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Cohomology and Stability of Generalized Sasakian Space-Forms

MARIA CÎRNU
Department of Mathematics and Informatics, University Transilvania of Braşov
Iuliu Maniu str., 50, Braşov, Romania
maria.cirnu@yahoo.com

Abstract. In this paper we study the geometry of distributions of semi-slant submanifolds of (α,β) trans-Sasakian manifolds, some problems concerning the stability of slant submanifolds of generalized Sasakian space-forms and we also investigate the first normal Chern class for integral submanifolds of (α,β) trans-Sasakian generalized Sasakian space-forms.

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1. Introduction

The geometry of distributions of slant and semi-slant submanifolds was studied by Cabrerizo, Carriazo, Fernández and Fernández [7, 8] in the case of K-contact and Sasakian manifolds. In this paper we study the geometry of distributions for semi-slant submanifolds of (α,β) trans-Sasakian manifolds. We obtain some results for cohomology groups and study remarkable forms associated to the Bott connection for semi-slant submanifolds in (α,β) trans-Sasakian manifolds. Ours results generalize those in [20] and [13]. Secondly, we study some aspects concerning variational problems for slant submanifolds in generalized Sasakian space-forms. Finally, studying the structure equations for integral submanifolds of (α,β) trans-Sasakian generalized Sasakian space-forms, we find certain conditions so that the first normal Chern class be trivial.

2. Preliminaries

Let $\widetilde{M}$ be an almost contact manifold, C^∞-differentiable with dimension $2m+1$. Let (F,ξ,η,g) be its almost contact structure, where F is a tensor field of type $(1,1)$, ξ is the Reeb vector field, η is a 1-form and g is a Riemannian metric on $\widetilde{M}$, all these tensors satisfying the following conditions:

$$F^2=-I+\eta\otimes\xi;\quad \eta(\xi)=1;\quad g(FX,FY)=g(X,Y)-\eta(X)\eta(Y), \tag{2.1}$$

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for all $X,Y \in \chi(\widetilde{M})$. Here $\chi(\widetilde{M})$ is the set of all vector fields on $\widetilde{M}$. We denote by Ω the fundamental (or the Sasaki) 2-form of $\widetilde{M}$, given by $\Omega(X,Y) = g(X,FY)$.

In [17], Oubina introduced the notion of a trans-Sasakian manifold. An almost contact metric manifold $\widetilde{M}$ is a *trans-Sasakian* manifold if there exist two functions α and β on $\widetilde{M}$ such that

$$(\widetilde{\nabla}_X F)Y = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(FX,Y)\xi - \eta(Y)FX], \tag{2.2}$$

for any $X,Y \in \chi(\widetilde{M})$. In particular, from (2.2) it is easy to see that the following equations hold on a trans-Sasakian manifold

$$\widetilde{\nabla}_X \xi = -\alpha FX + \beta[X - \eta(X)\xi]; \qquad d\eta = \alpha\Omega. \tag{2.3}$$

Moreover, if $\beta = 0$ then $\widetilde{M}$ is to said to be an *α-Sasakian manifold*. Sasakian manifolds appear as examples of α-Sasakian manifolds, with $\alpha = 1$, and Kenmotsu manifolds appear as examples of β-Kenmotsu manifolds, with $\beta = 1$. Another important kind of trans-Sasakian manifolds is that of *cosymplectic manifolds*, obtained for $\alpha = \beta = 0$. Marrero showed in [16] that a trans-Sasakian manifold of dimension greater than or equal to 5 is either α-Sasakian, β-Kenmotsu or cosymplectic manifold.

Given an almost contact metric manifold $\widetilde{M}$, we say that $\widetilde{M}$ is a *generalized Sasakian-space-form*, [1], if there exit three functions f_1, f_2, f_3 on $\widetilde{M}$ such that

$$\begin{aligned}\widetilde{R}(X,Y)Z =& f_1[g(Y,Z)X - g(X,Z)Y] \\ &+ f_2[g(X,FZ)FY - g(Y,FZ)FX + 2g(X,FY)FZ] \\ &+ f_3[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi],\end{aligned} \tag{2.4}$$

for any vector fields X,Y,Z on $\widetilde{M}$, where $\widetilde{R}$ denotes the curvature tensor of $\widetilde{M}$. In such a case, we will write $\widetilde{M}(f_1, f_2, f_3)$.

We also observe that this kind of manifold appears as a natural generalization of the well known Sasakian space-forms $\widetilde{M}(c)$, which can be obtained as particular cases of generalized Sasakian space-forms, by taking $f_1 = (c+3)/4$ and $f_2 = f_3 = (c-1)/4$ and as a generalization of Kenmotsu space-forms, by taking $f_1 = (c-3)/4$ and $f_2 = f_3 = (c+1)/4$.

Let M be a submanifold of the Riemannian manifold $\widetilde{M}$, ∇ the Levi-Civita connection induced by $\widetilde{\nabla}$ on M, $\nabla^\perp$ the connection in the normal bundle $T^\perp(M)$, h the second fundamental form of M and $A_{\vec{n}}$ the Weingarten operator. We recall the Gauss-Weingarten formulas on M

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y); \quad \widetilde{\nabla}_X \vec{n} = -A_{\vec{n}} X + \nabla^\perp_X \vec{n}, \tag{2.5}$$

for all $X,Y \in \chi(M)$ and $\vec{n} \in \chi^\perp(M)$.

A submanifold M of an almost contact metric manifold $\widetilde{M}$ is *totally contact geodesic*, [3], if

$$h(X,Y) = \eta(X)h(Y,\xi) + \eta(Y)h(X,\xi), \tag{2.6}$$

for all X,Y vector fields on M. From (2.6) it results that on a totally contact geodesic submanifold M, $h(\xi,\xi) = 0$.

A submanifold M of an almost contact metric manifold $\widetilde{M}$ is *parallel* if

$$(\widetilde{\nabla}_X h)(Y,Z) = 0, \tag{2.7}$$

where $(\widetilde{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X(h(Y,Z)) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z)$ and X,Y,Z are vector fields on M.

An integral submanifold M of the contact distribution $\mathscr{D} = \ker\eta$ is an *integral manifold* and such a submanifold is characterized by any of

(1) $\eta = 0, \quad d\eta = 0$;
(2) $FX \in \chi^{\perp}(M)$ for all X in $\chi(M)$.

The submanifold M of $\widetilde{M}$, tangent to ξ, is a *slant submanifold*, [15], if

$$\theta = \angle(FX_x, T_xM) = constant,$$

for all $x \in M$, $X_x \in T_xM$, X_x non-collinear with ξ. Taking into account the definition of the angle between a vector and a subspace in the Euclidean space, this is equivalent with

$$\cos\theta = \frac{g(FX,Y)}{\|FX\|\,\|Y\|} = constant,$$

for all $Y \in \chi(M)$, $X \in D$, X, Y nowhere zero, where D is the orthogonal distribution of ξ in $\chi(M)$. In this case, θ is the *slant angle* of the submanifold M and the distribution D is *the slant distribution* of M.

The submanifold M of a Riemannian manifold $\widetilde{M}$ is a *semi-slant submanifold*, [8], if there are D_1, D_2 two distributions on M so that

(i) $\chi(M) = D_1 \oplus D_2 \oplus \langle\xi\rangle$;
(ii) D_1 is invariant, i.e. $FD_1 = D_1$;
(iii) D_2 is slant with the slant angle θ.

For M a slant or semi-slant submanifold in a Riemannian manifold $\widetilde{M}$, we consider the decompositions

$$FX = TX + NX; \quad F\vec{n} = t\vec{n} + n\vec{n}, \tag{2.8}$$

for all $X \in \chi(M)$, $\vec{n} \in \chi^{\perp}(M)$, where TX is the tangent component and NX the normal component of FX and $t\vec{n}$ is the tangent component and $n\vec{n}$ is the normal component of $F\vec{n}$ in $\chi(\widetilde{M})$.

Moreover, if M is a semi-slant submanifold of a Riemannian manifold $\widetilde{M}$, then we consider

$$X = P_1X + P_2X + \eta(X)\xi, \tag{2.9}$$

for all $X \in \chi(M)$, where P_1 is the projection on D_1 and P_2 is the projection on D_2.

We recall some known results for slant and semi-slant submanifolds, [7, 8]:

Proposition 2.1. *Let M be a submanifold of the almost contact manifold $\widetilde{M}$ tangent to the Reeb vector field $\xi \in \chi(M)$. Then M is slant if and only if there is $\lambda \in [0,1]$ so that*

$$T^2 = -\lambda(I - \eta \otimes \xi).$$

Moreover, in this case, the slant angle θ of M satisfies the condition $\lambda = \cos^2\theta$.

Proposition 2.2. *Let M be a slant submanifold in an almost contact manifold $\widetilde{M}$ with the slant angle θ. Then*

$$g(TX,TY) = \cos^2\theta[g(X,Y) - \eta(X)\eta(Y)]$$

and

$$g(NX,NY) = \sin^2\theta[g(X,Y) - \eta(X)\eta(Y)],$$

for all $X,Y \in \chi(M)$.

Proposition 2.3. *Let M be a semi-slant submanifold of the almost contact manifold* $\widetilde{M}$ *with the slant angle* θ. *Then*

$$g(TX,TP_2Y) = \cos^2\theta g(X,P_2Y); \qquad g(NX,NP_2Y) = \sin^2\theta g(X,P_2Y),$$

for all $X,Y \in \chi(M)$.

3. Geometry of distributions on semi-slant submanifolds in (α,β) trans-Sasakian manifolds

Let M be a semi-slant submanifold of the (α,β) trans-Sasakian manifold $\widetilde{M}$. Denote by $T_i = P_i \circ T$, $i = 1,2$ and taking into account (2.9) and the fact that D_1 is invariant we have

$$FP_1X = TP_1X; \qquad NP_1X = 0; \qquad TP_2X \in D_2,$$

for all X vector fields on M. Using (2.2), (2.5), the definition of slant angle and these last equalities, we have

$$\begin{aligned} h(X,FP_1Y) + h(X,TP_2Y) + \nabla^{\perp}_X(NP_2Y) =& NP_2\nabla_XY + [Fh(X,Y)]^{\perp} \\ & - \beta\eta(Y)NP_2X, \end{aligned} \tag{3.1}$$

where X,Y are vector fields on M, $[Fh(X,Y)]^{\perp}$ is the normal component of $Fh(X,Y)$ in $\chi(\widetilde{M})$.

Now, let M be a p-dimensional semi-slant submanifold of (α,β) trans-Sasakian manifold $\widetilde{M}$ so that

$$\begin{aligned} (\nabla_XT)Y =& \alpha[g(P_1X,Y)\xi - \eta(Y)P_1X] \\ & + \alpha\cos^2\theta[g(P_2X,Y)\xi - \eta(Y)P_2X] \\ & + \beta[g(TX,Y)\xi - \eta(Y)TX], \end{aligned} \tag{3.2}$$

for all X,Y vector fields on M. Because D_1 is invariant, using (2.2) and (2.5) it results that

$$\begin{aligned} (\nabla_XT)Y =& \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(TX,Y)\xi - \eta(Y)TX] \\ & + A_{NP_2Y}X + [Fh(X,Y)]^T. \end{aligned}$$

Taking into account (3.2) we obtain

$$A_{NP_2Y}X = -[Fh(X,Y)]^T - \alpha\sin^2\theta[g(P_2X,Y)\xi - \eta(Y)P_2X],$$

$P_1A_{NP_2Y}X = 0$, for all $X,Y \in \chi(M)$ and $[Fh(X,Y)]^T = \alpha\sin^2\theta\eta(Y)P_2X$, for all $X \in \chi(M)$, $Y \in D_1 \oplus \langle\xi\rangle$. From these last relations, (3.1) and the fact that $TP_2X \in D_2$, we have that $g(NP_2\nabla_XY, Fh(X,Y)) = g(A_{NP_2\nabla_XY}X,Y)$
$= 0$ and then

$$\nabla_XY \in D_1 \oplus \langle\xi\rangle; \qquad \nabla_XZ \in D_2 \oplus \langle\xi\rangle, \tag{3.3}$$

for all $X \in \chi(M)$, $Y \in D_1$ and $Z \in D_2$.

We also have the following results:

Proposition 3.1. *Let M be a p-dimensional semi-slant submanifold of a $2m+1$-dimensional α-Sasakian manifold or β-Kenmotsu manifold $\widetilde{M}$, with $m \geq 2$ so that*

$$(\nabla_X T)Y = \alpha[g(P_1X,Y) - \eta(Y)P_1X] + \alpha\cos^2\theta[g(P_1X,Y)\xi - \eta(Y)P_2X]$$

in α-Sasakian case and

$$(\nabla_X T)Y = \beta[g(TX,Y)\xi - \eta(Y)TX]$$

in β-Kenmotsu case.

Then the invariant distribution D_1 is minimal for α-Sasakian case and is not minimal for β-Kenmotsu case.

Proof. We consider $\{X_i, FX_i\}$, $i = 1,...,q$, an orthonormal basis of the invariant distribution D_1 so that $2q < p$. The mean curvature vector of the distribution D_1 is

$$H_{D_1} = \frac{1}{2q}\sum_{i=1}^{q}(\nabla_{X_i}X_i + \nabla_{FX_i}FX_i)^{\perp},$$

where $(\nabla_{X_i}X_i + \nabla_{FX_i}FX_i)^{\perp}$ represents the orthogonal complement of $(\nabla_{X_i}X_i + \nabla_{FX_i}FX_i)$ in $D_2 \oplus \langle\xi\rangle$. For α-Sasakian case we obtain that $g(\nabla_{X_i}X_i, Z) = g(\nabla_{FX_i}FX_i, Z) = g(\nabla_{X_i}X_i, \xi)$ $= g(\nabla_{FX_i}FX_i, \xi) = 0$ and for β-Kenmotsu case $g(\nabla_{X_i}X_i, \xi) = g(\nabla_{FX_i}FX_i\xi) = -\beta$ and $g(\nabla_{X_i}X_i, Z) = g(\nabla_{FX_i}FX_i, Z) = 0$, where $Z \in D_2$. ∎

Denote by $\overset{o}{\nabla}: D_2 \times D_1 \to D_1$ *the Bott connection* defined by

$$\overset{o}{\nabla}_X U = P_1([X,U]),$$

for all $X \in D_2$ and $U \in D_1$. Let $S_{D_1} : D_1 \times D_1 \to D_2$ be defined by

$$S_{D_1} = P_2(\nabla_X Y + \nabla_Y X),$$

for $X, Y \in D_1$. If $S_{D_1} = 0$ then D_1 is a *totally geodesic plane field*, [22]. Let $\{\omega^1, ..., \omega^{2q}\}$ be the dual basis of the local orthonormal basis $\{X_1, ..., X_q, FX_1 = X_{q+1}, ..., FX_q = X_{2q}\}$ of D_1 and we extend it to whole $\chi(M)$. This means that

$$\omega^i(X_j) = \delta^i_j; \quad \omega^i_{/D_2} = 0; \quad \omega^i(\xi) = 0, \quad i,j = \overline{1,2q}.$$

We obtain the global defined $2q$-form $\omega = \omega^1 \wedge ... \wedge \omega^{2q}$ and it is a volume form of the distribution D_1.

Theorem 3.1. Let M be a p-dimensional semi-slant submanifold of the (α,β) trans-Sasakian manifold $\widetilde{M}$. Then

(i) the metric g of submanifold M is parallel with $\overset{o}{\nabla}$ if and only if D_1 is a totally geodesic plane field.

(ii) if M is compact and (3.2) holds for all $X, Y \in \chi(M)$, then ω is parallel with $\overset{o}{\nabla}$.

Proof.

(i) We consider $X \in D_2$ and $Y, Z \in D_1$. Taking into account the definition of $\overset{o}{\nabla}$, the properties of Levi-Civita connection, we obtain

$$(\overset{o}{\nabla}_X g)(Y,Z) = -g(X, S_{D_1}(Y,Z)),$$

for all $X \in D_2, Y, Z \in D_1$ and then i).

(ii) We have to prove that $(\overset{o}{\nabla}_X \omega)(X_1,...,X_{2q}) = 0$ for all $X \in D_2$. We have $\omega(X_1,X_2,..,X_{2q}) = 1$. From the definitions of $\overset{o}{\nabla}_X \omega$, 1-forms ω^i and (3.3) we have that $(\overset{0}{\nabla}_X \omega)(X_1,..., X_{2q}) = 0$. ∎

We denote by $\{X_{2q+1},...,X_{p-1}\}$ a local orthonormal basis in D_2 with its dual basis $\{\theta^{2q+1},...,\theta^{p-1}\}$ so that

$$\{X_1,...,X_q,X_{q+1} = FX_1,...,X_{2q} = FX_q,X_{2q+1},...,X_{p-1},\xi\}$$

is a local orthonormal basis in $\chi(M)$. Let $\theta = \theta^{2q+1} \wedge ... \wedge \theta^{p-1} \wedge \theta^p$ be a $(p-2q)$-form, where $\theta^p = \eta$. We extend θ^α, $\alpha = \overline{2q+1,p-1}$ at $\chi(M)$ so that $\theta^\alpha(X_\beta) = \delta^\alpha_\beta$; $\theta^\alpha(\xi) = 0$; $\theta^\alpha_{/D_1} = 0$, $\alpha,\beta = \overline{2q+1,p-1}$.

Proposition 3.2. *Let M be a p-dimensional semi-slant submanifold of the (α,β) trans-Sasakian manifold $\widetilde{M}$ so that (3.2) holds for all X,Y vector fields on M. Then*

(i) *the $2q$-form $\omega = \omega^1 \wedge ... \wedge \omega^{2q}$ is closed;*
(ii) *θ is closed;*
(iii) $*\omega = \theta$.

Proof.

(i) We have $d\omega = 0$ if and only if

$$(d\omega)(Y_1,X_1,...,X_{2q}) = 0; \quad (d\omega)(Y_1,Y_2,X_1,...,X_{2q-1}) = 0$$
$$(d\omega)(\xi,X_1,...,X_{2q}) = 0; \quad (d\omega)(\xi,Y_1,X_1,...,X_{2q-1}) = 0,$$

for all $Y_1,Y_2 \in D_2$. But these equalities follow by a straightforward computation using the definition of $d\omega$, ω^i, the property of Levi-Civita connection and (3.3).

(ii) Now, let $\{Y_{2q+1},Y_{2q+2},...,Y_{p-1},Y_p = \xi\}$ be a local orthonormal frame in $D_2 \oplus \langle\xi\rangle$. Then $d\theta = 0$ if and only if

$$(d\theta)(X_1,Y_{2q+1},...,Y_p) = 0; \quad (d\theta)(X_1,X_2,Y_{2q+1},...,Y_{p-1}) = 0,$$

for all $X_1,X_2 \in D_1$.
These two last equalities result from an analogous computation as that used at i), using the definitions of θ, θ^α, the property of Levi-Civita connection.

(iii) Results from the definition and the properties of the Hodge operator *. ∎

Theorem 3.2. Let M be a compact semi-slant submanifold of the (α,β) trans-Sasakian manifold $\widetilde{M}$ so that (3.2) holds for all X,Y vector fields on M. Then

$$b_{2k}(M) \geq 1,$$

where $k = \overline{1,q}$, $\dim D_1 = 2q$ and $b_{2k}(M)$ is the $2k^{th}$ Betti number of the submanifold M.

Proof. From the definition of Ω we consider $\Omega_M(X,Y) = g(X,FY)$, for all $X,Y \in \chi(M)$. It is easy to see that

$$\Omega^r_M(X_1,...,X_r) = (-1)^r r!; \qquad \Omega^r_M = 0$$

in other cases, where $r = \overline{1,q}$, $\Omega^r_M = \Omega_M \wedge ... \wedge \Omega_M$ and $\wedge$ is the exterior product. Moreover, $\Omega^q_M(X_1,...,X_q,FX_1,...,FX_q) = (-1)^q q!\omega$ and $\Omega^q_M = 0$, in other cases. From these last equalities, Proposition 3.1, the properties of the operators δ, * and the Hodge-de Rham decomposition, we have $b_{2q}(M) \geq 1$. ∎

4. Stability of slant submanifolds in generalized Sasakian space-forms

If M is a slant submanifold of the generalized Sasakian space-form $\widetilde{M}(f_1, f_2, f_3)$, with D the slant distribution with slant angle θ, then we consider $\{e_1, ..., e_{n-1}\}$ an orthonormal basis in D and

$$e_{n+1} = \frac{Ne_1}{\sin\theta}, ..., e_{2n-1} = \frac{Ne_{n-1}}{\sin\theta}.$$

Taking into account Proposition 2.2, we deduce that $\{e_{n+1}, ..., e_{2n-1}\}$ are orthonormal vectors. Let ΓNFD be the subspace spanned by $\{e_{n+1}, ..., e_{2n-1}\}$ and $\Gamma(\tau(M))$ the orthogonal complement of ΓNFD in $\chi^{\perp}(M)$, so that
$\{e_{2n}, ..., e_{2m+1}\}$ is an orthonormal basis in $\Gamma(\tau(M))$. We consider the dual 1-form to the vector $\vec{n} \in \chi^{\perp}(M)$, defined by

$$\alpha_{\vec{n}} : \chi(M) \to F(M), \quad \alpha_{\vec{n}}(X) = g(F\vec{n}, X), \tag{4.1}$$

for all $X \in \chi(M)$ and we denote by $\mathbf{L} = \{\vec{n} \in \chi^{\perp}(M) : d\alpha_{\vec{n}} = 0\}$ the set of *Legendre variations*, by $\mathbf{E} = \{\vec{n} \in \chi^{\perp}(M) : (\exists) f \in F(M) : \alpha_{\vec{n}} = df\}$ the set of *Hamiltonian variations* and by $\mathbf{H} = \{\vec{n} \in \chi^{\perp}(M) : d\alpha_{\vec{n}} = \delta\alpha_{\vec{n}} = 0\}$ the set of *harmonic variations*.

The first variation of the volume form of M, relative to the normal vector field $\vec{n}$ (that is the value at $t = 0$ of the first derivative of $V(\vec{n})$) can be expressed under the form [9]

$$V'(\vec{n}) = -n \int_M g(\vec{n}, H) dv, \tag{4.2}$$

where H is the mean curvature vector field of M. Then M is

(i) *l-minimal* if $V'(\vec{n}) = 0$ for all $\vec{n} \in \mathbf{L}$
(ii) *e-minimal* if $V'(\vec{n}) = 0$ for all $\vec{n} \in \mathbf{E}$
(iii) *h-minimal* if $V'(\vec{n}) = 0$ for all $\vec{n} \in \mathbf{H}$.

We also observe that:

(a) If the slant submanifold M is minimal, then M is l, e and h-minimal.
(b) If the slant submanifold M is e-minimal or h-minimal, then M is l-minimal.

Using similar arguments like those in [20], we have the following results:

Proposition 4.1. Let M be a slant submanifold with slant angle θ of the generalized Sasakian space-form $\widetilde{M}(f_1, f_2, f_3)$. Then:

(i) $\Gamma(\tau(M)) \subset \mathbf{L}$;
(ii) $\mathbf{H} \subset \mathbf{L}$;
(iii) $\mathbf{E} \subset \mathbf{L}$;
(iv) $F(\mathrm{grad} f)_{|\Gamma NFD} \subset \mathbf{E}$.

Theorem 4.1. Let M be a compact slant submanifold of the generalized Sasakian space-form $\widetilde{M}(f_1, f_2, f_3)$. Then:

(i) M is l-minimal if and only if $H \in \Gamma NFD$ and $\alpha_H = \sum_\mu f_\mu \Phi_\mu$, where $f_\mu \in F(M)$ and Φ_μ are co-exact 1-forms.
(ii) M is e-minimal if and only if $H \in \Gamma NFD$ and α_H is co-closed.
(iii) M is h-minimal if and only if $H \in \Gamma NFD$ and α_H is the sum of an exact 1-form and a co-exact 1-form.

Theorem 4.2. Let M be a compact slant submanifold of the generalized Sasakian space-form $\widetilde{M}(f_1, f_2, f_3)$. Then

(i) M is l-minimal if and only if M is minimal
(ii) M is e-minimal if and only if H is an harmonic variation.
(iii) M is h-minimal if and only if H is a hamiltonian variation.

Let $V''(\vec{n})$ be the second variation of the volume form of a n-dimensional slant submanifold M in the (α,β) generalized Sasakian-space-form $\widetilde{M}(f_1,f_2,f_3)$. By [9] this is given by

$$V\text{"}(\vec{n}) = \int_M \left\{ \left\| \nabla^{\perp}\vec{n} \right\|^2 - \|A_{\vec{n}}\|^2 \right\} dv$$

$$+ \int_M \left\{ n^2 g^2(H,\vec{n}) - ng(H,\widetilde{\nabla}_{\vec{n}}\vec{n}) - \sum_{a=1}^{n} \widetilde{R}(\vec{n},e_a,\vec{n},e_a) \right\} dv \tag{4.3}$$

where $\vec{n} \in \chi^{\perp}(M)$ and $\widetilde{R}$ is the Riemann Christoffel tensor of the manifold $\widetilde{M}$.
Then:

(i) M is *stable* if $V\text{"}(\vec{n}) \geq 0$ for all $\vec{n} \in \chi^{\perp}(M)$;
(ii) M is *l-stable* if $V\text{"}(\vec{n}) \geq 0$ for all $\vec{n} \in \mathbf{L}$;
(iii) M is *e-stable* if $V\text{"}(\vec{n}) \geq 0$ for all $\vec{n} \in \mathbf{E}$;
(iv) M is *h-stable* if $V\text{"}(\vec{n}) \geq 0$ for all $\vec{n} \in \mathbf{H}$.

Proposition 4.2. Let M be a n-dimensional slant submanifold with θ-the slant angle in the generalized Sasakian-space form $\widetilde{M}(f_1,f_2,f_3)$. If M is tangent to the Reeb vector field ξ then

$$\sum_{a=1}^{n} \widetilde{R}(\vec{n},e_a,\vec{n},e_a) = nf_1 - f_3, \tag{4.4}$$

for all $\vec{n} \in \Gamma(\tau(M))$.

Proof. From (2.4) and the fact that

$$\|\mathrm{proj}_{\Gamma NFD}\vec{n}\|^2 = \frac{1}{\sin^2\theta} \sum_{a=1}^{n-1} g^2(\vec{n},Ne_a)$$

we have

$$\sum_{a=1}^{n} \widetilde{R}(\vec{n},e_a,\vec{n},e_a) = nf_1 - f_3 + 3f_2 \sin^2\theta \, \|\mathrm{proj}_{\Gamma NFD}\vec{n}\|^2, \tag{4.5}$$

where $\vec{n}$ is a normal vector field on M and $\mathrm{proj}_{\Gamma NFD}\vec{n}$ is the projection of $\vec{n}$ on ΓNFD and then (4.4) ∎

Proposition 4.3. Let M be a n-dimensional minimal slant totally contact geodesic submanifold with θ-the slant angle in the generalized Sasakian-space-form $\widetilde{M}(f_1,f_2,f_3)$ so that M is tangent to the Reeb vector field ξ and

$$nf_1 - f_3 + 3f_3 \sin^2\theta \, \|proj_{\Gamma NFD}\vec{n}\|^2 \leq 0,$$

for all $\vec{n}$ normal vector fields on M. If $\vec{n}$ is parallel with the Levi-Civita connection $\widetilde{\nabla}$, then

$$V\text{"}(\vec{n}) \geq 0.$$

Proof. From (2.6), the properties of the Levi-Civita connection and the fact that $\vec{n}$ is parallel with respect the Levi-Civita connection $\widetilde{\nabla}$, we obtain that

$$\|A_{\vec{n}}\|^2 = 4\sum_{b=1}^{n} g^2(\xi, \widetilde{\nabla}_{e_b}\vec{n}) = 0.$$

From (4.3), the fact that M is minimal we have $V''(\vec{n}) \geq 0$. ∎

Example 4.1. Let $\widetilde{M} = \mathbb{R}^5$ be with local coordinates (x^1, x^2, y^1, y^2, z) and the Sasaki structure given by

$$\eta = \frac{1}{2}(dz - y^1 dx^1 - y^2 dx^2); \quad \xi = 2\frac{\partial}{\partial z};$$

$$g = \eta \otimes \eta + \frac{1}{4}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dy^1 \otimes dy^1 + dy^2 \otimes dy^2);$$

and $F : \chi(R^5) \to \chi(R^5)$ a tensor field of type (1,1) so that

$$F(\frac{\partial}{\partial x^1}) = -\frac{\partial}{\partial y^1}; \quad F(\frac{\partial}{\partial x^2}) = -\frac{\partial}{\partial y^2}; \quad F(\frac{\partial}{\partial z}) = 0;$$

$$F(\frac{\partial}{\partial y^1}) = \frac{\partial}{\partial x^1} + y^1\frac{\partial}{\partial z}; \quad F(\frac{\partial}{\partial y^2}) = \frac{\partial}{\partial x^2} + y^2\frac{\partial}{\partial z},$$

where $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial z}\right\}$ is a basis of $\chi(R^5)$. We observe that $\widetilde{M}$ is a generalized Sasakian space-form with $f_2 = f_3 = -1$ and $f_1 = 0$.

For $\theta \in [0, \frac{\pi}{2}]$ we consider the submanifold [3]

$$M : x(u, v, t) = (2u\cos\theta, 2u\sin\theta, 2v, 0, 2t).$$

From [3] and [13], it results that M is a minimal totally contact geodesic slant submanifold with the slant angle θ and slant distribution D, with the orthonormal basis

$$\left\{\vec{v}_1 = \frac{\partial}{\partial v}; \vec{v}_2 = \frac{\partial}{\partial u} + 2v\cos\theta\frac{\partial}{\partial t}\right\}$$

and

$$\left\{\vec{n}_1 = 2\frac{\partial}{\partial y^2}; \vec{n}_2 = 2\sin\theta\frac{\partial}{\partial x^1} - 2\cos\theta\frac{\partial}{\partial x^2} + 4v\sin\theta\frac{\partial}{\partial z}\right\}$$

the orthonormal basis in $\chi^\perp(M)$, $\vec{n}_1, \vec{n}_2$ in ΓNFD. We also have $\sum_{a=1}^{2}\widetilde{R}(\vec{n}_1, e_a, \vec{n}_1, e_a) = 1 - 3\sin^2\theta$ and $V''(\vec{n}_1) \geq 0$, for $\theta \in [0, \arcsin\frac{1}{\sqrt{3}}]$.

5. Chern classes of integral submanifolds of (α, β) trans-Sasakian generalized space-forms

In this section we give the structure equations of an integral submanifold M in an (α, β) trans-Sasakian generalized Sasakian space-form $\widetilde{M}(f_1, f_2, f_3)$ and we study the geometry of the maximal invariant normal subbundle $\tau(M)$. We also prove that the first Chern class of $\tau(M)$ is zero under certain conditions.

Taking into account Marrero's classification of the (α, β) trans-Sasakian manifolds with dimensions greater or equal with 5, [16], we recall some results obtained in [2] about this kind of manifolds.

Proposition 5.1. Let $\widetilde{M}(f_1,f_2,f_3)$ be an α-Sasakian generalized-space-form. Then α does not depend on the direction of ξ and the following equation holds

$$f_1 - f_3 = \alpha^2.$$

Moreover, if M is connected or $\dim\widetilde{M}(f_1,f_2,f_3) \geq 5$ then α is constant, respectively, f_1,f_2,f_3 are constant, related as follows

(i) If $\alpha = 0$, then $f_1 = f_2 = f_3$ and M is a cosymplectic manifold of constant F-sectional curvature.
(ii) If $\alpha \neq 0$, then $f_1 - \alpha = f_2 = f_3$.

Proposition 5.2. Let $\widetilde{M}(f_1,f_2,f_3)$ be a β-Kenmotsu generalized space-form. Then β does not depend on the direction of ξ and the following equation holds

$$f_1 - f_3 + \xi(\beta) + \beta^2 = 0.$$

Moreover, if $\dim\widetilde{M}(f_1,f_2,f_3) \geq 5$ then f_1,f_2,f_3 depend only on the direction of ξ and the following equations hold

$$\xi(f_1) + 2\beta f_3 = 0; \qquad \xi(f_2) + 2\beta f_2 = 0.$$

Proposition 5.3. Let M be a 3-dimensional (α,β) trans-Sasakian manifold such that α,β depend only the direction of ξ. Then M is a generalized Sasakian space-form $\widetilde{M}(f_1,f_2,f_3)$ with functions

$$f_1 = 3\rho - 2(\alpha^2 - \xi(\beta) - \beta^2); \quad f_2 = 0; \quad f_3 = 3\rho - 3(\alpha^2 - \xi(\beta) - \beta^2),$$

where ρ is the scalar curvature of M.

Now, let M be a n-dimensional integral submanifold of an (α,β) trans-Sasakian generalized Sasakian space-form $\widetilde{M}(f_1,f_2,f_3)$, with dimension $2m+1$. From the properties of the integral submanifolds, [6], we have $n \leq m$ and we consider on $\widetilde{M}(f_1,f_2.f_3)$ a local orthonormal basis $B = \{e_1,...,e_n,e_{n+1},..,e_m,e_{1^*} = Fe_1,...,e_{n^*} = Fe_n,e_{(n+1)^*} = Fe_{n+1},...,e_{m^*} = Fe_m,$ $\xi\}$, so that $\{e_1,...,e_n\}$ is a local orthonormal basis on M. Denote by $e_{(m+1)^*} = \xi$ and we will use the following convention on indices: $j = \overline{1,m}$; $j^* = j+m$; $a,b,c = \overline{1,n}$; $a^* = a+m$; $b^* = b+m$; $c^* = c+m$; $\lambda,\mu,\nu = \overline{n+1,m}$; $\lambda^* = \lambda+m$; $\alpha,\beta,\gamma,\delta = \overline{1,2m+1}$.
If $B^* = \left\{\omega^1,...,\omega^n,\omega^{n+1},...,\omega^m,...,\omega^{1^*},...,\omega^{n^*},...,\omega^{(n+1)^*},...,\omega^{m^*},\omega^{(m+1)^*} = \eta\right\}$ is the dual basis of B, then, at the points of M we locally have

$$\omega^\lambda = \omega^{j^*} = \omega^{(m+1)^*} = 0. \tag{5.1}$$

On the other hand, if we consider ω_α^β the connection forms of $\widetilde{\nabla}$, expressed with respect to B, on the submanifold M, we obtain:

$$\omega^a_{(m+1)^*} = \beta\omega^a; \quad \omega^\lambda_{(m+1)^*} = \omega^{\lambda^*}_{(m+1)^*} = 0; \quad \omega^{(m+1)^*}_{a^*} = \alpha\omega^a; \tag{5.2}$$

$$\omega^{j^*}_a = \omega^{a^*}_j; \quad \omega^{j^*}_{a^*} = \omega^j_a; \quad \omega^{j^*}_\lambda = \omega^{\lambda^*}_j; \quad \omega^{j^*}_{\lambda^*} = \omega^j_\lambda. \tag{5.3}$$

The curvature forms of $\widetilde{M}(f_1,f_2,f_3)$ and M are, respectively,

$$\widetilde{\Omega}^\alpha_\beta = \frac{1}{2}\sum_{\alpha,\beta=1}^{2m+1} \widetilde{R}^\alpha_{\beta\gamma\delta}\omega^\gamma \wedge \omega^\delta; \quad \Omega^a_b = \frac{1}{2}\sum_{c,d=1}^{n} R^a_{bcd}\omega^c \wedge \omega^d, \tag{5.4}$$

where $\widetilde{R}^{\alpha}_{\beta\gamma\delta}$ and R^{a}_{bcd} are the components with respect to B of the curvature tensors of $\widetilde{M}(f_1, f_2, f_3)$ and M, respectively. Then, at the points of M, we have

$$\Omega^a_b = \widetilde{\Omega}^a_b - \sum_{\lambda=n+1}^{m} \omega^a_\lambda \wedge \omega^\lambda_b - \sum_{j=1}^{m} \omega^a_{j^*} \wedge \omega^{j^*}_b, \tag{5.5}$$

$$\Omega^\lambda_\mu = \widetilde{\Omega}^\lambda_\mu - \sum_{a=1}^{n} \omega^\lambda_\alpha \wedge \omega^a_\mu = \frac{1}{2} \sum_{a,b=1}^{n} R^\lambda_{\mu ab} \omega^a \wedge \omega^b, \tag{5.6}$$

where $R^\lambda_{\mu ab}$ are the components of the curvature tensor of $\nabla^\perp$. From (5.1), (5.2), (5.3) and from the general form of the structure equations, [14], we have the following structure equations of the integral submanifold M, under the form

$$d\omega^a = -\sum_{b=1}^{n} \omega^a_b; \quad d\omega^a_b = -\sum_{c=1}^{n} \omega^a_c \wedge \omega^c_b + \Omega^a_b, \tag{5.7}$$

$$d\omega^\lambda_\mu = -\sum_{\nu=n+1}^{m} \omega^\lambda_\nu \wedge \omega^\nu_\mu - \sum_{j=1}^{m} \omega^\lambda_{j^*} \wedge \omega^{j^*}_\mu + \Omega^\lambda_\mu. \tag{5.8}$$

Let $\vec{n}$ be a normal vector field to the integral submanifold M of the (α, β) trans-Sasakian generalized Sasakian space-form $\widetilde{M}(f_1, f_2, f_3)$. We consider 1-form $\alpha_{\vec{n}}$ defined in (4.1) and 1-form $\theta = \sum_{a=1}^{n} \omega^{a^*}_a$. We obtain, using similar technics as those in [21], the following results:

Proposition 5.4. The forms $\alpha_{\vec{n}}$ and θ have the following properties:

(i) $\alpha_\xi = 0$ and $\theta = -n\alpha_H$, where H is the mean curvature vector of M

(ii) $\alpha_{\vec{n}}$ is closed if and only if

$$g(\nabla^\perp_X \vec{n}, FY) = g(\nabla^\perp_Y \vec{n}, FX),$$

for all X, Y vector fields on M.

(iii) The exterior derivative of θ is given by

$$d\theta = \sum_{b,c=1}^{n} (\widetilde{S}_{bc^*} - \sum_{\lambda} R^{\lambda^*}_{\lambda bc} - \frac{1}{2} \sum_{a=1}^{n} \widetilde{R}^{a^*}_{abc}) \omega^b \wedge \omega^c,$$

where $\widetilde{S}$ is the Ricci tensor of $\widetilde{M}(f_1, f_2, f_3)$.

The normal space $T^\perp_x M$ at each point x of M has the following orthogonal decomposition

$$T^\perp_x M = F(T_x M) \oplus \tau_x(M) \oplus \langle \xi_x \rangle, \tag{5.9}$$

where $\langle \xi_x \rangle$ is the normal subspace generated by ξ_x and $\tau_x(M)$ is the $2(m-n)$-dimensional subspace of $T_x\widetilde{M}$, orthogonal to $F(T_x M) \oplus \langle \xi_x \rangle$. Then $\tau(M) = \cup_{x \in M} \tau_x(M)$ is the total space of the subbundle of $T^\perp(M)$ and $B_\tau = \{e_{n+1}, ..., e_m, e_{(n+1)^*}, ..., e_{m^*}\}$ is a local basis in the module $\Gamma(\tau)$ of its sections. We also denote this bundle by $\tau(M)$ and it is called *the maximal invariant normal bundle* of the integral submanifold M.

Proposition 5.5. Let M be an integral submanifold of the (α, β) trans-Sasakian generalized-space-form $\widetilde{M}(f_1, f_2, f_3)$. Then its maximal invariant normal bundle $\tau(M)$ has the following properties:

(i) $\tau(M)$ is invariant by F, that is, $F(T_x(M)) = \tau_x(M)$ for each point x of M.

(ii) $\tau(M)$ has a natural structure of complex vector bundle.

Proof.

(i) Follows from (5.9).

(ii) Let $B_\tau = \left\{e_{n+1}, ..., e_m, e_{(n+1)^*}, ..., e_{m^*}\right\}$ be an orthonormal basis on $\Gamma(\tau)$. For $\vec{n} \in \Gamma(\tau)$ we consider $\left\{n^\lambda, n^{\lambda^*}\right\}$ the components of the vector $\vec{n}$ relative to the basis B_τ and $P : \tau(M) \to M$ be the natural projection. Then, using the classical notations, the vector charts

$$\Phi : P^{-1}(U) \to U \times \mathbf{C}^{m-n}, \Phi(\vec{n}_x) = (x, (n^\lambda + in^{\lambda^*})),$$

for $x \in U$, define on $\tau(M)$ a complex vector bundle structure. ∎

Because $g(\nabla^\perp_X \vec{n}, \xi) = 0$, for all X vector fields on M and $\vec{n} \in \Gamma(\tau)$, the normal vector field $\nabla^\perp_X \vec{n}$ has the following decomposition

$$\nabla^\perp_X \vec{n} = B_{\vec{n}} X + \nabla^\tau_X \vec{n}, \tag{5.10}$$

where $B_{\vec{n}} X \in \Gamma(FTM)$ and $\nabla^\tau_X \vec{n} \in \Gamma(\tau)$. Moreover, the maps $B : \Gamma(\tau) \times \chi(M) \to \Gamma(FTM)$ and $\nabla^\tau : \chi(M) \times \Gamma(\tau) \to \Gamma(\tau)$ have the following properties:

Proposition 5.6.

(i) ∇^τ is an almost complex connection on the maximal invariant normal bundle of the integral submanifold M, that is , $(\nabla^\tau_x F)\vec{n} = 0$.

(ii) $B_{\vec{n}} X = FA_{F\vec{n}} X$, for all $X \in \chi(M)$ and $\vec{n} \in \Gamma(\tau)$.

As a complex vector bundle, the basic characteristic classes of the maximal invariant normal bundle $\Gamma(\tau)$ are the *Chern classes* $[\gamma_k(\tau)]$, represented by Chern forms

$$\gamma_k = \frac{i^k}{(2\pi)^k k!} \delta^{\mu_1 ... \mu_k}_{\lambda_1 ... \lambda_k} \Omega^{\tau\lambda_1}_{\mu_1} \wedge ... \wedge \Omega^{\tau\lambda_k}_{\mu_k}, \tag{5.11}$$

where $\Omega^{\tau\lambda}_\mu$ are the curvature forms of ∇^τ and $\delta^{\mu_1 \cdots \mu_k}_{\lambda_1 \ldots \lambda_k}$ are the multiindex Kronecker symbol. $\gamma_k(\tau)$ is called *the kth normal Chern form* of the submanifold M. From a similar argument as that used in [21], we have the following

Theorem 5.1. The first normal Chern of an n-dimensional integral submanifold in the (α, β) trans-Sasakian generalized Sasakian space-form $\widetilde{M}(f_1, f_2, f_3)$ of dimension $2m + 1, m > n$, is given by

$$\gamma_1(\tau) = \frac{1}{2\pi} \sum_{\lambda=n+1}^{m} \Omega^{\lambda^*}_{\lambda.} \tag{5.12}$$

Theorem 5.2. Let M be an integral submanifold of the (α, β) trans-Sasakian generalized Sasakian space-form $\widetilde{M}(f_1, f_2, f_3)$. If the mean curvature vector of M is parallel, then its first normal Chern form $\gamma_1(\tau)$ is zero.

Proof. From (2.4) we have $\widetilde{R}^{a^*}_{abc} = 0$ and $\widetilde{S}_{bc^*} = 0$. Then, taking into account Proposition 5.4 and Theorem 5.1 we obtain the result. ∎

Proposition 5.7. Let M be a totally umbilical integral submanifold of the (α, β) trans-Sasakian generalized Sasakian space-form $\widetilde{M}(f_1, f_2, f_3)$. If M is parallel, then its first normal Chern form $\gamma_1(\tau)$ is zero.

Proof. Because M is totally umbilical, we have $h(X,Y) = g(X,Y)H$ and from (2.7) we obtain that the mean curvature vector H of M is parallel. Then we apply Theorem 5.2. ∎

References

[1] P. Alegre, D. E. Blair and A. Carriazo, Generalized Sasakian-space-forms, *Israel J. Math.* **141** (2004), 157–183.

[2] P. Alegre and A. Carriazo, Structures on generalized Sasakian-space-forms, *Differential Geom. Appl.* **26** (2008), no. 6, 656–666.

[3] A. Carriazo, On generalized Sasakian-space-forms, in *Proceedings of the Ninth International Workshop on Differential Geometry*, 31–39, Kyungpook Nat. Univ., Taegu, 2005.

[4] P. Alegre and A. Carriazo, Submanifolds of generalized Sasakian space forms, *Taiwanese J. Math.* **13** (2009), no. 3, 923–941.

[5] P. Alegre and A. Carriazo, Generalized Sasakian space forms and conformal changes of the metric, *Results Math.* **59** (2011), no. 3–4, 485–493.

[6] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics, Vol. 509, Springer, Berlin, 1976.

[7] J. L. Cabrerizo, A. Carriazo, L. M. Fernández and M. Fernández, Slant submanifolds in Sasakian manifolds, *Glasg. Math. J.* **42** (2000), no. 1, 125–138.

[8] J. L. Cabrerizo, A. Carriazo, L. M. Fernández and M. Fernández, *Semi-Slant Submanifolds of a Sasakian Manifold*, Geom. Dedicata, **78**(1999), 183–199.

[9] B. Chen, *Geometry of Submanifolds and its Applications*, Sci. Univ. Tokyo, Tokyo, 1981.

[10] B. Chen, *Geometry of Slant Submanifolds*, Katholieke Univ. Leuven, Louvain, 1990.

[11] B. Chen, Cohomology of CR-submanifolds, *Ann. Fac. Sci. Toulouse Math. (5)* **3** (1981), no. 2, 167–172.

[12] M. Cîrnu, Topological properties of semi-slant submanifolds in Sasaki manifolds, *Bull. Transilv. Univ. Braşov Ser. III* **1(50)** (2008), 79–86.

[13] M. Cîrnu, Stability of slant and semi-slant submanifolds in Sasaki manifolds, *Acta Univ. Apulensis Math. Inform. No. 20* (2009), 63–78.

[14] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry. Vol I*, Interscience Publishers, a division of John Wiley & Sons, New York, 1963.

[15] A. Lotta, Slant submanifolds in contact geometry, *Bull. Math. Soc. Sci. Math. Roumanie* **39** (1996), 183–198.

[16] J. C. Marrero, The local structure of trans-Sasakian manifolds, *Ann. Mat. Pura Appl. (4)* **162** (1992), 77–86.

[17] J. A. Oubiña, New classes of almost contact metric structures, *Publ. Math. Debrecen* **32** (1985), no. 3-4, 187–193.

[18] Y.-G. Oh, Second variation and stabilities of minimal Lagrangian submanifolds in Kähler manifolds, *Invent. Math.* **101** (1990), no. 2, 501–519.

[19] G. Pitiş, Feuilletages et sous-variétés d'une classe de variétés de contact, *C. R. Acad. Sci. Paris Sér. I Math.* **310** (1990), no. 4, 197–202.

[20] G. Pitiş, Stability of integral submanifolds in a Sasakian manifold, *Kyungpook Math. J.* **41** (2001), no. 2, 381–392.

[21] G. Pitiş, Chern classes of integral submanifolds of some contact manifolds, *Int. J. Math. Math. Sci.* **32** (2002), no. 8, 481–490.

[22] B. L. Reinhart, *Differential Geometry of Foliations*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 99, Springer, Berlin, 1983.