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On Topological Congruences of a Topological Semigroup

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Abstract. We continue the study of the topological Rees congruences on a topological semigroup. First, we give some necessary and sufficient conditions for the question "when the quotient space of S over one of its ideal I is a k_ω-space?", then we use it to generalize the Rees version of Lawson and Madison's well-known theorem about topological congruences on S.

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1. Introduction and preliminaries

In this paper all spaces are assumed to be Hausdorff. Let S be a topological semigroup and θ be a closed congruence on it. It is a famous problem that *"When the quotient semigroup S/θ with the quotient topology is a topological semigroup?"*. This problem was studied by Wallace in [14] and in fact, he was one of the first mathematicians who worked on this problem. Latter Lawson and Madison in [9] studied this question and more specially they raised and studied the Rees version of this question *"When the quotient semigroup S/I with the quotient topology is a topological semigroup?"* (they also studied this question for a locally compact topological semigroup). In the following, we mention some of the earlier works about this question (for other similar results see [8, 10, 11, 12]). Wallace in [14] proved that if S is a compact topological semigroup and ρ is a closed congruence on it, then S/ρ is a compact topological semigroup. This result was generalized by Lawson and Madison in [9] and they proved the following theorem.

Lawson and Madison's Theorem. Let S be a locally compact, σ-compact Hausdorff topological semigroup and θ be a closed congruence on S. Then S/θ is a topological semigroup.

Later, Gonzalez studied σ-compact locally compact spaces and he presented a new proof for Lawson and Madison's theorem in [5]. Gutik and Pavlyk in [6] showed that if S is a topological semigroup and I is a compact ideal of S, then S/I is a topological semigroup. It is a known fact that a closed Rees congruences on a topological semigroup is not necessarily

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topological and there are examples of H-closed metrizable topological semigroups which have Rees congruences that are not topological (see [6, 7]). Almost all the works about this question use some strong topological assumption and they used rarely the algebraic structures of topological semigroups (see [6, 9]). However, we need to consider both of these structures for a complete answer. In this paper, we investigate conditions which are closely related to ideals and topological structures of S. First, we give necessary and sufficient condition on S and a closed ideal I of S such that S/I is a k_w-space, then we use this result to generalize Lawson and Madison's well known theorem. As a consequence of our results, we show that for a topological semigroup S and a closed ideal I, if $S \setminus I^\circ$ is σ-compact, then for every closed ideal J in S such that $I \subseteq J$, S/J is a topological semigroup.

We recall that a semigroup S with a topology τ_S is a *topological semigroup* if the multiplication $\lambda : S \times S \to S$ is jointly continuous, where $S \times S$ has the product topology. A Hausdorff space X is called a *k-space*, if it has the weak topology determined by the family of its compact subsets. A locally compact space is *σ-compact*, if it can be expressed as the union of at most countably many compact spaces K_n (so that a subset $A \subseteq X$ is closed whenever $A \cap K_n$ is closed in K_n for all n). A space X is a *k_w-space*, if it is the union of a countable collection $\{K_n\}$ of compact subsets so that a subset $A \subseteq X$ is closed whenever $A \cap K_n$ is closed in K_n for all n (therefore, any σ-compact space is k_w-space). For an ideal I in S, the *Rees congruence* ρ_I is equal to $\Delta_S \cup (I \times I)$, where $\Delta_S = \{(s,s) : s \in S\}$. For a congruence θ on S, if we consider the set of equivalence classes of θ, $\{[t] : t \in S\}$, and define $[t][s] := [ts]$, then $\{[t] : t \in S\}$ with this multiplication is a semigroup denoted by S/θ. For simplicity, we denote S/ρ_I by S/I. If ρ is a congruence in S, which is closed in $S \times S$, it is called a *closed congruence* (see [5]). A closed ideal I of S is called *regular*, if for any $s \in S \setminus I$, there exist open neighborhoods V_s and W_I of s and I, respectively, such that $V_s \cap W_I = \emptyset$. A *topological congruence* on a topological semigroup S is a semigroup congruence θ (that is, if $s\theta s'$ where $s, s' \in S$, then for any $t \in S$, $ts\theta ts'$ and $st\theta s't$) such that the semigroup S/θ with the quotient topology is a topological semigroup (Note that we assume that topological semigroups are Hausdorff). A closed congruence which is topological, is called a *closed topological congruence*. It is a known fact that for a closed ideal I of S, the quotient space S/I is Hausdorff if and only if I is regular in S. For a closed subset A of S, we denote the boundary of A in S by $\partial(A)$. Finally, we recall the Second Isomorphism Theorem for topological semigroups (we can find the general form of this theorem for any congruence in [2, Theorem 2.1]).

Theorem 1.1 (Second Isomorphism Theorem for Rees Congruences)**.** *Let S be a topological semigroup and let I be a closed ideal of S. Then every topological Rees congruences on S/I has the form J/I, where J is a closed ideal of S such that ρ_J is a topological congruence on S and $I \subseteq J$.*

2. Topological Rees congruences

Before we begin our investigation, we invite the reader to the following remark which gives the structure of open sets in S/I for some closed ideal I of S. Furthermore, we present the following notations needed in the sequel.

Remark 2.1. Let S be a topological semigroup, I be a closed ideal of S and π_I be the natural quotient map from S to S/I. It is straightforward to see that the family of open sets in S/I is equal to

$$\tau_I := \{\pi_I(O) : \ O \text{ is an open set in } S \text{ such that either } O \cap I = \emptyset \text{ or } I \subseteq O.\}$$

(Recall that a set O is open in the quotient topology, if the inverse image of O under π_I is open in S.)

Notation 1.

(i) Let X be a topological space and $\{Y_\alpha\}_{\alpha \in J}$ be a family of its subspaces. If X has the weak topology induced by $\{Y_\alpha\}_{\alpha \in J}$, then we denote it by $X = \Sigma_{\alpha \in J} Y_\alpha$ (if X has the weak topology induced by two subspaces Z and Y, then we denote it by $X = Z \oplus Y$). Recall that if X has the weak topology induced by a family $\{Y_\alpha\}_{\alpha \in J}$ of its subspaces, then a set O is open in X if and only if $O \cap Y_\alpha$ is open in Y_α for any $\alpha \in J$ (for more details about the weak topology, see [3]).

(ii) Let S be a topological semigroup and I be a closed ideal of S. Let τ be the following topology on $S \setminus I^\circ$ which is defined by

$$\tau := \{O | \ O \text{ is open in } S \text{ and either } O \cap I = \emptyset, \text{ or } I \subseteq O\}.$$

From now on, we denote the underlying set of $S \setminus I^\circ$ with the topology τ by S_I.

Theorem 2.1. *Let S be a topological semigroup and I be a closed ideal of S. The following conditions are equivalent*

(i) *S/I is a Hausdorff k_ω-space;*

(ii) *I is regular in S and $S = I \oplus (\Sigma_{n \in J} K'_n)$, where $J \subseteq \mathbb{N}$ and either $K'_n \cap I = \emptyset$ and K'_n is compact in S, or $K'_n = I \cup K_n$, where K_n is compact in S_I.*

Proof. **(i) ⇒ (ii)** Let S/I be a Hausdorff k_ω-space and π be the natural quotient map from S to S/I. Since $I = S$ is a trivial case, suppose that $I \neq S$. Since S/I is a k_ω-space, it is equal to the weak topology induced by a family of its compact subsets $\{\widetilde{K}_n\}_{n \in J}$, where $J \subseteq \mathbb{N}$. Clearly, since S/I is Hausdorff, I is regular in S. Define $K'_n := \pi^{-1}(\widetilde{K}_n)$. Note that clearly for any $n \in J$, K'_n is closed in S and if $[y] \notin \widetilde{K}_n$, where $y \in I$, then K'_n is compact in S. We show that $S = I \oplus (\Sigma_{n \in J} K'_n)$. Let A be a subset of S such that $A \cap K'_n$ is closed in K'_n for any $n \in J$ and $A \cap I$ is closed in I. First, we show that $\pi(A)$ is closed in S/I, then we show that A is closed in S.

Case (i): If $A \cap I = \emptyset$, then since for any $n \in J$, $\pi(A) \cap \widetilde{K}_n = \pi(A) \cap \pi(K'_n) = \pi(A \cap K'_n)$, $\pi(A) \cap \widetilde{K}_n$ is closed in $\widetilde{K}_n$, for any $n \in J$. Hence $\pi(A)$ is closed in S/I. Since $\pi|_A$ is one-one and π is a quotient map, A is a closed subset of S.

Case (ii): Let $A \cap I \neq \emptyset$ and consider an arbitrary K'_n and fix it. If $K'_n \cap I = \emptyset$, then $\pi(A \cap K'_n)$ is closed in $\widetilde{K}_n$, because $A \cap K'_n$ is compact in S and π is continuous.

If $K'_n \cap I \neq \emptyset$, then $A \cap I \neq \emptyset$ and $I \subseteq K'_n$. Clearly, since $\pi^{-1}(\pi(A) \cap \widetilde{K}_n) = (A \cup I) \cap K'_n = I \cup (A \cap K'_n)$, and since π is a quotient map, $\pi(A) \cap \widetilde{K}_n$ is closed in $\widetilde{K}_n$. Therefore, by the first and the second part of the proof in Case (ii), since K'_n is arbitrary, $\pi(A)$ is closed in S/I. Hence $A \cup I$ is closed in S. Clearly $(A \cup I) \setminus I^\circ$ is closed in S. Since we know that $A \cap I$ is closed, to prove our assertion, we show that $A \setminus (A \cap I^\circ)$ is closed in S (since $A = (A \cap I) \cup (A \setminus (A \cap I^\circ))$). Note that $\overline{(A \setminus (A \cap I^\circ))} \subseteq (A \cup I) \setminus I^\circ = (A \setminus I^\circ) \cup \partial I$. Therefore, if $x \in \partial(\overline{(A \setminus (A \cap I^\circ))})$, then $x \in A \setminus I^\circ$ or $x \in \partial I$. For the nontrivial case, let $x \in \partial I$. Since

S/I is Hausdorff, there exists a nontrivial open set O in S which contains I, and since S/I is a k_ω-space, $\pi(O)\cap\widetilde{K}_n$ is open in any $\widetilde{K}_n$. Therefore, x belongs to $\pi^{-1}(\widetilde{K}_n^\circ)\subseteq(K_n')^\circ$. Let $(a_\alpha)_{\alpha\in D}$ be a net in $A\setminus(A\cap I^\circ)$ which converges to x. Since $a_\alpha\to x$, there exists a $\beta\in D$ such that for any $\alpha\geq\beta$, $a_\alpha\in K_n'\cap A$. Since $K_n'\cap A$ is closed in K_n', x belongs to $K_n'\cap A\subseteq A$. Therefore A is a closed subset of S.

(ii) $\Rightarrow$ **(i)** Let $S=I\oplus(\Sigma_{n\in J}K_n')$, for some subset $J\subseteq\mathbb{N}$ and $\{K_n'\}_{n\in J}$ which fulfil the condition in (ii). Define $\widetilde{K}_n:=\pi(K_n')$. Clearly, by Remark 2.1, $\widetilde{K}_n$ is compact for any $n\in J$. Let $\widetilde{O}$ be a subset of S/I such that $\widetilde{O}\cap\widetilde{K}_n$ is open in $\widetilde{K}_n$ for any $n\in J$. Therefore, for any $n\in J$, $\pi^{-1}(\widetilde{O}\cap\widetilde{K}_n)\subseteq K_n'$ and $\pi^{-1}(\widetilde{O})$ is open in K_n'. Hence since $\pi^{-1}(\widetilde{O})\cap K_n'=\pi^{-1}(\widetilde{O}\cap\widetilde{K}_n)$, $\pi^{-1}(\widetilde{O})$ is open in S. Therefore $\widetilde{O}$ is open in S/I. Hence S/I is a k_ω-space. ∎

Remark 2.2. Note that the above theorem is true for the equivalence relation generated by the closed subset I of a topological space.

Proposition 2.1. *Let S be a topological semigroup and I be a regular closed ideal of S. If $S=I\oplus(\Sigma_{n\in Y}K_n')$ where $Y\subseteq\mathbb{N}$ and either $K_n'\cap I=\emptyset$ and K_n' is compact in S, or $K_n'=I\cup K_n$, where K_n is compact in S_I, then S/I is a topological semigroup.*

Proof. By Theorem 2.1, S/I is a Hausdorff k_w-space. Therefore by [9, Proposition 2.3 (a)], $S/I\times S/I$ is a Hausdorff k_w-space. Therefore by [9, Proposition 2.2], $\pi_I\times\pi_I$ is a quotient map. Now by [9, Proposition 2.1], S/I is a topological semigroup. ∎

Finally, to show an application of our results in this note, we prove the following theorem which is a generalization of Lawson and Madison's Theorem.

Theorem 2.2. *Let S be a topological semigroup which is a k-space and I be a regular closed ideal of S.*

(i) *If $S\setminus I^\circ$ is a k_ω-space, then S/I is topological.*

(ii) *If $S\setminus I^\circ$ is σ-compact,*

then for any closed ideal J in S where $I\subseteq J$, S/J is a topological semigroup.

Proof. First we prove (i). Let S be a topological semigroup and I be a closed ideal of S such that $S\setminus I^\circ$ is a k_ω-space. Therefore, there exists a countable family of compact sets in $S\setminus I^\circ$ like, $\{X_n\}_{n\in Y}$, where $Y\subseteq\mathbb{N}$, such that $S\setminus I^\circ=\Sigma_{n\in Y}X_n$. Define

$$X_n':=\begin{cases} X_n, & \text{if } X_n\cap I=\emptyset \\ X_n\cup I, & \text{otherwise}\end{cases}$$

It is straightforward to see that $S=I\oplus(\Sigma_{n\in Y}X_n')$ and X_n' satisfies the conditions in Theorem 2.1. Therefore S/I is a k_ω-space. Since S and $S/I\times S/I$ are both Hausdorff k-spaces, then by [9, Proposition 2.2], $\pi_I\times\pi_I$ is a quotient map. Now by [9, Proposition 2.1], the induced multiplication on S/I is continuous and ρ_I is a topological congruence.

To prove part (ii), first note that since $S\setminus I^\circ$ is σ-compact, S/I is a Hausdorff k_w-space. On the other hand, by part (i), S/I is a topological semigroup. Therefore by [9, Corollary 2.4] for every ideal $I\subseteq J$, $(S/I)/(J/I)$ is a topological semigroup. Now by the Second Isomorphism Theorem, the result is obvious. ∎

To illustrate better the application of the above theorem, we apply it in the next example.

Before we state our next result, we recall that an element z is called *zero* in a semigroup S, if for any $s\in S$ we have $zs=sz=z$.

Example 2.1. Let $S = S^0$ be a locally compact σ-compact topological semigroup with an adjoined zero 0_S as an isolated point. Let $T = T^0$ be a σ-compact non-locally compact topological semigroup with an adjoined zero 0_T as an isolated point. Define the equivalence relation ρ on the discrete sum of S^0 and T^0 as follows

$$\rho := \{(0_S, 0_T)\} \cup \Delta_S \cup \Delta_T \cup \{(0_T, 0_S)\}.$$

It is straightforward to see that

$$S' := (S^0 \oplus T^0)/\rho$$

with multiplication $\lambda : S' \times S' \to S'$ defined by

$$\lambda(x,y) := \begin{cases} [x \cdot_S y] & ,(x,y \in S) \\ [x \cdot_T y] & ,(x,y \in T) \\ [0] & ,\text{otherwise} \end{cases}$$

is a topological semigroup. Now clearly S' is Hausdorff locally compact, but it is not σ-compact. Let I be a closed ideal of S. Clearly, $I \cup \{0_S\}$ is an ideal of S^0. It is straightforward to see that $\pi(I \cup \{0_S\} \cup T^0)$ is a closed ideal of S', where π is the natural quotient map from $S^0 \oplus T^0$ to $(S^0 \oplus T^0)/\rho$. Now by Theorem 2.2, the corresponding Rees congruence of ideal $\pi(I \cup \{0_S\} \cup T^0)$ is topological and for any Rees congruence ρ_J, where $I \subseteq J$ and J is a closed ideal of S, ρ_J is topological.

As another consequence of the results in this note, we have

Corollary 2.1. *Let S be a topological semigroup which is a k-space and let I be a minimal ideal of S and $S \setminus I^\circ$ be σ-compact. If $S = I \oplus (\Sigma_{n \in J} K'_n)$ where $J \subseteq \mathbb{N}$ and either $K'_n \cap I = \emptyset$ and K'_n is compact in S, or $K'_n = I \cup K_n$, where K_n is compact in S_I, then all the closed Rees congruences on S are topological.*

Proof. By Theorem 2.2 part (ii), S/I is a topological semigroup and all the closed Rees congruences of S which contain ρ_I are topological congruences. But since I is a minimal ideal of S, all the Rees congruences of S are topological congruences. ∎

Finally in this note, we continue Lawson and Madison's study of the relation between separation axioms on S/I and the continuity of its multiplication (see the question on page 20 of [9]).

Let $\mathscr{A}$ be a class of topological semigroups. A semigroup $S \in \mathscr{A}$ is called *H-closed in $\mathscr{A}$*, if S is a closed subsemigroup of any topological semigroup $T \in \mathscr{A}$ which contains S as a subsemigroup. If $\mathscr{A}$ coincides with the class of all topological semigroups, then the semigroup S is called *H-closed*. A topological semigroup $S \in \mathscr{A}$ is called absolutely H-closed in the class $\mathscr{A}$, if any continuous homomorphic image of S into $T \in \mathscr{A}$ is H-closed in $\mathscr{A}$. Let $\{A_\alpha | \alpha \in \mathscr{A}\}$ and $\{B_\beta | \beta \in \mathscr{B}\}$ be two covers of a space Y. $\{A_\alpha\}$ is said to *refine* (or be a *refinement of*) $\{B_\beta\}$ if for each A_α there is some B_β with $A_\alpha \subseteq B_\beta$. A Hausdorff topological space X is called *paracompact* if every open cover of X has a locally finite open refinement.

Remark 2.3. Hryniv in [7] showed that there is a locally compact topological semigroup with a closed ideal I such that S/I is not a topological semigroup. Later, Gutik and Pavlyk in [6] presented another example and showed that there exists an absolutely H-closed ideal I of an absolutely H-closed countable metrizable topological semigroup S such that S/I is not a topological semigroup. We use Hryniv's example to study the relation between separation axioms and the continuity of the multiplication of S/I. This relation was studied by Lawson and Madison in [9]. By Hryniv's example (or similarly by Gutik and Pavlyk's example [6, Theorem 15]) we know that there exists a locally compact metrizable topological semigroup S and a closed ideal $I \leq S$ such that S/I is not topological. Since by [3, Theorem IX.5.3], every metric space is paracompact, and since by [3, Theorem VIII.2.2], every paracompact space is normal, S/I is a normal space; however, we know that S/I is not a topological semigroup. Therefore the continuity of the multiplication of S/I is independent from the normality of the space S/I.

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