1 + (1-\alpha)e^{i\theta}}\right) < -\frac{1}{2-\alpha} \quad \left(z = e^{i\theta}\right).$$

Therefore, we conclude that $f \in \mathscr{NS}^*(1/(2-\alpha))$.

In order to establish our main results, we need the following lemmas.

Lemma 1.1. (Jack's lemma [7]) *Let φ be a non-constant regular function in $\mathbb{U}$. If $|\varphi|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, then*

$$z_0\varphi'(z_0) = k\varphi(z_0),$$

where $k \geq 1$ is a real number.

Lemma 1.2. [9] *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that Φ is a mapping from $\mathbb{C}^2 \times \mathbb{U}$ to $\mathbb{C}$ which satisfies $\Phi(ix,y;z) \notin \Omega$ for $z \in \mathbb{U}$, and for all real x,y such that $y \leq -(1+x^2)/2$. If the function $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in $\mathbb{U}$ and $\Phi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{U}$, then* Re $(p(z)) > 0$.

Lemma 1.3. [23] *Let $\rho(z) = 1 + b_1 z + b_2 z^2 + \cdots$ be analytic in $\mathbb{U}$ and η be analytic and starlike (with respect to the origin) univalent in $\mathbb{U}$ with $\eta(0) = 0$. If*

$$z\rho'(z) \prec \eta(z),$$

then

$$\rho(z) \prec 1 + \int_0^z \frac{\eta(t)}{t} dt.$$

In recent years, several authors studied meromorphic starlike functions and starlike functions of reciprocal order (see details, [1–6, 8, 10–12, 14–22, 24]). Nunokawa *et al.* [13] obtained some argument properties of starlike functions of reciprocal order. In the present investigation, we give some sufficient conditions for the functions belonging to the class $\mathcal{NS}^*(\alpha)$.

2. Main results

We begin by presenting the following coefficient sufficient condition for functions belonging to the class $\mathcal{NS}^*(\alpha)$.

Theorem 2.1. *If $f \in \Sigma$ satisfies*

$$\sum_{k=0}^{\infty} (1+k\alpha)|a_k| \leq \frac{1}{2}(1 - |1-2\alpha|), \tag{2.1}$$

then $f \in \mathcal{NS}^*(\alpha)$, for $0 < \alpha < 1$.

Proof. By virtue of the condition (1.4), we only need to show that

$$\left| \frac{2\alpha z f'(z)}{f(z)} + 1 \right| < 1 \quad (z \in \mathbb{U}). \tag{2.2}$$

We first observe that

$$\begin{aligned} \left| \frac{2\alpha z f'(z) + f(z)}{f(z)} \right| &= \left| \frac{(1-2\alpha) + \sum_{k=0}^{\infty}(1+2k\alpha) a_k z^{k+1}}{1 + \sum_{k=0}^{\infty} a_k z^{k+1}} \right| \\ &\leq \frac{|1-2\alpha| + \sum_{k=0}^{\infty}(1+2k\alpha)|a_k| |z|^{k+1}}{1 - \sum_{k=0}^{\infty} |a_k| |z|^{k+1}} \\ &< \frac{|1-2\alpha| + \sum_{k=0}^{\infty}(1+2k\alpha)|a_k|}{1 - \sum_{k=0}^{\infty} |a_k|}. \end{aligned}$$

Now, by using the inequality (2.1), we have

$$\frac{|1-2\alpha| + \sum_{k=0}^{\infty}(1+2k\alpha)|a_k|}{1 - \sum_{k=0}^{\infty} |a_k|} < 1, \tag{2.3}$$

which, in conjunction with (2.2), completes the proof of Theorem 2.1. ∎

Example 2.1. The function $f(z)$ given by

$$f(z)=\frac{1}{z}+\sum_{k=2}^{\infty}\frac{1-|1-2\alpha|}{k(k+1)(1+k\alpha)}z^k$$

belongs to the class $\mathcal{NS}^*(\alpha)$.

By using Jack's lemma, we now derive the following result for the class $\mathcal{NS}^*(\alpha)$.

Theorem 2.2. *If $f\in\Sigma$ satisfies*

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right|<1-\alpha, \tag{2.4}$$

then $f\in\mathcal{NS}^*(\alpha)$, for $1/2\leq\alpha<1$.

Proof. Let

$$\omega(z)=\frac{1+\frac{\alpha zf'(z)}{f(z)}}{1-\alpha}-1 \quad \left(\frac{1}{2}\leq\alpha<1;\ z\in\mathbb{U}\right). \tag{2.5}$$

Then the function ω is analytic in $\mathbb{U}$ with $\omega(0)=0$. We easily find from (2.5) that

$$\frac{zf'(z)}{f(z)}=\frac{(1-\alpha)\omega(z)-\alpha}{\alpha} \quad (z\in\mathbb{U}). \tag{2.6}$$

Differentiating both sides of (2.6) logarithmically, we obtain

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}=\frac{(1-\alpha)z\omega'(z)}{(1-\alpha)\omega(z)-\alpha}, \tag{2.7}$$

by virtue of (2.4) and (2.7), we find that

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right|=|1-\alpha|\left|\frac{z\omega'(z)}{(1-\alpha)\omega(z)-\alpha}\right|<1-\alpha.$$

Next, we claim that $|\omega(z)|<1$. Indeed, if not, there exists a point $z_0\in\mathbb{U}$ such that

$$\max_{|z|\leq|z_0|}=|\omega(z_0)|=1. \tag{2.8}$$

Applying Jack's lemma to $\omega(z)$ at the point z_0, we have

$$\omega(z_0)=e^{i\theta} \quad \text{and} \quad \frac{z_0\omega'(z_0)}{\omega(z_0)}=k \quad (k\geq 1).$$

This gives us that

$$\left|1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0f'(z_0)}{f(z_0)}\right|=|1-\alpha|\left|\frac{k}{(1-\alpha)-\alpha e^{-i\theta}}\right|\geq|1-\alpha|\left|\frac{1}{(1-\alpha)-\alpha e^{-i\theta}}\right|.$$

This implies that

$$\left|1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0f'(z_0)}{f(z_0)}\right|^2\geq\frac{(1-\alpha)^2}{(1-\alpha)^2+\alpha^2-2\alpha(1-\alpha)\cos\theta}. \tag{2.9}$$

Since the right hand side of (2.9) takes it minimum value for $\cos\theta=-1$, we have that

$$\left|1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0f'(z_0)}{f(z_0)}\right|^2\geq\frac{(1-\alpha)^2}{(1-\alpha+\alpha)^2}=(1-\alpha)^2.$$

This contradicts our condition (2.4) of Theorem 2.2. Therefore, we conclude that $|\omega(z)| < 1$, which shows that

$$\left| \frac{1+\frac{\alpha z f'(z)}{f(z)}}{1-\alpha} - 1 \right| < 1 \quad \left(\frac{1}{2} \leq \alpha < 1;\ z \in \mathbb{U}\right).$$

This implies that

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| < \frac{1}{\alpha} - 1 \quad \left(\frac{1}{2} \leq \alpha < 1;\ z \in \mathbb{U}\right), \tag{2.10}$$

then, we have

$$\left| \frac{zf'(z)}{f(z)} + \frac{1}{2\alpha} \right| \leq \left| \frac{zf'(z)}{f(z)} + 1 \right| + \left| \frac{1}{2\alpha} - 1 \right| < \frac{1}{\alpha} - 1 + 1 - \frac{1}{2\alpha} = \frac{1}{2\alpha} \quad \left(\frac{1}{2} \leq \alpha < 1;\ z \in \mathbb{U}\right),$$

Therefore, we conclude that $f \in \mathcal{NS}^*(\alpha)$. ∎

Example 2.2. Let us consider the function $f \in \Sigma$ given by

$$f(z) = \frac{1}{z} + a_0 \quad (z \in \mathbb{U}^*)$$

with

$$a_0 = \frac{1-\alpha}{2-\alpha}$$

for some α $(1/2 \leq \alpha < 1)$, then we see that $0 < a_0 \leq 1/3$.

Note that

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| = \left| \frac{-a_0 z}{1+a_0 z} \right| < \frac{a_0}{1-a_0} = 1 - \alpha.$$

Moreover

$$\mathrm{Re}\left(\frac{f(z)}{zf'(z)}\right) = \mathrm{Re}\,(-1 - a_0 z) \leq a_0 - 1 = \frac{1}{\alpha - 2} < -\alpha \quad \left(\frac{1}{2} \leq \alpha < 1;\ z \in \mathbb{U}\right).$$

Therefore, $f \in \mathcal{NS}^*(\alpha)$.

Theorem 2.3. *If $f \in \Sigma$ satisfies*

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) < \begin{cases} \frac{\alpha}{2(1-\alpha)} & \left(0 \leq \alpha \leq \frac{1}{2}\right), \\ \frac{1-\alpha}{2\alpha} & \left(\frac{1}{2} \leq \alpha < 1\right), \end{cases} \tag{2.11}$$

then $f \in \mathcal{NS}^*(\alpha)$, for $0 \leq \alpha < 1$.

Proof. Suppose that

$$g(z) := \frac{-\frac{f(z)}{zf'(z)} - \alpha}{1-\alpha} \quad (0 \leq \alpha < 1;\ z \in \mathbb{U}). \tag{2.12}$$

Then g is analytic in $\mathbb{U}$. It follows from (2.12) that

$$-1 - \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{f(z)} = \frac{(1-\alpha)zg'(z)}{\alpha + (1-\alpha)g(z)} = \Phi\left(g(z), zg'(z); z\right), \tag{2.13}$$

where

$$\Phi(r,s;t) = \frac{(1-\alpha)s}{\alpha + (1-\alpha)r}.$$

For all real x and y satisfying $y \leq -(1+x^2)/2$, we have

$$\begin{aligned}\text{Re}\,\left(\Phi(g(z),zg'(z);z)\right) &= \frac{(1-\alpha)\alpha y}{\alpha^2+(1-\alpha)^2x^2} \leq -\frac{(1-\alpha)\alpha}{2}\cdot\frac{1+x^2}{\alpha^2+(1-\alpha)^2x^2} \\ &\leq \begin{cases} -\frac{(1-\alpha)\alpha}{2}\cdot\frac{1}{(1-\alpha)^2} & \left(0 \leq \alpha \leq \frac{1}{2}\right), \\ -\frac{(1-\alpha)\alpha}{2}\cdot\frac{1}{\alpha^2} & \left(\frac{1}{2} \leq \alpha < 1\right). \end{cases}\end{aligned}$$

We now put

$$\Omega = \left\{\xi :\ \text{Re}\,(\xi) > \begin{cases} \frac{\alpha}{2(\alpha-1)} & \left(0 \leq \alpha \leq \frac{1}{2}\right) \\ \frac{\alpha-1}{2\alpha} & \left(\frac{1}{2} \leq \alpha < 1\right) \end{cases}\right\},$$

then $\Phi(ix,y;z) \notin \Omega$ for all real x,y such that $y \leq -(1+x^2)/2$. Moreover, in view of (2.11), we know that $\Phi(g(z),zg'(z);z) \in \Omega$. Thus, by Lemma 1.2, we deduce that

$$\text{Re}\,(g(z)) > 0 \quad (z \in \mathbb{U}),$$

which shows that the desired assertion of Theorem 2.3 holds. ∎

Theorem 2.4. *If $f \in \Sigma$ satisfies*

$$\text{Re}\,\left(\frac{f(z)}{zf'(z)}\left(1+\beta\frac{zf''(z)}{f'(z)}\right)\right) < \frac{1}{2}\beta(\alpha+3)-\alpha, \tag{2.14}$$

then $f \in \mathcal{NS}^*(\alpha)$, for $0 \leq \alpha < 1$ and $\beta \geq 0$.

Proof. We define the function $h(z)$ by

$$h(z) := \frac{-\frac{f(z)}{zf'(z)}-\alpha}{1-\alpha} \quad (0 \leq \alpha < 1;\ z \in \mathbb{U}). \tag{2.15}$$

Then h is analytic in $\mathbb{U}$. It follows from (2.15) that

$$1+\beta\frac{zf''(z)}{f'(z)} = \frac{\beta[(1-\alpha)zh'(z)-1]}{(1-\alpha)h(z)+\alpha}+1-\beta. \tag{2.16}$$

Combining (2.15) and (2.16), we get

$$\begin{aligned}-\frac{f(z)}{zf'(z)}\left(1+\beta\frac{zf''(z)}{f'(z)}\right) &= \beta(1-\alpha)zh'(z)+(1-\beta)(1-\alpha)h(z)+(1-\beta)\alpha-\beta \\ &= \Phi\left(h(z),zh'(z);z\right),\end{aligned}$$

where

$$\Phi(r,s;t) = \beta(1-\alpha)s+(1-\beta)(1-\alpha)r+(1-\beta)\alpha-\beta.$$

For all real x and y satisfying $y \leq -(1+x^2)/2$, we have

$$\begin{aligned}\text{Re}\,(\Phi(ix,y;z)) &= \beta(1-\alpha)y+(1-\beta)\alpha-\beta \\ &\leq -\frac{\beta(1-\alpha)}{2}(1+x^2)+(1-\beta)\alpha-\beta \\ &\leq -\frac{\beta(1-\alpha)}{2}+(1-\beta)\alpha-\beta\end{aligned}$$

$$= \alpha - \frac{1}{2}\beta(\alpha+3) \quad (0 \leq \alpha < 1).$$

If we set

$$\Omega = \left\{ \xi : \text{Re}\,(\xi) > \alpha - \frac{1}{2}\beta(\alpha+3) \right\},$$

then $\Phi(ix, y; z) \notin \Omega$ for all real x, y such that $y \leq -(1+x^2)/2$. Furthermore, by virtue of (2.14), we know that $\Phi(h(z), zh'(z); z) \in \Omega$. Thus, by Lemma 1.2, we conclude that

$$\text{Re}\,(h(z)) > 0 \quad (z \in \mathbb{U}),$$

which implies that the assertion of Theorem 2.4 holds true. ∎

Theorem 2.5. *If $f \in \Sigma$ satisfies*

$$\left| \left(1 + \frac{2\alpha z f'(z)}{f(z)}\right)' \right| \leq \beta\,|z|^{\gamma}, \tag{2.17}$$

then $f \in \mathcal{NS}^*(\alpha)$, for $0 < \alpha < 1$, $0 < \beta \leq \gamma + 1$ and $\gamma \geq 0$.

Proof. For $f \in \Sigma$, we define the function $\psi(z)$ by

$$\psi(z) = z\left(1 + \frac{2\alpha z f'(z)}{f(z)}\right) \quad (z \in \mathbb{U}).$$

Then $\psi(z)$ is regular in $\mathbb{U}$ and $\psi(0) = 0$. The condition of the theorem gives us that

$$\left| \left(1 + \frac{2\alpha z f'(z)}{f(z)}\right)' \right| = \left| \left(\frac{\psi(z)}{z}\right)' \right| \leq \beta\,|z|^{\gamma} \quad (z \in \mathbb{U}).$$

It follows that

$$\left| \left(\frac{\psi(z)}{z}\right)' \right| = \left| \int_0^z \left(\frac{\psi(t)}{t}\right)' dt \right| \leq \int_0^{|z|} \beta\,|t|^{\gamma}\,d\,|t| = \frac{\beta}{\gamma+1}\,|z|^{\gamma+1} \quad (z \in \mathbb{U}).$$

This implies that

$$\left| \left(\frac{\psi(z)}{z}\right)' \right| \leq \frac{\beta}{\gamma+1}\,|z|^{\gamma+1} < 1 \quad (0 < \beta \leq \gamma+1,\ \gamma \geq 0;\ z \in \mathbb{U}).$$

Therefore, by the definition of $\psi(z)$, we conclude that

$$\left| \frac{2\alpha z f'(z)}{f(z)} + 1 \right| < 1 \quad (0 < \alpha < 1;\ z \in \mathbb{U}),$$

which is equivalent to

$$\left| \frac{z f'(z)}{f(z)} + \frac{1}{2\alpha} \right| < \frac{1}{2\alpha} \quad (0 < \alpha < 1;\ z \in \mathbb{U}).$$ ∎

Theorem 2.6. *If $f \in \Sigma$ satisfies*

$$\left| \frac{z f'(z)}{f(z)} \left(1 + \frac{z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)}\right) \right| < \frac{1}{\alpha} - 1, \tag{2.18}$$

then $f \in \mathcal{NS}^*(\alpha)$, for $1/2 < \alpha < 1$.

Proof. Let

$$q(z) := -\frac{f(z)}{zf'(z)} \quad (z \in \mathbb{U}). \tag{2.19}$$

Then the function $q(z)$ is analytic in $\mathbb{U}$. It follows from (2.19) that

$$z\left(\frac{1}{q(z)}\right)' = -\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \quad (z \in \mathbb{U}). \tag{2.20}$$

Combining (2.18) and (2.20), we find that

$$z\left(\frac{1}{q(z)}\right)' \prec \frac{(1-\alpha)z}{\alpha} \quad (z \in \mathbb{U}). \tag{2.21}$$

An application of Lemma 1.3 to (2.21) yields

$$q(z) \prec \frac{\alpha}{\alpha+(1-\alpha)z} =: F(z) \quad (z \in \mathbb{U}). \tag{2.22}$$

By noting that

$$\text{Re}\left(1+\frac{zF''(z)}{F'(z)}\right) = \text{Re}\left(\frac{\alpha-(1-\alpha)z}{\alpha+(1-\alpha)z}\right) \geq \frac{\alpha-(1-\alpha)}{\alpha+(1-\alpha)} > 0 \quad \left(\frac{1}{2} < \alpha < 1;\ z \in \mathbb{U}\right),$$

which implies that the region $F(\mathbb{U})$ is symmetric with respect to the real axis and F is convex univalent in $\mathbb{U}$. Therefore, we have

$$\text{Re}\,(F(z)) \geq F(1) \geq 0 \quad (z \in \mathbb{U}). \tag{2.23}$$

Combining (2.19), (2.22) and (2.23), we deduce that

$$\text{Re}\left(\frac{f(z)}{zf'(z)}\right) < -\alpha \quad \left(\frac{1}{2} < \alpha < 1;\ z \in \mathbb{U}\right).$$

This evidently completes the proof of Theorem 2.6. ∎

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