**BULLETIN** *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

*Bull. Malays. Math. Sci. Soc. (2)* **35**(2) (2012), 277–285

# Characterizations of Term-Rank Preservers over Boolean Matrices

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**Abstract.** In this paper we obtain some characterizations of linear operators that preserve term rank of Boolean matrices. With certain conditions, we prove that for a linear operator $T$ on the Boolean matrix space, $T$ preserves term rank if and only if $T$ preserves two consecutive term ranks if and only if $T$ strongly preserves just one term rank.

2010 Mathematics Subject Classification: 15A86, 15A03, 15A33

Keywords and phrases: Boolean matrix, term rank, covering, linear operator.

## 1. Introduction

Let $\mathbb{M}_{m,n}$ be the set of all $m \times n$ matrices with entries in the *Boolean algebra* $\mathbb{B} = \{0,1\}$. If $m = n$, we will use the notation $\mathbb{M}_n$ instead of $\mathbb{M}_{n,n}$. Arithmetic in $\mathbb{B}$ follows the usual rules except that $1+1=1$. The usual definitions for adding and multiplying matrices apply to Boolean matrices as well.

The *term rank* of a matrix $X$ in $\mathbb{M}_{m,n}$ is the least number of lines (rows and columns) needed to include all the nonzero entries in $X$, and denoted by $t(X)$. Term ranks play a central role in the combinatorial matrix theory and have many applications in network and graph theory (see [4]).

An operator $T : \mathbb{M}_{m,n} \to \mathbb{M}_{m,n}$ is called *linear* if $T(aX+bY) = aT(X)+bT(Y)$ for all $X,Y \in \mathbb{M}_{m,n}$ and for all $a,b \in \mathbb{B}$. Let $T$ be a linear operator on $\mathbb{M}_{m,n}$. If $f$ is a function defined on $\mathbb{M}_{m,n}$, then $T$ *preserves* $f$ if $f(T(X)) = f(X)$ for all $X$. There are many papers on linear operators that preserve matrix functions over $\mathbb{B}$ ([1]–[5] and therein). But there are few papers on term-rank preservers of Boolean matrices.

Hereafter, unless otherwise specified, we will assume that $2 \le m \le n$. It follows that $1 \le t(X) \le m$ for all nonzero $X$ in $\mathbb{M}_{m,n}$.

For a linear operator $T$ on $\mathbb{M}_{m,n}$, we say that $T$

(1) *preserves term rank* $k$ if $t(T(X)) = k$ whenever $t(X) = k$ for all $X$;
(2) *strongly preserves term rank* $k$ if $t(T(X)) = k$ if and only if $t(X) = k$ for all $X$;
(3) *preserves term rank* if $T$ preserves term rank $k$ for every $k \le m$.

*Communicated by* Rosihan M. Ali, Dato'.
*Received:* May 12, 2010; *Revised:* July 20, 2010.

In [3], Beasley and Pullman have studied linear operators on $\mathbb{M}_{m,n}$ that preserve term rank, and obtained the following main results:

**Theorem 1.1.** *For a linear operator $T$ on $\mathbb{M}_{m,n}$, $T$ preserves term rank if and only if $T$ preserves term ranks* 1 *and* 2.

**Theorem 1.2.** *For a linear operator $T$ on $\mathbb{M}_{m,n}$, $T$ preserves term rank if and only if $T$ strongly preserves term rank* 1 *or $m$.*

We continue their work in this paper. Let $T$ be a linear operator on $\mathbb{M}_{m,n}$. As a generalization of Theorem 1.1, we obtain that $T$ preserves term rank if and only if $T$ preserves two consecutive term ranks $k$ and $k+1$ in a restricted condition, where $1 \leq k \leq m-1$. Also, we generalize Theorem 1.2 as follow: $T$ preserves term rank if and only if $T$ strongly preserves term rank $k$ in a restricted condition, where $1 \leq k \leq m$.

## 2. Preliminaries

A matrix in $\mathbb{M}_{m,n}$ with only one entry equal to 1 is called a *cell*. If the nonzero entry occurs in the $i$th row and the $j$th column, we denote the cell by $E_{i,j}$.

**Definition 2.1.** *A matrix $L$ in $\mathbb{M}_{m,n}$ is called a full line matrix if $L = \sum_{s=1}^{n} E_{i,s}$ for some $i$ or $L = \sum_{t=1}^{m} E_{t,j}$ for some $j$: $R_i = \sum_{s=1}^{n} E_{i,s}$ is the $i$th full row matrix and $C_j = \sum_{t=1}^{m} E_{t,j}$ is the $j$th full column matrix.*

**Definition 2.2.** *Let $X$ and $Y$ be matrices in $\mathbb{M}_{m,n}$. Then $X$ dominates $Y$ (denoted by $Y \sqsubseteq X$) if $y_{i,j} \neq 0$ implies $x_{i,j} \neq 0$ for all $i$ and $j$.*

**Definition 2.3.** *Let $X$ be a matrix in $\mathbb{M}_{m,n}$ with $t(X) = k$. Then there are $r$ full row matrices $R_{i_1}, \ldots, R_{i_r}$ and $k-r$ full column matrices $C_{j_1}, \ldots, C_{j_{k-r}}$ such that $X \sqsubseteq (R_{i_1} + \cdots + R_{i_r}) + (C_{j_1} + \cdots + C_{j_{k-r}})$, where $0 \leq r \leq k$. We say that*

$$\operatorname{cov}(X) = \left\{R_{i_1}, \ldots, R_{i_r}, C_{j_1}, \ldots, C_{j_{k-r}}\right\}$$

*is a covering of $X$. If the possible value of $r$ are only 0 or $k$, we say that $X$ does not have a proper covering; Otherwise $X$ has a proper covering.*

*For example, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has a proper covering $\{R_1, C_3\}$, while $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ does not have a proper covering.*

*The $m \times n$ zero matrix is denoted by $O_{m,n}$, and we will suppress the subscript when the order is evident from the context. The matrix $I_n$ is the $n \times n$ identity matrix.*

**Proposition 2.1.** *Let $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a partitioned matrix in $\mathbb{M}_{m,n}$, where $A \in \mathbb{M}_k$ with $I_k \sqsubseteq A$ and $1 \leq k < m$. If $B \neq O$, $\begin{bmatrix} C & D \end{bmatrix} \neq O$ and $\begin{bmatrix} A & B \end{bmatrix}$ does not have a proper covering, then $t(X) \geq k+1$.*

*Proof.* By hypothesis, $t(X) \geq k$ and $\{R_1, \ldots, R_k\}$ is the only covering of $\begin{bmatrix} A & B \end{bmatrix}$. If $k = 1$, obviously $t(X) \geq 2$. For $k \geq 2$, suppose that $t(X) = k$. If $\operatorname{cov}(X)$ is a covering of $X$, then $\operatorname{cov}(X)$ cannot be composed of $k$ full row matrices or $k$ full column matrices. Thus $\operatorname{cov}(X)$ must be a proper covering. But then $\operatorname{cov}(X)$ is a proper covering of $\begin{bmatrix} A & B \end{bmatrix}$. This contradiction shows that $t(X) \geq k+1$. ∎

**Lemma 2.1.** *Let $X$ be a matrix in $\mathbb{M}_{m,n}$ with $t(X) = k(\geq 2)$. If $X$ has a proper covering, for some $r \in \{1,\dots,k-1\}$, by permuting rows and columns of $X$, we can assume that*

$$X = \begin{bmatrix} X_1 & X_2 & X_3 \\ X_4 & O & O \\ X_7 & O & O \end{bmatrix}, \quad I_r \sqsubseteq X_2 \quad \text{and} \quad I_{k-r} \sqsubseteq X_4,$$

*where $X_2 \in \mathbb{M}_r$ and $X_4 \in \mathbb{M}_{k-r}$; the rest $X_i$ are matrices of suitable sizes.*

*Proof.* Since $X$ has a proper covering, $\text{cov}(X)$, there is an integer $r$ in $\{1,\dots,k-1\}$ such that $\text{cov}(X)$ is composed of $r$ full row matrices and $k-r$ full column matrices. Hence by permuting rows and columns of $X$ (if necessary), we can assume that $\text{cov}(X) = \{R_1,\dots,R_r,C_1,\dots,C_{k-r}\}$ so that $X = \begin{bmatrix} X_1 & X_2 & X_3 \\ X_4 & O & O \\ X_7 & O & O \end{bmatrix}$, where $X_2 \in \mathbb{M}_r$ with $I_r \sqsubseteq X_2$ and $X_4 \in \mathbb{M}_{k-r}$ with $I_{k-r} \sqsubseteq X_4$; the rest $X_i$ are matrices of suitable sizes. ∎

**Definition 2.4.** *Let $X$ be a matrix in $\mathbb{M}_{m,n}$ partitioned as in Lemma 2.2. Then we say that $X$ has a pure proper covering if neither $\begin{bmatrix} X_2 & X_3 \end{bmatrix}$ nor $\begin{bmatrix} X_4 \\ X_7 \end{bmatrix}$ has a proper covering.*

*For example, consider two matrices $X = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Then $t(X) = t(Y) = 3$. But $X$ has a pure proper covering, while $Y$ does not.*

**Lemma 2.2.** *Let $X$ be a matrix in $\mathbb{M}_{m,n}$ with $t(X) = k$, where $k \geq 2$. Assume that $X$ has a pure proper covering and is partitioned as in Lemma 2.1. For every $Y \in \mathbb{M}_{m,n}$, if $t(X+Y) = k$ and $\text{cov}(X+Y)$ is a covering of $X+Y$, then $\text{cov}(X+Y) \in \{S_1,S_2,S_3,S_4\}$, where*

$$S_1 = \{R_1,\dots,R_k\}, \quad S_2 = \{R_{r+1},\cdots,R_k,C_{k-r+1},\cdots C_k\},$$
$$S_3 = \{C_1,\dots,C_k\} \quad \text{and} \quad S_4 = \{R_1,\cdots,R_r,C_1,\cdots C_{k-r}\}.$$

*Proof.* Assume that $t(X+Y) = k$ and $\text{cov}(X+Y)$ is a covering of $X+Y$. Since $I_r \sqsubseteq X_2$ and $I_{k-r} \sqsubseteq X_4$, we can easily show that $\text{cov}(X+Y)$ contains neither $R_i$ nor $C_j$ for all $i,j > k$. Let $D_1 = \{R_1,\dots,R_r\}$, $D_2 = \{R_{r+1},\dots,R_k\}$, $D_3 = \{C_1,\dots,C_{k-r}\}$ and $D_4 = \{C_{k-r+1},\dots,C_k\}$.

First, suppose that $\text{cov}(X+Y)$ contain no member in $D_1$. Then $X_3 = O$. It follows from $I_r \sqsubseteq X_2$ that $\text{cov}(X+Y)$ must contain all members in $D_4$. Since $\begin{bmatrix} X_4 \\ X_7 \end{bmatrix}$ does not have a proper covering, $\text{cov}(X+Y)$ must contain all members in either $D_3$ or $D_2$ (in this case, $X_1 = O$ and $X_7 = O$). It follows that $\text{cov}(X+Y) = D_4 \cup D_3 = S_3$ or $\text{cov}(X+Y) = D_4 \cup D_2 = S_2$.

Next, suppose that $\text{cov}(X+Y)$ contains $t$ members in $D_1$, where $1 \leq t \leq r$. If $t < r$, then $\text{cov}(X+Y)$ contains at least $r-t$ members in $D_4$ because $I_r \sqsubseteq X_2$. Therefore $\text{cov}(X+Y)$ contains at most $k-r$ members in $D_2 \cup D_3$. Since $\begin{bmatrix} X_4 \\ X_7 \end{bmatrix}$ does not have a proper covering, $\text{cov}(X+Y)$ must contain all members in either $D_2$ or $D_3$. Thus $\text{cov}(X+Y)$ contains exactly $r-t$ members in $D_4$ so that $\begin{bmatrix} X_2 & X_3 \end{bmatrix}$ is dominated by $t$ members in $D_1$ and $r-t$ members in $D_4$, a contradiction to the fact that $\begin{bmatrix} X_2 & X_3 \end{bmatrix}$ does not have a proper covering. Thus $t = r$. Since $\begin{bmatrix} X_4 \\ X_7 \end{bmatrix}$ does not have a proper covering, it follows that $\text{cov}(X+Y) = D_1 \cup D_3 = S_4$ or $\text{cov}(X+Y) = D_1 \cup D_2 = S_1$ (in this case, $X_7 = O$). ∎

## 3. Term-rank preservers

A matrix $M$ in $\mathbb{M}_{m,n}$ is called a *monomial* if it has exactly $m$ 1's with no two of the 1's on a line. That is, there is an $n\times n$ permutation matrix $P$ such that $MP=[I_m \mid O_{m,n-m}]$. If $N\sqsubseteq M$ and $M$ is a monomial, we call $N$ a *submonomial*.

**Lemma 3.1.** [4] *If $X$ is an nonzero matrix in $\mathbb{M}_{m,n}$, there is a submonomial $N(\sqsubseteq X)$ such that $t(X)=t(N)$.*

**Lemma 3.2.** *If $X$ and $Y$ are matrices in $\mathbb{M}_{m,n}$, then $t(X+Y)\le t(X)+t(Y)$.*

*Proof.* Obvious. ∎

Notice that an invertible linear operator need not preserve term-rank. For example, define the linear operator $T:\mathbb{M}_2\to\mathbb{M}_2$ by $T\left(\begin{bmatrix} x & y\\ z & w\end{bmatrix}\right)=\begin{bmatrix} x & y\\ w & z\end{bmatrix}$. Then $T$ is an invertible linear operator that does not preserve term-rank.

**Example 3.1.** Consider the operator $T:\mathbb{M}_{m,n}\to\mathbb{M}_{m,n}$ defined by

$$T(X)=\left(\sum_{i=1}^{m}\sum_{j=1}^{n}x_{i,j}\right)\begin{bmatrix} I_k & O\\ O & O\end{bmatrix}\quad \text{for all } X\in\mathbb{M}_{m,n}.$$

Then $T$ preserves term rank $k$, while it does not preserve term rank.

The number of nonzero entries of a matrix $X$ in $\mathbb{M}_{m,n}$ is denoted by $\sharp(X)$.

**Proposition 3.1.** [3] *Let $T$ be a linear operator on $\mathbb{M}_{m,n}$. For a nonzero $X\in\mathbb{M}_{m,n}$, suppose that $N\sqsubseteq T(X)$, where $N$ is a submonomial of term rank $k(<m)$. If $t(X)>k$, there is a matrix $Y(\sqsubseteq X)$ such that $N\sqsubseteq T(Y)$ and $\sharp(Y)\le k$.*

**Lemma 3.3.** *If $T$ is a linear operator on $\mathbb{M}_{m,n}$ preserving term ranks $k$ and $k+1$, where $2\le k\le m-1$, then $t(T(X))=k$ or $k-1$ for all $X$ in $\mathbb{M}_{m,n}$ with $t(X)=k-1$.*

*Proof.* Let $X$ be a matrix in $\mathbb{M}_{m,n}$ with $t(X)=k-1$. If $t(T(X))\ge k+1$, take a cell $E$ such that $t(X+E)=k$. But then $k=t(T(X+E))\ge t(T(X))\ge k+1$, impossible. So $t(T(X))\le k$. Assume that $t(T(X))\le k-2$. By Lemma 3.1, there is a submonomial $N_1(\sqsubseteq X)$ such that $t(N_1)=k-1$. Furthermore $t(T(N_1))\le k-2$ because $T(N_1)\sqsubseteq T(X)$.

Take a submonomial $N_2$ with $t(N_2)=2$ such that $N_1+N_2$ is a submonomial with $t(N_1+N_2)=k+1$. By hypothesis, $t(T(N_1+N_2))=k+1$. Thus $t(T(N_2))\ge(k+1)-t(T(N_1))\ge 3$ by Lemma 3.2. Since $t(T(N_1+N_2))=k+1$, there is a submonomial $G(\sqsubseteq T(N_1+N_2))$ such that $t(G)=k+1$ by Lemma 3.1. Write $G=G_1+G_2$ for some two submonomials $G_1$ and $G_2$, where $G_1\sqsubseteq T(N_1)$ and $G_2\sqsubseteq T(N_2)$. Hence $t(G_1)\le t(T(N_1))\le k-2$. By Proposition 3.1, there is a matrix $Y(\sqsubseteq N_1)$ such that $G_1\sqsubseteq T(Y)$ and $\sharp(Y)\le k-2$. By Lemma 3.2, $t(Y+N_2)\le t(Y)+t(N_2)\le k$. Also

$$G=G_1+G_2\sqsubseteq T(Y)+T(N_2)=T(Y+N_2)$$

and hence $t(T(Y+N_2))\ge k+1$. Since $t(Y+N_2)\le k$, we can choose a matrix $Z$ such that $t(Y+N_2+Z)=k$. But then $k=t(T(Y+N_2+Z))\ge t(T(Y+N_2))\ge k+1$, impossible. Therefore $t(T(X))=k$ or $k-1$ for all $X$ in $\mathbb{M}_{m,n}$ with $t(X)=k-1$. ∎

Let $X$ be a partitioned matrix in $\mathbb{M}_{m,n}$ as $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$, where $X_1 \in \mathbb{M}_{p,q}$. For a cell $E_{i,j}$, we say that $E_{i,j}$ is *in* $X_1$ if $x_{i,j} = 1$ with $1 \le i \le p$ and $1 \le j \le q$. Similarly we can define that $E_{i,j}$ is a cell *in* $X_l$ for all $l \in \{1,2,3,4\}$.

The matrix in $\mathbb{M}_{m,n}$ whose entries are all 1, is denoted by $J$. That is, $J = \sum_{i=1}^{m} \sum_{j=1}^{n} E_{i,j}$.

**Lemma 3.4.** *Suppose that $T$ is a linear operator on $\mathbb{M}_{m,n}$ preserving term ranks $k$ and $k+1$, where $2 \le k \le m-1$. Let $A$ be a matrix in $\mathbb{M}_{m,n}$ with $t(A) = k-1$. If $T(A)$ does not have a proper covering, then $t(T(A)) = k-1$.*

*Proof.* Clearly $t(T(A)) = k$ or $k-1$ by Lemma 3.3. Now if $t(T(A)) = k$, we will get a contradiction. And then $t(T(A)) = k-1$. Suppose that $t(T(A)) = k$. Since $t(A) = k-1$, there is a matrix $X$ that is a sum of $k-1$ distinct full line matrices such that $A \sqsubseteq X$. Obviously $t(X) = k-1$. It follows from Lemma 3.3 that $t(T(X)) = k$ because $t(T(A)) = k$ and $T(A) \sqsubseteq T(X)$. Since $T(A)$ does not have a proper covering, neither does $T(X)$.

If the covering of $T(X)$ was composed of $k$ full row matrices, by permuting rows and columns of $T(X)$, we can assume that $T(X) = \begin{bmatrix} X_1 & X_2 \\ O & O \end{bmatrix}$, where $X_1 \in \mathbb{M}_k$ with $I_k \sqsubseteq X_1$. For a cell $E_{i,j}$ with $i, j > k$, suppose that $E_{i,j} \sqsubseteq T(E)$ for some cell $E$. Then $E \not\sqsubseteq X$ and so $t(X+E) = k$, while $t(T(X+E)) \ge k+1$, a contradiction. Hence $T(J)$ is of the form $T(J) = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & O \end{bmatrix}$ with $X_1 \sqsubseteq Y_1$. Since $T$ preserves term rank $k+1$, we have $Y_2 \ne O$ and $Y_3 \ne O$. So there are two cells $F_2$ and $F_3$ in $Y_2$ and $Y_3$, respectively such that $F_2 \sqsubseteq T(E_{a,b})$ and $F_3 \sqsubseteq T(E_{c,d})$ for some cells $E_{a,b}$ and $E_{c,d} (\not\sqsubseteq X)$.

If $X_2 \ne O$, it follows from Proposition 2.1 and $F_3 \sqsubseteq T(E_{c,d})$ that $t(T(X+E_{c,d})) = k+1$, while $t(X+E_{c,d}) = k$, a contradiction. So $X_2 = O$ and hence $T(X) = \begin{bmatrix} X_1 & O \\ O & O \end{bmatrix}$ and $E_{a,b} \not\sqsubseteq X$. If $a = c$ or $b = d$, then $t(X+E_{a,b}+E_{c,d}) = k$, while $t(T(X+E_{a,b}+E_{c,d})) \ge k+1$ by Proposition 2.1, a contradiction. Hence $a \ne c$ and $b \ne d$.

If $T(E_{a,d})$ dominates a cell in $Y_2$, then $t(T(X+E_{a,d}+E_{c,d})) = k+1$ by Proposition 2.1, while $t(X+E_{a,d}+E_{c,d}) = k$, a contradiction. Hence $T(E_{a,d})$ cannot dominate a cell in $Y_2$. Similarly it cannot dominate a cell in $Y_3$. Hence $T(E_{a,d})$ only dominates a cell in $Y_1$. A parallel argument shows that $T(E_{c,b})$ only dominates a cell in $Y_1$. It follows that $t(T(X+E_{a,d}+E_{c,b})) = k$, while $t(X+E_{a,d}+E_{c,b}) = k+1$, a contradiction. Similarly if the covering of $T(X)$ was composed of $k$ full column matrices, we get a contradiction. ∎

**Proposition 3.2.** *Suppose that $T$ is a linear operator on $\mathbb{M}_{m,n}$ preserving term ranks $k$ and $k+1$, where $2 \le k \le m-1$, and let $X \in \mathbb{M}_{m,n}$ be a sum of $k-1$ distinct full line matrices. For cells $E_{a_1,b_1}, \ldots, E_{a_s,b_s}$ that are not dominated by $X$, we have that* $\mathrm{cov}(T(X+R_{a_i})) \ne \mathrm{cov}(T(X+C_{b_j}))$ *for all $i, j \in \{1, \ldots, s\}$.*

*Proof.* Clearly $t(T(X+R_{a_i})) = t(T(X+C_{b_j})) = k$ for all $i, j \in \{1, \ldots, s\}$ because $t(X+R_{a_i}) = t(X+C_{b_j}) = k$. If $\mathrm{cov}(T(X+R_{a_i})) = \mathrm{cov}(T(X+C_{b_j}))$ for some $i, j \in \{1, \ldots, s\}$, then $t(T(X+R_{a_i}+C_{b_j})) = k$, while $t(X+R_{a_i}+C_{b_j}) = k+1$, a contradiction. Hence the result follows. ∎

**Lemma 3.5.** *Suppose that $T$ is a linear operator on $\mathbb{M}_{m,n}$ preserving term ranks $k$ and $k+1$, where $2 \le k \le m-2$. Let $A$ be a matrix in $\mathbb{M}_{m,n}$ with $t(A) = k-1$. If $T(A)$ has a pure proper covering, then $t(T(A)) = k-1$.*

*Proof.* Clearly $t(T(A)) = k$ or $k-1$ by Lemma 3.3. Suppose that $t(T(A)) = k$. Since $t(A) = k-1$, there is a matrix $X$ that is a sum of $k-1$ distinct full line matrices such that $A \sqsubseteq X$. Obviously $t(X) = k-1$. It follows from Lemma 3.3 that $t(T(X)) = k$ because $t(T(A)) = k$ and $T(A) \sqsubseteq T(X)$. Since $T(A)$ has a pure proper covering, it follows that $T(X)$ does not have a proper covering or has a pure proper covering. But if $T(X)$ does not have a proper covering, then $t(T(X)) = k-1$ by Lemma 3.4, a contradiction. Thus $T(X)$ has a pure proper covering. It follows from Lemma 2.2 that by permuting rows and columns of $T(X)$, we can assume that

$$T(X) = \begin{bmatrix} X_1 & X_2 & X_3 \\ X_4 & O & O \\ X_7 & O & O \end{bmatrix}, \quad I_r \sqsubseteq X_2 \quad \text{and} \quad I_{k-r} \sqsubseteq X_4,$$

where $1 \le r \le k-1$, $X_2 \in \mathbb{M}_r$ and $X_4 \in \mathbb{M}_{k-r}$; Furthermore both $\begin{bmatrix} X_2 & X_3 \end{bmatrix}$ and $\begin{bmatrix} X_4 \\ X_7 \end{bmatrix}$ have no proper covering. Given a cell $E_{i,j}$ with $i, j > k$, if $E_{i,j} \sqsubseteq T(E)$ for some cell $E$, then $E \not\sqsubseteq X$ and so $t(X+E) = k$, while $t(T(X+E)) \ge k+1$, a contradiction. Thus $T(J)$ is of the form

$$T(J) = \begin{bmatrix} Y_1 & Y_2 & Y_3 \\ Y_4 & Y_5 & Y_6 \\ Y_7 & Y_8 & O \end{bmatrix},$$

where the sizes of partitions of $T(X)$ and $T(J)$ are equal. Let

$$S_1 = \{R_1, \ldots, R_k\}, \quad S_2 = \{R_{r+1}, \cdots, R_k, C_{k-r+1}, \cdots C_k\},$$
$$S_3 = \{C_1, \ldots, C_k\} \quad \text{and} \quad S_4 = \{R_1, \cdots, R_r, C_1, \cdots C_{k-r}\}.$$

**Case 1.** $X_3 \ne O$ or $X_7 \ne O$: Suppose that $X_3 \ne O$. If $Y_8 \ne O$, there is a cell $F$ in $Y_8$ such that $F \sqsubseteq T(E)$ for some cell $E (\not\sqsubseteq X)$. Then $t(X+E) = k$, while $t(T(X+E)) \ge k+1$ by Lemma 2.2, a contradiction. Hence $Y_8 = O$. It follows that $\begin{bmatrix} Y_5 & Y_6 \end{bmatrix} \ne O$ and $Y_7 \ne O$ because $T$ preserves term rank $k+1$. So there are two cells $F_1$ and $F_2$ in $\begin{bmatrix} Y_5 & Y_6 \end{bmatrix}$ and $Y_7$, respectively such that $F_1 \sqsubseteq T(E_{a,b})$ and $F_2 \sqsubseteq T(E_{c,d})$ for some cells $E_{a,b} (\not\sqsubseteq X)$ and $E_{c,d}$. By Lemma 2.2, $t(T(X + E_{a,b} + E_{c,d})) \ge k+1$. Hence $E_{c,d} \not\sqsubseteq X$ and $t(E_{a,b} + E_{c,d}) = 2$. So $a \ne c$ and $b \ne d$. If $T(E_{a,d})$ dominates a cell in $\begin{bmatrix} Y_5 & Y_6 \end{bmatrix}$, then $t(T(X + E_{a,d} + E_{c,d})) \ge k+1$ by Lemma 2.2, while $t(X + E_{a,d} + E_{c,d}) = k$, a contradiction. That is, $T(E_{a,d})$ cannot dominate a cell in $\begin{bmatrix} Y_5 & Y_6 \end{bmatrix}$. Similarly $T(E_{c,b})$ cannot dominate a cell in $\begin{bmatrix} Y_5 & Y_6 \end{bmatrix}$. It follows from $Y_8 = O$ that $S_4$ is a covering of $T(X + E_{a,d} + E_{c,b})$. Hence $t(T(X + E_{a,d} + E_{c,b})) = k$, while $t(X + E_{a,d} + E_{c,b}) = k+1$, a contradiction. By a parallel argument, we get a contradiction for the case of $X_7 \ne O$.

**Case 2.** $X_3 = O$ and $X_7 = O$: That is, $T(X) = \begin{bmatrix} X_1 & X_2 & O \\ X_4 & O & O \\ O & O & O \end{bmatrix}$. Now we will show that $Y_6 = O$ or $Y_7 = O$. If not, there are two cells $F_6$ and $F_7$ in $Y_6$ and $Y_7$, respectively such that $F_6 \sqsubseteq T(E_{e,f})$ and $F_7 \sqsubseteq T(E_{g,h})$ for some cells $E_{e,f} (\not\sqsubseteq X)$ and $E_{g,h} (\not\sqsubseteq X)$. By Lemma 2.2, we have

$$\operatorname{cov}(T(X + L_1)) \in \{S_1, S_2\} \quad \text{and} \quad \operatorname{cov}(T(X + L_2)) \in \{S_3, S_4\},$$

where $L_1 = R_e$ or $C_f$, and $L_2 = R_g$ or $C_h$. Notice that if $\mathrm{cov}(T(X+L_1)) = S_2$, then $X_1 = O$. By Proposition 3.2, we loss no generality in assuming that $\mathrm{cov}(T(X+R_e)) = S_1$, $\mathrm{cov}(T(X+R_g)) = S_3$ and

$$\mathrm{cov}(T(X+C_f)) = S_2 \quad \text{and} \quad \mathrm{cov}(T(X+C_h)) = S_4. \tag{3.1}$$

Since $t(X) = k-1$ and $k \le m-2$, we can choose a full row matrix $R_l$ that is not dominated by $X+R_e+R_g$. Clearly $\mathrm{cov}(T(X+R_l)) \in \{S_1,S_2,S_3,S_4\}$ because $t(X+R_l) = t(T(X+R_l)) = k$. But then $\mathrm{cov}(T(X+R_l)) \in \{S_1,S_3\}$ by (3.1) and Proposition 3.2. Say that $\mathrm{cov}(T(X+R_l)) = S_1$. But then $t(T(X+R_l+R_e)) = k$, while $t(X+R_l+R_e) = k+1$, a contradiction.

Consequently we have established $Y_6 = O$ or $Y_7 = O$. A parallel argument shows that $Y_3 = O$ or $Y_8 = O$. Suppose that $Y_6 = O$. Then $Y_3 \neq O$ because $T$ preserves term rank $k+1$. Hence $Y_8 = O$ and so $T(J) = \begin{bmatrix} Y_1 & Y_2 & Y_3 \\ Y_4 & Y_5 & O \\ Y_7 & O & O \end{bmatrix}$. Clearly $Y_5 \neq O$ and $Y_7 \neq O$. Hence there are three cells $F_3$, $F_5$ and $F_7$ in $Y_3$, $Y_5$ and $Y_7$, respectively such that

$$F_3 \sqsubseteq T(E_{a_3,b_3}), \quad F_5 \sqsubseteq T(E_{a_5,b_5}) \quad \text{and} \quad F_7 \sqsubseteq T(E_{a_7,b_7})$$

for some cells $E_{a_i,b_i}$ that are not dominated by $X$. By Lemma 2.2 and Proposition 3.2, we loss no generality in assuming that

$$\mathrm{cov}(T(X+R_{a_3})) = S_1 \quad \text{and} \quad \mathrm{cov}(T(X+C_{b_3})) = S_4. \tag{3.2}$$

Again by Lemma 2.2, Proposition 3.2 and (3.2), we have

$$\mathrm{cov}(T(X+R_{a_7})) = S_3 \quad \text{and} \quad \mathrm{cov}(T(X+C_{b_7})) = S_4. \tag{3.3}$$

If $b_3 \neq b_7$, then $t(T(X+C_{b_3}+C_{b_7})) = k$, while $t(X+C_{b_3}+C_{b_7}) = k+1$, a contradiction. Hence $b_3 = b_7$. But then $b_5 \neq b_3$ by Lemma 2.2. Thus $\mathrm{cov}(T(X+C_{b_5})) = S_2$ by (3.2), (3.3) and Proposition 3.2. Since $t(X) = k-1$ and $k \le m-2$, we can choose a full row matrix $R_t$ that is not dominated by $X+R_{a_3}+R_{a_7}$. It follows from (3.2), $\mathrm{cov}(T(X+C_{b_5})) = S_2$ and Proposition 3.2 that $\mathrm{cov}(T(X+R_t)) = S_1$ or $S_3$. Say that $\mathrm{cov}(T(X+R_t)) = S_1$. But then $t(T(X+R_{a_3}+R_t)) = k$, while $t(X+R_{a_3}+R_t) = k+1$, a contradiction. Similarly, we get a contradiction for the case of $Y_7 = O$. ∎

**Theorem 3.1.** *Let $T$ be a linear operator on $\mathbb{M}_{m,n}$, where $m \ge 3$. Then $T$ preserves term rank if and only if $T$ preserves term ranks 2 and 3.*

*Proof.* Suppose that $T$ preserves term ranks 2 and 3. Let $X$ be a matrix in $\mathbb{M}_{m,n}$ with $t(X) = 1$. Then $t(T(X)) = 1$ or 2 by Lemma 3.3. Hence $T(X)$ does not have a proper covering or has a pure proper covering. It follows from Lemmas 3.4 and 3.5 that $t(T(X)) = 1$. Thus $T$ preserves term rank 1. Therefore $T$ preserves term rank by Theorem 1.1. The converse is obvious. ∎

**Theorem 3.2.** *Let $T$ be a linear operator on $\mathbb{M}_{m,n}$. Then $T$ preserves term rank if and only if $T$ preserves term ranks $m-1$ and $m$.*

*Proof.* Assume that $T$ preserves term ranks $m-1$ and $m$. Let $X$ be a matrix in $\mathbb{M}_{m,n}$ with $t(X) = m$. Then $t(T(X)) = m$ by hypothesis. Oppositely, let $t(T(X)) = m$ for some $X$. If $t(X) < m$, take a matrix $Y$ such that $t(X+Y) = m-1$. But then $m-1 = t(T(X+Y)) \ge t(T(X)) = m$, impossible. It follows that $t(T(X)) = m$ if and only if $t(X) = m$ for all $X$. Thus $T$ strongly preserves term rank $m$. Therefore $T$ preserves term rank by Theorem 1.2. The converse is obvious. ∎

Let $\mathbb{P}_{m,n}$ be the set of all matrix $X$ in $\mathbb{M}_{m,n}$ which either do not have a proper covering or have a pure proper covering. Clearly, $X \in \mathbb{P}_{m,n}$ for all $X \in \mathbb{M}_{m,n}$ with $t(X) \leq 2$. But if $X \in \mathbb{M}_{m,n}$ and $t(X) \geq 3$, then $X \in \mathbb{P}_{m,n}$ may be true or false. For examples, consider two matrices $X = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \oplus O_{m-3,n-3}$ and $Y = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \oplus O_{m-3,n-3}$, where $A \oplus B$ denotes the direct sum of matrices $A$ and $B$. Then $t(X) = t(Y) = 3$ and $X \in \mathbb{P}_{m,n}$, while $Y \notin \mathbb{P}_{m,n}$.

**Lemma 3.6.** *Suppose that $T$ is a linear operator on $\mathbb{M}_{m,n}$. If $T$ preserves $\mathbb{P}_{m,n}$ and term ranks $k$ and $k+1$, where $2 \leq k \leq m-1$, then $T$ preserves term rank $k-1$.*

*Proof.* If $k = m-1$, then $T$ preserves term rank $m-2$ by Theorem 3.2. Suppose that $k \leq m-2$. Let $A$ be a matrix in $\mathbb{M}_{m,n}$ with $t(A) = k-1$. Then $A \sqsubseteq X$ for some $X$ in $\mathbb{M}_{m,n}$, where $X$ is a sum of $k-1$ distinct full line matrices. Clearly $X \in \mathbb{P}_{m,n}$ and hence $T(X) \in \mathbb{P}_{m,n}$ by hypothesis. Thus $T(X)$ does not have a proper covering or has a pure proper covering. Hence $t(T(X)) = k-1$ by Lemmas 3.4 and 3.5. Since $t(T(A)) \leq t(T(X))$, it follows from Lemma 3.3 that $t(T(A)) = k-1$. Therefore $T$ preserves term rank $k-1$. ∎

**Theorem 3.3.** *Let $T$ be a linear operator on $\mathbb{M}_{m,n}$. Then $T$ preserves term rank if and only if $T$ preserves $\mathbb{P}_{m,n}$ and term ranks $k$ and $k+1$, where $1 \leq k \leq m-1$.*

*Proof.* Assume that $T$ preserves $\mathbb{P}_{m,n}$ and term ranks $k$ and $k+1$, where $1 \leq k \leq m-1$. If $k = 1$, then $T$ preserves term rank by Theorem 1.1. For the case of $k \geq 2$, $T$ preserves term ranks 1 and 2 by applying Lemma 3.6 $k$ times. Thus $T$ preserves term rank by Theorem 1.1. The converse is obvious. ∎

The following is an immediate consequence of Theorem 3.3.

**Corollary 3.1.** *Let $T$ be a linear operator on $\mathbb{P}_{m,n}$. Then $T$ preserves term rank if and only if $T$ preserves term ranks $k$ and $k+1$, where $1 \leq k \leq m-1$.*

**Theorem 3.4.** *Let $T$ be a linear operator on $\mathbb{M}_{m,n}$. Then $T$ preserves term rank if and only if $T$ strongly preserves term rank $m-1$.*

*Proof.* Assume that $T$ strongly preserves term rank $m-1$. First suppose that $X$ is a matrix in $\mathbb{M}_{m,n}$ with $t(X) = m$. Take a matrix $Y$ such that $Y \sqsubseteq X$ and $t(Y) = m-1$. Then $t(T(X)) \geq t(T(Y)) = t(Y) = m-1$. Hence $t(T(X)) = m$ by hypothesis. Next, suppose that $t(T(X)) = m$ for some $X$ in $\mathbb{M}_{m,n}$. By hypothesis, $t(X) = m$ or $t(X) \leq m-2$. If $t(X) \leq m-2$, take a matrix $Z$ such that $t(X+Z) = m-1$. But then $m = t(T(X)) \leq t(T(X+Z)) = m-1$, which is impossible. Hence $t(X) = m$. Therefore $T$ strongly preserves term rank $m$. Hence $T$ preserves term rank by Theorem 1.2.

The converse is obvious. ∎

**Theorem 3.5.** *Let $T$ be a linear operator on $\mathbb{M}_{m,n}$. Then $T$ preserves term rank if and only if $T$ preserves $\mathbb{P}_{m,n}$ and strongly preserves term rank $k$, where $1 \leq k \leq m$.*

*Proof.* Suppose that $T$ preserves $\mathbb{P}_{m,n}$ and strongly preserves term rank $k$, where $1 \leq k \leq m$. If $k = 1$ or $m$, then $T$ preserves term rank by Theorem 1.2. Assume that $2 \leq k \leq m-1$. Let $X$ be a matrix in $\mathbb{M}_{m,n}$ with $t(X) = k-1$. By the same pattern of the proof in Lemma 3.3, we have $t(T(X)) = k-1$ or $k$. But then $t(T(X)) = k-1$ by hypothesis. Therefore $T$ preserves term rank $k-1$. Thus $T$ preserves term rank by Theorem 3.3. The converse is obvious. ∎

The following is an immediate consequence of Theorem 3.5.

**Corollary 3.2.** *Let $T$ be a linear operator on $\mathbb{P}_{m,n}$. Then $T$ preserves term rank if and only if $T$ strongly preserves term rank $k$, where $1 \leq k \leq m$.*

**Acknowledgement.** This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No.2011-0006095). The second author, Seok-Zun Song, is the corresponding author.

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