BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **35**(2) (2012), 287–301

Inversion of the Dunkl-Hermite Semigroup

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Abstract. Let $\{e^{-c\mathscr{H}^\alpha}/\Re c \geq 0\}$ be the Dunkl-Hermite semigroup on the real line $\mathbb{R}$, defined by

$$[e^{-c\mathscr{H}^\alpha} f](x) = \int_{\mathbb{R}} \mathscr{K}_c^\alpha(x,\xi) f(\xi) d\mu_\alpha(\xi)\ ,\ \ x \in \mathbb{R}\ ,$$

where $\mathscr{K}_c^\alpha(x,\xi) = \sum_{n=0}^\infty e^{-cn} H_n^\alpha(x) H_n^\alpha(\xi)$. Here, $H_n^\alpha, n \in \mathbb{N}$, are the Dunkl-Hermite polynomials which are the eigenfunctions of the operator $D_\alpha^2 - 2xd/dx$, D_α being the Dunkl operator on the real line. For $\Re c > 0$, we give a representation for inverting the semigroup. Next, we extend $e^{-c\mathscr{H}^\alpha}$ and we give an integral representation of it for $\Re c < 0$. Moreover, in this last case, we characterize the domain in which $e^{-c\mathscr{H}^\alpha}$ is well defined.

2010 Mathematics Subject Classification: 47D03, 47B38, 46E20, 33C45

Keywords and phrases: Dunkl operator, Dunkl-Hermite polynomials, Dunkl-Hermite semigroup.

1. Introduction

The study of Dunkl operators has known a considerable growth during the last two decades due to their relevance in various fields of mathematics and in physical applications. Also they give the way to build a parallel to the theory of harmonic analysis based on finite root systems and depending on a set of real parameters.
In this work we are interesting in the rank one case. Let $\alpha \geq -1/2$, the Dunkl operator D_α acting on smooth functions f on $\mathbb{R}$, is defined by

$$D_\alpha f(x) = f'(x) + \frac{2\alpha+1}{x}\Big[\frac{f(x)-f(-x)}{2}\Big].$$

This operator is associated with the Dunkl-Hermite operator

$$D_\alpha^2 - 2x\frac{d}{dx}.$$

Its spectral decomposition is given by the Dunkl-Hermite polynomials H_n^α, namely we have (see [9])

$$(D_\alpha^2 - 2xd/dx)H_n^\alpha = -2nH_n^\alpha.$$

Communicated by Saburou Saitoh.
Received: October 16, 2009; *Revised:* May 18, 2010.

These polynomials are given by

$$H_n^\alpha(x) = 2^{-\frac{n}{2}}\sqrt{b_n(\alpha)}\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k}{k!b_{n-2k}(\alpha)}(2x)^{n-2k},$$

where $b_n(\alpha)$ is the generalized factorial defined by Rosenblum in [8],

$$b_n(\alpha) = \frac{2^n([\frac{n}{2}])!}{\Gamma(\alpha+1)}\Gamma\left(\left[\frac{n+1}{2}\right]+\alpha+1\right),$$

where $\left[n/2\right]$ is the integral part of $n/2$.We point out that these polynomials are expressed in terms of Laguerre polynomials [8]. More precisely we have

$$H_n^\alpha(x) = (-1)^{[\frac{n}{2}]}\frac{2^{\frac{n}{2}}([\frac{n}{2}])!}{\sqrt{b_n(\alpha)}}x^{\theta_n}L_{[\frac{n}{2}]}^{\alpha+\theta_n}(x^2),$$

where θ_n is defined to be 0 if n is even and 1 if n is odd. It is well known that the system $\{H_n^\alpha\}_{n\geq 0}$ is complete and orthonormal in $L^2(\mu_\alpha)$, where μ_α is the Gaussian-Dunkl measure defined on $\mathbb{R}$, by

$$d\mu_\alpha(x) = \frac{1}{\Gamma(\alpha+1)}|x|^{2\alpha+1}e^{-x^2}dx.$$

The system $\{H_n^\alpha\}_{n\geq 0}$ generates a semigroup of linear operators denoted by $e^{-c\mathscr{H}^\alpha}$, $c \geq 0$, on $L^2(\mu_\alpha)$ and is defined by

$$[e^{-c\mathscr{H}^\alpha}f](x) = \sum_{n=0}^{\infty} a_n^\alpha e^{-cn}H_n^\alpha(x),\ x \in \mathbb{R},$$

when f is expanded in $L^2(\mu_\alpha)$ as $f = \sum_{n=0}^{\infty} a_n^\alpha H_n^\alpha$, $a_n^\alpha = \int_{\mathbb{R}} f(x)H_n^\alpha(x)d\mu_\alpha(x)$. Obviously we can extend the operator $e^{-c\mathscr{H}^\alpha}$ for every complex number c with $\Re c > 0$ or $c = 0$.

In this paper, we prove that for $\Re c > 0$, the operator $e^{-c\mathscr{H}^\alpha}$ possesses the following integral representation

$$[e^{-c\mathscr{H}^\alpha}f](x) = \int_{\mathbb{R}} \mathscr{K}_c^\alpha(x,\xi)f(\xi)d\mu_\alpha(\xi),$$

where

$$\mathscr{K}_c^\alpha(x,\xi) = \sum_{n=0}^{\infty} e^{-cn}H_n^\alpha(x)H_n^\alpha(\xi).$$

Next, for $c \in \mathbb{C}$ with $\Re c > 0$, we characterize the range of $L^2(\mu_\alpha)$ under the operator $e^{-c\mathscr{H}^\alpha}$, as Fock type space, furthermore, we give two representations for inverting the operator $e^{-c\mathscr{H}^\alpha}$ in terms of integrals. These inverse transforms inspire us an extension of $e^{-c\mathscr{H}^\alpha}$ for $\Re c < 0$. Also, we establish a characterization of the elements in $\mathscr{D}(e^{-c\mathscr{H}^\alpha})$, the domain of $e^{-c\mathscr{H}^\alpha}$. Finally, when a $\mathscr{C}^\infty$-function $f \in L^2(\mu_\alpha)$ is an element in $\mathscr{D}(e^{-c\mathscr{H}^\alpha})$, with $\Re c < 0$, we give a representation of its analytic extension $\widehat{f}$ in terms of f.

We conclude this introduction by giving the organization of this paper. In the next section, we recall some notations and results related to the Dunkl operator on the real line. The third section deals with the inverse of $e^{-c\mathscr{H}^\alpha}$ for $\Re c > 0$. The last section is devoted to the study of the extension $e^{-c\mathscr{H}^\alpha}$ for $\Re c < 0$.

2. Preliminaries

In this section, we recall some notations and results related to the Dunkl operator on the real line, given by

$$D_\alpha f(x) = f'(x) + \frac{2\alpha+1}{x}\left[\frac{f(x)-f(-x)}{2}\right], \; \alpha \geq -\frac{1}{2}.$$

For every $\xi \in \mathbb{C}$, the equation

$$D_\alpha f(x) = \xi f(x), \;\; f(0) = 1,$$

has a unique analytic solution $E_\alpha(\xi x)$, called Dunkl Kernel (see [5]), defined by

$$E_\alpha(\xi x) = j_\alpha(\xi x) + \frac{\xi x}{2(\alpha+1)} j_{\alpha+1}(\xi x)$$

where j_β is the modified spherical Bessel function of order β given, for $\beta \geq -1/2$, by

$$j_\beta(z) = \Gamma(\beta+1)\sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\beta+1)}(z/2)^{2n} \quad .$$

We define $E_\alpha^e(z) = j_\alpha(z)$ the even part of $E_\alpha(z)$ and

$$E_\alpha^o(z) = \frac{z}{2(\alpha+1)} j_{\alpha+1}(z)$$

the odd part of $E_\alpha(z)$.
We note that

$$E_\alpha(z) = V_\alpha(\exp)(z),$$

where V_α is the intertwining operator between D_α and the usual derivative d/dx given by

$$V_\alpha(f)(x) = \frac{2^{-2\alpha-1}\Gamma(2\alpha+2)}{\Gamma(\alpha+\frac{1}{2})\Gamma(\alpha+\frac{3}{2})}\int_{-1}^{1} f(xt)(1-t^2)^{\alpha-\frac{1}{2}}(1+t)dt,$$

this operator is an isomorphism on the space of polynomials.
We remark that

$$H_n^\alpha = \sqrt{\frac{b_n(\alpha)}{n!}}V_\alpha(H_n),$$

where $\{H_n\}_{n\geq 0}$ is the set of classical normalized Hermite polynomials in $L^2(\frac{1}{\sqrt{\pi}}e^{-x^2}dx)$.
We recall the following formulas given in [8]. The formula in (2.1) is the Mehler formula

$$\sum_{n=0}^{\infty} H_n^\alpha(x)H_n^\alpha(y)z^n = \left(\frac{1}{1-z^2}\right)^{\alpha+1}\exp\left(-\frac{(x^2+y^2)z^2}{1-z^2}\right)E_\alpha\left(\frac{(2xy)z}{1-z^2}\right), \;\; |z|<1. \tag{2.1}$$

$$\int_{\mathbb{R}} E_\alpha(xt)E_\alpha(yt)\exp(-\lambda t^2)|t|^{2\alpha+1}dt = \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}}\exp\left(\frac{x^2+y^2}{4\lambda}\right)E_\alpha\left(\frac{xy}{2\lambda}\right), \; x,y\in\mathbb{C} \text{ and } \Re\lambda>0. \tag{2.2}$$

We denote by m_α the measure defined on $\mathbb{C}$ by

$$dm_\alpha(z) = \frac{1}{\pi 2^\alpha \Gamma(\alpha+1)}|z|^{2\alpha+2}K_\alpha(|z|^2)dxdy \;\; , \; z = x+iy,$$

where K_α is the Macdonald function, (see [6]), defined by

$$K_\alpha(z)=\frac{\pi}{2}\frac{I_{-\alpha}(z)-I_\alpha(z)}{\sin(\alpha\pi)}\ ,\ \alpha\in\mathbb{C}\backslash\mathbb{Z}\ ,\ |\arg(z)|<\pi,$$

for integer n,

$$K_n(z)=\lim_{\alpha\to n}K_\alpha(z)$$

and

$$I_\alpha(z)=\frac{1}{\Gamma(\alpha+1)}\left(\frac{z}{2}\right)^\alpha j_\alpha(z).$$

Since $K_\alpha(|z|^2)$ is positive, the measure m_α is nonnegative.
For $\alpha>-\frac{1}{2}$, we note

- $\mathscr{F}_{\alpha,e}$ is the Hilbert space of even entire functions on $\mathbb{C}$, with the inner product defined by $\langle f,g\rangle_{\alpha,e}=\int_{\mathbb{C}}f(z)\overline{g(z)}dm_\alpha(z)$.
- $\mathscr{F}_{\alpha,o}$ is the Hilbert space of odd entire functions on $\mathbb{C}$, with the inner product defined by $\langle f,g\rangle_{\alpha,o}=2(\alpha+1)\int_{\mathbb{C}}f(z)\overline{g(z)}|z|^{-2}dm_{\alpha+1}(z)$.

Meanwhile, referring to Sifi and Soltani [13] defined, for $\alpha>-1/2$, the generalized Fock space $\mathscr{F}_\alpha$ as the direct sum of $\mathscr{F}_{\alpha,e}$ and $\mathscr{F}_{\alpha,o}$, admitting the inner product

$$\langle f,g\rangle_\alpha=\langle f_e,g_e\rangle_{\alpha,e}+\langle f_o,g_o\rangle_{\alpha,o}\ ,$$

where

$$f_e(z)=\frac{f(z)+f(-z)}{2}\quad\text{and}\quad f_o(z)=\frac{f(z)-f(-z)}{2}\ .$$

It is also given in [13] that the kernel $\mathscr{Z}_\alpha$ given for $z,\xi\in\mathbb{C}$ by

$$\mathscr{Z}_\alpha(\xi,z)=E_\alpha(\overline{\xi}z),$$

is a reproducing kernel for the generalized Fock space $\mathscr{F}_\alpha$, that is

(i) For every $\xi\in\mathbb{C}$, the function $z\longrightarrow\mathscr{Z}_\alpha(\xi,z)$ belongs to $\mathscr{F}_\alpha$.

(ii) The reproducing property : For every $\xi\in\mathbb{C}$ and $f\in\mathscr{F}_\alpha$, we have

$$\langle f,\mathscr{Z}_\alpha(\xi,.)\rangle_\alpha=f(\xi).$$

3. The Dunkl-Hermite semigroup for $\Re c>0$

Let $f\in L^2(\mu_\alpha)$, $f=\sum_{n=0}^\infty a_n^\alpha H_n^\alpha$, for any complex number c with $\Re c>0$, we define the semigroup $e^{-c\mathscr{H}^\alpha}$ of operators by

$$\begin{aligned}[e^{-c\mathscr{H}^\alpha}f](x)&=\sum_{n=0}^\infty e^{-cn}a_n^\alpha H_n^\alpha(x)\ ,\ x\in\mathbb{R}\\ &=\sum_{n=0}^\infty e^{-cn}H_n^\alpha(x)\int_{\mathbb{R}}f(y)H_n^\alpha(y)d\mu_\alpha(y)\ .\end{aligned}$$

Let $r>0$, using the estimates for the classical Hermite polynomials, see [7], the intertwining operator V_α and the Stirling formula, (see [3]), we deduce the following: there exists a positive constant $C(r)$ depending on r such that

$$|H_n^\alpha(x)|\le C(r)n^{\frac{\alpha}{2}+\frac{1}{6}}\ ,\ \ x\in[-r,r].$$

Since

$$\sum_{n=0}^{\infty} e^{-\Re cn} \int_{\mathbb{R}} |H_n^{\alpha}(x) H_n^{\alpha}(y) f(y)| d\mu_{\alpha}(y)$$
$$\leq \sum_{n=0}^{\infty} e^{-\Re cn} C(r) n^{\frac{\alpha}{2}+\frac{1}{6}} \|H_n^{\alpha}\|_{\alpha,2} \|f\|_{\alpha,2} < +\infty ,$$

we can interchange the order of summation and integration and so

$$[e^{-c\mathscr{H}^{\alpha}} f](x) = \int_{\mathbb{R}} \left(\sum_{n=0}^{\infty} e^{-cn} H_n^{\alpha}(x) H_n^{\alpha}(y) \right) f(y) d\mu_{\alpha}(y) \ , \ x \in [-r,r].$$

As r is arbitrary, this relation holds for all $x \in \mathbb{R}$. We put

$$\mathscr{K}_c^{\alpha}(x,y) = \sum_{n=0}^{\infty} e^{-cn} H_n^{\alpha}(x) H_n^{\alpha}(y).$$

Hence, we can write

$$[e^{-c\mathscr{H}^{\alpha}} f](x) = \int_{\mathbb{R}} \mathscr{K}_c^{\alpha}(x,y) f(y) d\mu_{\alpha}(y) \ .$$

In the following we are interesting in inverting these operators, more precisely, we shall give two representations of the inverse of the operator $e^{-c\mathscr{H}^{\alpha}}$, $c \in \mathbb{C}$, $\Re c > 0$, in terms of integrals.

Notation 1. We denote by

$$\mathscr{N}^{\alpha}(z,\overline{\xi},c) = \langle \mathscr{K}_c^{\alpha}(z,.), \mathscr{K}_c^{\alpha}(\xi,.) \rangle_{2,\alpha}$$
$$= \int_{\mathbb{R}} \mathscr{K}_c^{\alpha}(z,y) \overline{\mathscr{K}_c^{\alpha}(\xi,y)} d\mu_{\alpha}(y).$$

Lemma 3.1. *$\mathscr{N}^{\alpha}(z,\overline{\xi},c)$ determines uniquely a reproducing kernel Hilbert space $\mathbf{H}_c^{\alpha}$ admitting the reproducing kernel $\mathscr{N}^{\alpha}(z,\overline{\xi},c)$ given by*

(3.1)
$$\mathscr{N}^{\alpha}(z,\overline{\xi},c) = \left(\frac{1}{1-|\omega|^4} \right)^{\alpha+1} \exp\left(\frac{-|\omega|^4}{1-|\omega|^4} z^2 \right) \exp\left(\frac{-|\omega|^4}{1-|\omega|^4} \overline{\xi}^2 \right) E_{\alpha}\left(\frac{2|\omega|^2}{1-|\omega|^4} z\overline{\xi} \right),$$

where $\omega = e^{-c}$.

Proof. From the Mehler formula (2.1), we have

$$\mathscr{K}_c^{\alpha}(x,\xi) = \sum_{n=0}^{\infty} H_n^{\alpha}(x) H_n^{\alpha}(\xi) e^{-nc}$$
$$= \left(\frac{1}{1-\omega^2} \right)^{\alpha+1} \exp\left(-\frac{\omega^2}{1-\omega^2}(x^2+\xi^2) \right) E_{\alpha}\left(2\frac{\omega}{1-\omega^2} x\xi \right).$$

For $(z,\xi) \in \mathbb{C} \times \mathbb{C}$, we have

$$\mathscr{N}^{\alpha}(z,\overline{\xi},c) = \int_{\mathbb{R}} \mathscr{K}_c^{\alpha}(z,y) \overline{\mathscr{K}_c^{\alpha}(\xi,y)} d\mu_{\alpha}(y)$$
$$= \frac{1}{\Gamma(\alpha+1)\left((1-\omega^2)(1-\overline{\omega}^2)\right)^{\alpha+1}}$$

$$\times \int_{\mathbb{R}} \exp\left(-\frac{\omega^2}{1-\omega^2}(z^2+y^2)\right) E_\alpha\left(2\frac{\omega}{1-\omega^2}zy\right)$$
$$\exp\left(-\frac{\overline{\omega}^2}{1-\overline{\omega}^2}(\overline{\xi}^2+y^2)\right) E_\alpha\left(2\frac{\overline{\omega}}{1-\overline{\omega}^2}\overline{\xi}y\right)|y|^{2\alpha+1}e^{-y^2}dy$$
$$= \frac{1}{\Gamma(\alpha+1)\left((1-\omega^2)(1-\overline{\omega}^2)\right)^{\alpha+1}}$$
$$\times \exp\left(-\frac{\omega^2}{1-\omega^2}z^2\right)\exp\left(-\frac{\overline{\omega}^2}{1-\overline{\omega}^2}\overline{\xi}^2\right) \times A_\alpha(z,\overline{\xi},c)$$

with

$$A_\alpha(z,\overline{\xi},c) = \int_{\mathbb{R}} \exp\left(-\frac{1-|\omega|^4}{(1-\omega^2)(1-\overline{\omega}^2)}y^2\right) E_\alpha\left(2\frac{\omega}{1-\omega^2}zy\right)$$
$$E_\alpha\left(2\frac{\overline{\omega}}{1-\overline{\omega}^2}\overline{\xi}y\right)|y|^{2\alpha+1}e^{-y^2}dy$$

and by formula (2.2) we have

$$A_\alpha(z,\overline{\xi},c) = \Gamma(\alpha+1)\left(\frac{(1-\omega^2)(1-\overline{\omega}^2)}{1-|\omega|^4}\right)^{\alpha+1}$$
$$\times \exp\left\{\frac{(1-\omega^2)(1-\overline{\omega}^2)}{4(1-|\omega|^4)}\left(\left(\frac{2\omega}{1-\omega^2}z\right)^2+\left(\frac{2\overline{\omega}}{1-\overline{\omega}^2}\overline{\xi}\right)^2\right)\right\}$$
$$\times E_\alpha\left(\frac{2|\omega|^2}{1-|\omega|^4}z\overline{\xi}\right).$$

So

$$\mathscr{N}^\alpha(z,\overline{\xi},c) = \left(\frac{1}{1-|\omega|^4}\right)^{\alpha+1}\exp\left\{\frac{-|\omega|^4}{1-|\omega|^4}z^2\right\}\exp\left\{\frac{-|\omega|^4}{1-|\omega|^4}\overline{\xi}^2\right\}E_\alpha\left\{\frac{2|\omega|^2}{1-|\omega|^4}z\overline{\xi}\right\}.$$

Since $\mathscr{N}^\alpha(z,\overline{\xi},c)$ is a positive matrix on $\mathbb{C}$ in the sense of Moore [1], i.e. for $q\in\mathbb{N}$ and finite sets $\{\xi_n\}_{1\le n\le q}$, $\{\gamma_n\}_{1\le n\le q}$ in $\mathbb{C}$

$$\sum_{n=1}^{q}\sum_{m=1}^{q}\gamma_n\overline{\gamma}_m\mathscr{N}^\alpha(\xi_n,\overline{\xi}_m,c)\ge 0\,,$$

using the results in [1, p. 344] we deduce that it uniquely determines the reproducing kernel Hilbert space $\mathbf{H}_c^\alpha$ admitting the reproducing kernel $\mathscr{N}^\alpha(z,\overline{\xi},c)$. ∎

Notation 2. We denote by

- $\|f\|_{2,\alpha}^2 = \int_{\mathbb{R}}|f|^2d\mu_\alpha$.
- $\|g\|_{c,\alpha}^2 = \dfrac{4|\omega|^{2\alpha+4}}{\pi\Gamma(\alpha+1)(1-|\omega|^4)}\times$
  $\int_{\mathbb{C}}\left\{K_\alpha\left(\frac{2|\omega|^2}{1-|\omega|^4}|z|^2\right)|g_e(z)|^2+K_{\alpha+1}\left(\frac{2|\omega|^2}{1-|\omega|^4}|z|^2\right)|g_o(z)|^2\right\}\exp\left\{\frac{2|\omega|^4}{1-|\omega|^4}(x^2-y^2)\right\}|z|^{2\alpha+2}dxdy,$

where $z = x + iy$, K_α is the Macdonald function.

Theorem 3.1. *For $c \in \mathbb{C}$ with $\Re c > 0$, the range of $L^2(\mu_\alpha)$ under the operator $e^{-c\mathscr{H}^\alpha}$ coincides with the Hilbert space $\mathbf{H}_c^\alpha$ consisting of entire functions with finite norms $\|.\|_{c,\alpha}$. Moreover, the isometrical identity*

$$\|e^{-c\mathscr{H}^\alpha} f\|_{c,\alpha}^2 = \|f\|_{2,\alpha}^2$$

holds.

Proof. For any fixed complex number c with $\Re c > 0$, applying the dominated convergence theorem and Morera's theorem to the integral representation

$$[e^{-c\mathscr{H}^\alpha} f](x) = \int_{\mathbb{R}} \mathscr{K}_c^\alpha(x,\xi) f(\xi) d\mu_\alpha(\xi),$$

we see that every element in the range of the operator $e^{-c\mathscr{H}^\alpha}$ can be analytically extended to the complex plane $\mathbb{C}$.

Hence we shall consider the operator $e^{-c\mathscr{H}^\alpha}$ as the linear operator of $L^2(\mu_\alpha)$ into an entire function space. Then, following the method of characterizing the ranges of integral transforms established by Saitoh in [11] and Lemma 3.1, the space $\mathbf{H}_c^\alpha$ is the range of $L^2(\mu_\alpha)$ under the operator $e^{-c\mathscr{H}^\alpha}$. The family $\{\mathscr{K}_c^\alpha(z,\xi), z \in \mathbb{C}\}$ being complete in $L^2(\mu_\alpha)$, hence we have the isometrical identity.

Thus, it is sufficient to prove that the elements in $\mathbf{H}_c^\alpha$ are characterized as entire functions with finite norms. Using the well known results of Aronszajn (see [1]), if $g \in \mathbf{H}_c^\alpha$, then g can be expressed in the form

$$g(z) = \left(\frac{1}{1-|\omega|^4}\right)^{\alpha+1} \exp\left\{\frac{-|\omega|^4}{1-|\omega|^4} z^2\right\} g_1(z), \tag{3.2}$$

where $z \in \mathbb{C}$ and g_1 is a member in the reproducing kernel Hilbert space $\widetilde{\mathscr{F}_\alpha}$ admitting the reproducing kernel $E_\alpha\left\{\frac{2|\omega|^2}{1-|\omega|^4} z\overline{\xi}\right\}$. Moreover, the following isometrical identity holds

$$\|g\|_{c,\alpha}^2 = \left(\frac{1}{1-|\omega|^4}\right)^{\alpha+1} \|g_1\|_{\widetilde{\mathscr{F}_\alpha}}^2 . \tag{3.3}$$

By change of variable, we have

$$\begin{aligned}\|g_1\|_{\widetilde{\mathscr{F}_\alpha}}^2 = & \frac{4}{\pi\Gamma(\alpha+1)} \left(\frac{|\omega|^2}{1-|\omega|^4}\right)^{\alpha+2} \times \int_{\mathbb{C}} \left\{K_\alpha\left(\frac{2|\omega|^2}{1-|\omega|^4}|z|^2\right) |g_{1,e}(z)|^2\right. \\ & \left. + K_{\alpha+1}\left(\frac{2|\omega|^2}{1-|\omega|^4}|z|^2\right) |g_{1,o}(z)|^2\right\} |z|^{2\alpha+2} dxdy,\end{aligned}$$

which completes the proof. ∎

Remark 3.1. For two complex numbers c_1 , c_2 with $\Re c_1 > 0$ and $\Re c_2 > 0$, we shall discuss a relation between $\mathbf{H}_{c_1}^\alpha$ and $\mathbf{H}_{c_2}^\alpha$. Put $g_1 = e^{-c_1\mathscr{H}^\alpha} f$ and $g_2 = e^{-c_2\mathscr{H}^\alpha} f$ for some $f \in L^2(\mu_\alpha)$. If $\Re c_1 > \Re c_2$, then

$$g_1 = e^{-c_1\mathscr{H}^\alpha} f = e^{-(c_1-c_2)\mathscr{H}^\alpha} e^{-c_2\mathscr{H}^\alpha} f = e^{-(c_1-c_2)\mathscr{H}^\alpha} g_2 ,$$

and so we can directly obtain a representation of g_1 in terms of g_2 by using the integral representation of $e^{-c\mathscr{H}^\alpha}$. However, if $\Re c_1 \leq \Re c_2$, it is not obvious to represent g_1 in terms of g_2. Hence we are interested in this case.

Theorem 3.2. *For $\Re c_1 > 0$, $\Re c_2 > 0$ and $f \in L^2(\mu_\alpha)$, let $g_1 = e^{-c_1\mathscr{H}^\alpha} f$ and $g_2 = e^{-c_2\mathscr{H}^\alpha} f$. Then g_1 is expressible in the form*

$$\begin{aligned} g_1(\xi) = {} & \frac{4|\omega_2|^{2\alpha+4}}{\pi\Gamma(\alpha+1)(1-|\omega_2|^4)}\left(\frac{1}{1-\omega_1^2\overline{\omega}_2^2}\right)^{\alpha+1} \exp\left(-\frac{\omega_1^2\overline{\omega}_2^2}{1-\omega_1^2\overline{\omega}_2^2}\xi^2\right) \\ & \times \int_{\mathbb{C}} \left(K_\alpha\left(\frac{2|\omega_2|^2}{1-|\omega_2|^4}|z|^2\right) g_{2,e}(z) E_\alpha^e\left(\frac{2\omega_1\overline{\omega}_2}{1-\omega_1^2\overline{\omega}_2^2}\overline{z}\xi\right)\right. \\ & \left. + K_{\alpha+1}\left(\frac{2|\omega_2|^2}{1-|\omega_2|^4}|z|^2\right) g_{2,o}(z) E_\alpha^o\left(\frac{2\omega_1\overline{\omega}_2}{1-\omega_1^2\overline{\omega}_2^2}\overline{z}\xi\right)\right) \\ & \times \exp\left(-\frac{\omega_1^2\overline{\omega}_2^2}{1-\omega_1^2\overline{\omega}_2^2}\overline{z}^2\right) \exp\left(\frac{2|\omega_2|^4}{1-|\omega_2|^4}(x^2-y^2)\right)|z|^{2\alpha+2} dx dy, \ \xi \in \mathbb{R}, \end{aligned} \tag{3.4}$$

where $\omega_1 = e^{-c_1}$ and $\omega_2 = e^{-c_2}$.

Proof. We assume that $T_{c_1}^\alpha$ is the inverse operator of $e^{-c_1\mathscr{H}^\alpha}$ from $\mathbf{H}_{c_1}^\alpha$ into $L^2(\mu_\alpha)$. In addition, let S_{c_1,c_2}^α be the linear operator of $\mathbf{H}_{c_1}^\alpha$ into $\mathbf{H}_{c_2}^\alpha$ defined by

$$S_{c_1,c_2}^\alpha g = e^{-c_2\mathscr{H}^\alpha} T_{c_1}^\alpha g \quad , \quad g \in \mathscr{H}_{c_1}^\alpha$$

then we have

$$S_{c_1,c_2}^\alpha g_1 = g_2 \quad .$$

It follows

$$\|S_{c_1,c_2}^\alpha g_1\|_{c_2,\alpha} = \|e^{-c_2\mathscr{H}^\alpha} f\|_{c_2,\alpha} = \|f\|_{2,\alpha} = \|e^{-c_1\mathscr{H}^\alpha} f\|_{c_1,\alpha} = \|g_1\|_{c_1,\alpha}.$$

Hence the operator S_{c_1,c_2}^α is an isometry from $\mathbf{H}_{c_1}^\alpha$ onto $\mathbf{H}_{c_2}^\alpha$, so the adjoint operator $S_{c_1,c_2}^{*,\alpha}$ of S_{c_1,c_2}^α is its inverse. Thus, for $\xi \in \mathbb{R}$ we get the representation

$$\begin{aligned} g_1(\xi) = \left[S_{c_1,c_2}^{*,\alpha} g_2\right](\xi) &= \langle S_{c_1,c_2}^{*,\alpha} g_2, \mathscr{N}^\alpha(.,\xi,c_1)\rangle_{c_1,\alpha} \\ &= \langle g_2, S_{c_1,c_2}^\alpha \mathscr{N}^\alpha(.,\xi,c_1)\rangle_{c_2,\alpha}. \end{aligned}$$

Meanwhile, for $z \in \mathbb{C}$, the following is valid

$$\left[S_{c_1,c_2}^\alpha \mathscr{N}^\alpha(.,\xi,c_1)\right](z) = \left[e^{-c_2\mathscr{H}^\alpha} T_{c_1}^\alpha \mathscr{N}^\alpha(.,\xi,c_1)\right](z)$$

now

$$\mathscr{N}^\alpha(z,\xi,c_1) = \left[e^{-c_1\mathscr{H}^\alpha}\overline{\mathscr{K}_{c_1}^\alpha(\xi,.)}\right](z),$$

so

$$\overline{\mathscr{K}_{c_1}^\alpha(\xi,z)} = \left[T_{c_1}^\alpha \mathscr{N}^\alpha(.,\xi,c_1)\right](z)$$

and

$$\begin{aligned} \left[S_{c_1,c_2}^\alpha \mathscr{N}^\alpha(.,\xi,c_1)\right](z) &= \left[e^{-c_2\mathscr{H}^\alpha}\overline{\mathscr{K}_{c_1}^\alpha(\xi,.)}\right](z) \\ &= \int_{\mathbb{R}} \mathscr{K}_{c_2}^\alpha(z,y)\overline{\mathscr{K}_{c_1}^\alpha(\xi,y)} d\mu_\alpha(y) \end{aligned}$$

$$= \frac{1}{\Gamma(\alpha+1)}\left(\frac{1}{(1-\omega_2^2)(1-\overline{\omega_1}^2)}\right)^{\alpha+1}$$
$$\times \exp\left(-\frac{\overline{\omega}_1^2}{1-\overline{\omega}_1^2}\xi^2\right)\exp\left(-\frac{\omega_2^2}{1-\omega_2^2}z^2\right)B_\alpha(z,\xi,c)$$

with

$$B_\alpha(z,\xi,c) = \int_{\mathbb{R}} \exp\left(-\frac{1-\overline{\omega}_1^2\omega_2^2}{(1-\overline{\omega}_1^2)(1-\omega_2^2)}y^2\right)$$
$$\times E_\alpha\left(2\frac{\omega_2}{1-\omega_2^2}zy\right)E_\alpha\left(2\frac{\overline{\omega}_1}{1-\overline{\omega}_1^2}\xi y\right)|y|^{2\alpha+1}dy$$

from the formula (2.2), we have

$$B_\alpha(z,\xi,c) = \Gamma(\alpha+1)\left(\frac{(1-\overline{\omega}_1^2)(1-\omega_2^2)}{1-\overline{\omega}_1^2\omega_2^2}\right)^{\alpha+1}\exp\left\{\frac{\omega_2^2(1-\overline{\omega}_1^2)}{(1-\omega_2^2)(1-\overline{\omega}_1^2\omega_2^2)}z^2\right\}$$
$$\times \exp\left(\frac{\overline{\omega}_1^2(1-\omega_2^2)}{(1-\overline{\omega}_1^2)(1-\overline{\omega}_1^2\omega_2^2)}\xi^2\right)E_\alpha\left\{\frac{2\overline{\omega}_1\omega_2}{1-\overline{\omega}_1^2\omega_2^2}z\xi\right\},$$

so

$$\left[S^\alpha_{c_1,c_2}\mathscr{N}^\alpha(.,\xi,c_1)\right](z) = \left(\frac{1}{1-\overline{\omega}_1^2\omega_2^2}\right)^{\alpha+1}\exp\left\{-\frac{\overline{\omega}_1^2\omega_2^2}{1-\overline{\omega}_1^2\omega_2^2}\xi^2\right\}$$
$$\times \exp\left\{-\frac{\overline{\omega}_1^2\omega_2^2}{1-\overline{\omega}_1^2\omega_2^2}z^2\right\}E_\alpha\left(\frac{2\overline{\omega}_1\omega_2}{1-\overline{\omega}_1^2\omega_2^2}z\xi\right).$$

Thus

$$g_1(\xi) = \langle g_2, S^\alpha_{c_1,c_2}\mathscr{N}^\alpha(.,\xi,c_1)\rangle_{c_2,\alpha}$$
$$= \frac{4|\omega_2|^{2\alpha+4}}{\pi\Gamma(\alpha+1)(1-|\omega_2|^4)}\int_{\mathbb{C}}\left(K_\alpha\left(\frac{2|\omega_2|^2}{1-|\omega_2|^4}|z|^2\right)g_{2,e}(z)\overline{\left(S^\alpha_{c_1,c_2}\mathscr{N}^\alpha\right)_e}(z)\right.$$
$$\left.+K_{\alpha+1}\left(\frac{2|\omega_2|^2}{1-|\omega_2|^4}|z|^2\right)g_{2,o}(z)\overline{\left(S^\alpha_{c_1,c_2}\mathscr{N}^\alpha\right)_o}(z)\right)$$
$$\times \exp\left(\frac{2|\omega_2|^4}{1-|\omega_2|^4}(x^2-y^2)\right)|z|^{2\alpha+2}dxdy$$

with $\left(S^\alpha_{c_1,c_2}\mathscr{N}^\alpha\right)_e$ and $\left(S^\alpha_{c_1,c_2}\mathscr{N}^\alpha\right)_o$ are respectively the even and the odd part of $S^\alpha_{c_1,c_2}$ $\mathscr{N}^\alpha(.,\xi,c_1)$. So we obtain the identity (3.4). ∎

From the definition of $e^{-c_2\mathscr{H}^\alpha}$ and the identity (3.4), we deduce the following.

Corollary 3.1. *Under the same assumptions in the previous theorem, f is given by*

$$f(\xi) = \lim_{r\to 1}\frac{4|\omega_2|^{2\alpha+4}}{\pi\Gamma(\alpha+1)(1-|\omega_2|^4)}\left(\frac{1}{1-r^2\overline{\omega}_2^2}\right)^{\alpha+1}\exp\left(-\frac{r^2\overline{\omega}_2^2}{1-r^2\overline{\omega}_2^2}\xi^2\right)$$

$$\times \int_{\mathbb{C}} \Big\{ K_\alpha\left(\frac{2|\omega_2|^2}{1-|\omega_2|^4}|z|^2\right) g_{2,e}(z) E_\alpha^e\left(\frac{2r\overline{\omega}_2}{1-r^2\overline{\omega}_2^2}\overline{z}\xi\right)$$

$$+K_{\alpha+1}\left(\frac{2|\omega_2|^2}{1-|\omega_2|^4}|z|^2\right) g_{2,o}(z) E_\alpha^o\left(\frac{2r\overline{\omega}_2}{1-r^2\overline{\omega}_2^2}\overline{z}\xi\right)\Big\}$$

$$\times \exp\left(-\frac{r^2\overline{\omega}_2^2}{1-r^2\overline{\omega}_2^2}\overline{z}^2\right) \exp\left\{\frac{2|\omega_2|^4}{1-|\omega_2|^4}(x^2-y^2)\right\} |z|^{2\alpha+2} dxdy \ , \ \xi \in \mathbb{R}, \tag{3.5}$$

the convergence holds in $L^2(\mu_\alpha)$.

Remark 3.2. For any $n \in \mathbb{N}$, $H_n^\alpha(z)$ denotes the analytic extension of $H_n^\alpha(x)$ to $\mathbb{C}$, then we see that, for $\Re c > 0$, the family $\left\{e^{-cn}H_n^\alpha(z)\right\}_{n\geq 0}$ is a complete orthonormal system in $\mathbf{H}_c^\alpha$ because $e^{-c\mathscr{H}^\alpha} : L^2(\mu_\alpha) \longrightarrow \mathbf{H}_c^\alpha$ is an isometric isomorphism. Hence, the expression

$$\mathscr{K}_c^\alpha(z,\xi) = \sum_{n=0}^{\infty} e^{-cn} H_n^\alpha(z) H_n^\alpha(\xi)$$

and Theorem 3.1 suggest the representation of T_c^α, for fixed ξ , in the form

$$[T_c^\alpha g](\xi) = \langle g, \mathscr{K}_c^\alpha(.,\xi)\rangle_{c,\alpha}, \quad g \in \mathbf{H}_c^\alpha.$$

Indeed

$$\begin{aligned} T_c^\alpha g(\xi) &= \sum_{n=0}^{\infty} \langle T_c^\alpha g, H_n^\alpha\rangle_{2,\alpha} H_n^\alpha(\xi) \\ &= \sum_{n=0}^{\infty} \langle g, e^{-cH^\alpha} H_n^\alpha\rangle_{c,\alpha} H_n^\alpha(\xi) \\ &= \sum_{n=0}^{\infty} \langle g, e^{-cn} H_n^\alpha\rangle_{c,\alpha} H_n^\alpha(\xi) \\ &= \sum_{n=0}^{\infty} \langle g, H_n^\alpha\rangle_{c,\alpha} e^{-cn} H_n^\alpha(\xi) \\ &= \langle g, \mathscr{K}_c^\alpha(.,\xi)\rangle_{c,\alpha}. \end{aligned}$$

However, the integral in the right-hand side need not converge. Following the method given in [2, p. 202] (also one can see [10] and [12]), we can obtain another representation of (3.5)

$$\begin{aligned} f(\xi) = \lim_{\sigma\to\infty} & \frac{4|\omega_2|^{2\alpha+4}}{\pi\Gamma(\alpha+1)(1-|\omega_2|^4)} \\ & \times \int_{|z|\leq\sigma} \left(K_\alpha\left(\frac{2|\omega_2|^2}{1-|\omega_2|^4}|z|^2\right) g_{2,e}(z) \overline{(K_{c_2}^\alpha(z,\xi))}_e \right. \\ & \left. + K_{\alpha+1}\left(\frac{2|\omega_2|^2}{1-|\omega_2|^4}|z|^2\right) g_{2,o}(z) \overline{(K_{c_2}^\alpha(z,\xi))}_o \right) \\ & \times \exp\left(\frac{2|\omega_2|^4}{1-|\omega_2|^4}(x^2-y^2)\right) |z|^{2\alpha+2} dxdy \ , \ \xi \in \mathbb{R} \\ = \lim_{\sigma\to\infty} & \frac{4|\omega_2|^{2\alpha+4}}{\pi\Gamma(\alpha+1)(1-|\omega_2|^4)} \Big(\frac{1}{1-\overline{\omega}_2^2}\Big)^{\alpha+1} \exp\Big(-\frac{\overline{\omega}_2^2}{1-\overline{\omega}_2^2}\xi^2\Big) \end{aligned}$$

$$\times \int_{|z|\le\sigma} \Big\{ K_\alpha\Big(\frac{2|\omega_2|^2}{1-|\omega_2|^4}|z|^2\Big) g_{2,e}(z) E^e_\alpha\Big(\frac{2\overline{\omega}_2}{1-\overline{\omega}_2^2}\overline{z}\xi\Big)$$

$$+ K_{\alpha+1}\Big(\frac{2|\omega_2|^2}{1-|\omega_2|^4}|z|^2\Big) g_{2,o}(z) E^o_\alpha\Big(\frac{2\overline{\omega}_2}{1-\overline{\omega}_2^2}\overline{z}\xi\Big)\Big\}$$

$$\times \exp\Big(-\frac{\overline{\omega}_2^2}{1-\overline{\omega}_2^2}\overline{z}^2\Big) \exp\Big\{\frac{2|\omega_2|^4}{1-|\omega_2|^4}(x^2-y^2)\Big\}|z|^{2\alpha+2}dxdy\ ,\ \xi\in\mathbb{R}. \tag{3.6}$$

4. Extension of $e^{-c\mathscr{H}^\alpha}$ for $\Re c < 0$

The inverse transform (3.5) inspires us an extension of $e^{-c\mathscr{H}^\alpha}$ for $\Re c < 0$. Indeed, let $g_2(x) = \sum_{n=0}^\infty a_n^\alpha H_n^\alpha(x)$ then $f(x)$ has the representation $\sum_{n=0}^\infty a_n^\alpha e^{-(-c_2)n} H_n^\alpha(x)$ in $L^2(\mu_\alpha)$. Hence, for any c with $\Re c < 0$, we define the linear operator $e^{-c\mathscr{H}^\alpha}$ in the form

$$e^{-c\mathscr{H}^\alpha} f(x) = \sum_{n=0}^\infty a_n^\alpha e^{-cn} H_n^\alpha(x).$$

So, in the expression (3.5) replacing ω_2 by e^c , we obtain the representation of $e^{-c\mathscr{H}^\alpha}$. However, since the expression (3.5) requires the analytic extension form of a member in $L^2(\mu_\alpha)$, we shall give its representation in terms of real variable. For any fixed c with $\Re c < 0$, we first establish a characterization of the elements in $\mathscr{D}(e^{-c\mathscr{H}^\alpha})$, the domain of the operator. Since the family $\{e^{cn}H_n^\alpha(z)\}_{n\ge 0}$ is a complete orthonormal system in $\mathbf{H}^\alpha_{-c}$, for a given $f \in L^2(\mu_\alpha)$, $f \in \mathscr{D}(e^{-c\mathscr{H}^\alpha})$ if and only if f is almost everywhere equal to the restriction of an element in $\mathbf{H}^\alpha_{-c}$ to $\mathbb{R}$ with respect to μ_α.

Theorem 4.1. *For any c with $\Re c < 0$, let $\omega = e^c$.*
If f is a $\mathscr{C}^\infty$-function in $L^2(\mu_\alpha)$, then the following are equivalent:

(i) *f is an element in $\mathscr{D}(e^{-c\mathscr{H}^\alpha})$.*

(ii) *The integral*

$$\int_{\mathbb{C}} \Bigg(K_\alpha\Big(\frac{2|\omega|^6}{(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)}|z|^2\Big)$$
$$\Big|\int_{\mathbb{R}} f(\xi)\exp\Big(-\frac{1}{1-|\omega|^4}\xi^2\Big) E^e_\alpha\Big(\frac{2|\omega|^2}{1-|\omega|^4}z\xi\Big)|\xi|^{2\alpha+1}d\xi\Big|^2$$
$$+K_{\alpha+1}\Big(\frac{2|\omega|^6}{(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)}|z|^2\Big)$$
$$\Big|\int_{\mathbb{R}} f(\xi)\exp\Big(-\frac{1}{1-|\omega|^4}\xi^2\Big) E^o_\alpha\Big(\frac{2|\omega|^2}{1-|\omega|^4}z\xi\Big)|\xi|^{2\alpha+1}d\xi\Big|^2\Bigg)$$
$$\times\exp\Big(-\frac{2|\omega|^4(|\omega|^4+1)}{(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)}(x^2-y^2)\Big)|z|^{2\alpha+2}dxdy$$

is finite.

Proof. We prove that f is the restriction of an element in $\mathbf{H}^\alpha_{-c}$ to $\mathbb{R}$ if and only if f satisfies the condition (ii). For simplicity, let $\mathscr{X}_\alpha$ be the restriction operator to $\mathbb{R}$. Then $\mathscr{X}_\alpha$ is a bounded operator of $\mathbf{H}^{\alpha}_{-c}$ into $L^2(\mu_\alpha)$, in fact, we have $\|\mathscr{X}_\alpha\| = 1$. Moreover, the adjoint operator $\mathscr{X}^*_\alpha$ of $\mathscr{X}_\alpha$ has the following representation, for all $h \in L^2(\mu_\alpha)$

$$\begin{aligned}[\mathscr{X}^*_\alpha h](z) &= \langle \mathscr{X}^*_\alpha h, \mathscr{N}^\alpha(.,\bar{z},-c)\rangle_{-c,\alpha} \\ &= \langle h, \mathscr{X}_\alpha \mathscr{N}^\alpha(.,\bar{z},-c)\rangle_{2,\alpha}, \quad z \in \mathbb{C}\,. \end{aligned} \tag{4.1}$$

Since $\left\{e^{cn}H^{\alpha}_n(z)\right\}_{n\geq 0}$ is a complete orthonormal system in $\mathbf{H}^{\alpha}_{-c}$, the range $\mathscr{X}_\alpha(\mathbf{H}^{\alpha}_{-c})$ is dense in $L^2(\mu_\alpha)$, so the operator $\mathscr{X}^*_\alpha$ is one-to-one. Hence, f can be extended as the member in $\mathbf{H}^{\alpha}_{-c}$ if and only if $\mathscr{X}^*_\alpha f$ is contained in the range $\mathscr{X}^*_\alpha \mathscr{X}_\alpha(\mathbf{H}^{\alpha}_{-c})$. If g is an element in $\mathscr{X}^*_\alpha \mathscr{X}_\alpha(\mathbf{H}^{\alpha}_{-c})$, then g is expressible in the form

$$\begin{aligned} g(z) &= \langle \mathscr{X}^*_\alpha \mathscr{X}_\alpha G, \mathscr{N}^\alpha(.,\bar{z},-c)\rangle_{-c,\alpha} \\ &= \langle G, \mathscr{X}^*_\alpha \mathscr{X}_\alpha \mathscr{N}^\alpha(.,\bar{z},-c)\rangle_{-c,\alpha}, \quad z \in \mathbb{C} \end{aligned} \tag{4.2}$$

where G is in $\mathbf{H}^{\alpha}_{-c}$ with $\mathscr{X}^*_\alpha \mathscr{X}_\alpha G = g$.
Following a general method, given in [11], we shall give a characterization of the members in $\mathscr{X}^*_\alpha \mathscr{X}_\alpha(\mathbf{H}^{\alpha}_{-c})$.
From the representation (4.2) of $\mathscr{X}^*_\alpha \mathscr{X}_\alpha$, we calculate the kernel form

$$\begin{aligned} \widetilde{\mathscr{N}^\alpha}(z,\bar{u},c) &= \langle \mathscr{X}^*_\alpha \mathscr{X}_\alpha \mathscr{N}^\alpha(.,\bar{u},-c), \mathscr{X}^*_\alpha \mathscr{X}_\alpha \mathscr{N}^\alpha(.,\bar{z},-c)\rangle_{-c,\alpha} \\ &= \langle \mathscr{X}_\alpha \mathscr{N}^\alpha(.,\bar{u},-c), \mathscr{X}_\alpha \mathscr{X}^*_\alpha \mathscr{X}_\alpha \mathscr{N}^\alpha(.,\bar{z},-c)\rangle_{2,\alpha}. \end{aligned} \tag{4.3}$$

Meanwhile, for all $z' \in \mathbb{C}$, we have from (4.1)

$$\begin{aligned} &[\mathscr{X}^*_\alpha \mathscr{X}_\alpha \mathscr{N}^\alpha(.,\bar{z},-c)](z') \\ &= \langle \mathscr{X}_\alpha \mathscr{N}^\alpha(.,\bar{z},-c), \mathscr{X}_\alpha \mathscr{N}^\alpha(.,\overline{z'},-c)\rangle_{2,\alpha} \\ &= \left(\frac{1}{1-|\omega|^4}\right)^{2\alpha+2} \exp\left(-\frac{|\omega|^4}{1-|\omega|^4}(z')^2\right)\exp\left(-\frac{|\omega|^4}{1-|\omega|^4}\bar{z}^2\right) \\ &\quad\times \int_{\mathbb{R}} \exp\left(-\frac{2|\omega|^4}{1-|\omega|^4}\xi^2\right) E_\alpha\left(\frac{2|\omega|^2}{1-|\omega|^4}\xi\bar{z}\right) E_\alpha\left(\frac{2|\omega|^2}{1-|\omega|^4}\xi z'\right) d\mu_\alpha(\xi). \end{aligned}$$

By formula (2.2), we have

$$\begin{aligned} [\mathscr{X}^*_\alpha \mathscr{X}_\alpha \mathscr{N}^\alpha(.,\bar{z},-c)](z') = \left(\frac{1}{1-|\omega|^8}\right)^{\alpha+1} &\exp\left(\frac{-|\omega|^8}{1-|\omega|^8}(z')^2\right) \\ &\exp\left(\frac{-|\omega|^8}{1-|\omega|^8}\bar{z}^2\right) E_\alpha\left(\frac{2|\omega|^4}{1-|\omega|^8}z'\bar{z}\right). \end{aligned} \tag{4.4}$$

Hence, the desired kernel form is deduced from (4.3).

$$\begin{aligned} \widetilde{\mathscr{N}^\alpha}(z,\bar{u},c) &= \langle \mathscr{X}_\alpha \mathscr{N}^\alpha(.,\bar{u},-c), \mathscr{X}_\alpha \mathscr{X}^*_\alpha \mathscr{X}_\alpha \mathscr{N}^\alpha(.,\bar{z},-c)\rangle_{2,\alpha} \\ &= \frac{1}{((1-|\omega|^4)(1-|\omega|^8))^{\alpha+1}\Gamma(\alpha+1)} \exp\left(-\frac{|\omega|^4}{1-|\omega|^4}\bar{u}^2\right)\exp\left(\frac{-|\omega|^8}{1-|\omega|^8}z^2\right) \\ &\quad\times \int_{\mathbb{R}} \exp\left(-\frac{1+|\omega|^4+|\omega|^8}{1-|\omega|^8}\xi^2\right) E_\alpha\left(\frac{2|\omega|^2}{1-|\omega|^4}\bar{u}\xi\right) E_\alpha\left(\frac{2|\omega|^4}{1-|\omega|^8}z\xi\right)|\xi|^{2\alpha+1}d\xi. \end{aligned}$$

Also by formula (2.2), we have

$$\widetilde{\mathscr{N}^\alpha}(z,\bar{u},c) = \left(\frac{1}{(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)}\right)^{\alpha+1} \exp\left(\frac{-|\omega|^{12}}{(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)}z^2\right)$$

$$\times \exp\left(\frac{-|\omega|^{12}}{(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)}\overline{u}^2\right) E_\alpha\left(\frac{2|\omega|^6}{(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)}z\overline{u}\right). \tag{4.5}$$

Then the range $\mathscr{X}_\alpha^*\mathscr{X}_\alpha(\mathbf{H}_{-c}^\alpha)$ coincides with the reproducing kernel Hilbert space $\mathbf{H}_{1,c}^\alpha$ admitting the reproducing kernel $\widetilde{\mathscr{N}}^\alpha(z,\overline{u},c)$ and $\mathscr{X}_\alpha^*\mathscr{X}_\alpha$ is an isometry of $\mathbf{H}_{-c}^\alpha$ onto $\mathbf{H}_{1,c}^\alpha$. Let $\mathbf{H}_{2,c}^\alpha$ be the reproducing kernel Hilbert space determined by the positive matrix

$$E_\alpha\left(\frac{2|\omega|^6}{(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)}z\overline{u}\right)$$

as in the proof of Theorem 3.1 , we obtain the factorization for $g \in \mathscr{X}_\alpha^*\mathscr{X}_\alpha(\mathbf{H}_{-c}^\alpha)$

$$g(z) = \left(\frac{1}{(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)}\right)^{\alpha+1} \exp\left(\frac{-|\omega|^{12}}{(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)}z^2\right) g_1(z)$$

for an entire function $g_1 \in \mathbf{H}_{2,c}^\alpha$.
So, it gives the norm identity

$$\|g\|_{\mathbf{H}_{1,c}^\alpha}^2 = \left(\frac{1}{(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)}\right)^{\alpha+1} \|g_1\|_{\mathbf{H}_{2,c}^\alpha}^2,$$

where

$$\begin{aligned}\|g_1\|_{\mathbf{H}_{2,c}^\alpha}^2 = {} & \frac{4}{\pi\Gamma(\alpha+1)}\left(\frac{|\omega|^6}{(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)}\right)^{\alpha+2} \\ & \times \int_{\mathbb{C}} \left(K_\alpha\left(\frac{2|\omega|^6}{(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)}|z|^2\right)|g_{1,e}(z)|^2\right. \\ & \left. + K_{\alpha+1}\left(\frac{2|\omega|^6}{(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)}|z|^2\right)|g_{1,o}(z)|^2\right)|z|^{2\alpha+2}dxdy.\end{aligned}$$

Note that $\mathbf{H}_{2,c}^\alpha$ is the totality of entire functions with finite norms $\|g_1\|_{\mathbf{H}_{2,c}^\alpha}$. The space $\mathbf{H}_{1,c}^\alpha$ consists of entire functions with finite norms given by

$$\begin{aligned}\|g\|_{\mathbf{H}_{1,c}^\alpha}^2 = {} & \frac{4|\omega|^{6\alpha+12}}{\pi\Gamma(\alpha+1)(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)} \\ & \times \int_{\mathbb{C}} \left\{K_\alpha\left(\frac{2|\omega|^6}{(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)}|z|^2\right)|g_{1,e}(z)|^2\right. \\ & \left. + K_{\alpha+1}\left(\frac{2|\omega|^6}{(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)}|z|^2\right)|g_{1,o}(z)|^2\right\} \\ & \times \exp\left(\frac{2|\omega|^{12}(x^2-y^2)}{(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)}\right) \\ & |z|^{2\alpha+2}dxdy.\end{aligned} \tag{4.6}$$

By formula (4.1) $\mathscr{X}_\alpha^* f$ is expressible in the form

$$\begin{aligned}[\mathscr{X}_\alpha^* f](z) = {} & \frac{1}{\Gamma(\alpha+1)}\left(\frac{1}{1-|\omega|^4}\right)^{\alpha+1} \exp\left(-\frac{|\omega|^4}{1-|\omega|^4}z^2\right) \\ & \times \int_{\mathbb{R}} f(\xi)\exp\left(-\frac{1}{1-|\omega|^4}\xi^2\right)E_\alpha\left(\frac{2|\omega|^2}{1-|\omega|^4}z\xi\right)|\xi|^{2\alpha+1}d\xi.\end{aligned} \tag{4.7}$$

So $\mathscr{X}_\alpha^* f \in \mathbf{H}_{1,c}^\alpha$ if $\|\mathscr{X}_\alpha^* f\|_{\mathbf{H}_{1,c}^\alpha} < +\infty$, then our claim has been proved. ∎

Remark 4.1. When a $\mathscr{C}^\infty$-function $f \in L^2(\mu_\alpha)$ satisfies (ii) in Theorem 4.1, we shall give the representation of its analytic extension $\widehat{f}$ in terms of f. By the isometry $\mathscr{X}_\alpha^* \mathscr{X}_\alpha$ of $\mathbf{H}_{-c}^\alpha$ onto $\mathbf{H}_{1,c}^\alpha$, we have the following.

Theorem 4.2. *Let c, ω and f be as in Theorem 4.1. If f satisfies (ii) in Theorem 4.1 and $\widehat{f}$ denotes the analytic extension of f to $\mathbb{C}$, then $\widehat{f}$ is represented by*

$$\begin{aligned}
\widehat{f}(z) &= \frac{4|\omega|^{6\alpha+12}}{\pi\Gamma(\alpha+1)^2(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)} \left(\frac{1}{(1-|\omega|^4)(1-|\omega|^8)}\right)^{\alpha+1} \\
&\times \exp\left(-\frac{|\omega|^8}{1-|\omega|^8} z^2\right) \int_{\mathbb{C}} \Bigg\{ K_\alpha\left(\frac{2|\omega|^6}{(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)}|z'|^2\right) \\
&\times \left(\int_{\mathbb{R}} f(\xi) \exp\left(-\frac{1}{1-|\omega|^4}\xi^2\right) E_\alpha^e\left(\frac{2|\omega|^2}{1-|\omega|^4} z'\xi\right) |\xi|^{2\alpha+1} d\xi\right) E_\alpha^e\left(\frac{2|\omega|^4}{1-|\omega|^8}\overline{z'} z\right) \\
&+ K_{\alpha+1}\left(\frac{2|\omega|^6}{(1-|\omega|^4)(|\omega|^8+|\omega|^4+1)}|z'|^2\right) \\
&\times \left(\int_{\mathbb{R}} f(\xi) \exp\left(-\frac{1}{1-|\omega|^4}\xi^2\right) E_\alpha^o\left(\frac{2|\omega|^2}{1-|\omega|^4} z'\xi\right) |\xi|^{2\alpha+1} d\xi\right) E_\alpha^o\left(\frac{2|\omega|^4}{1-|\omega|^8}\overline{z'} z\right) \Bigg\} \\
&\times \exp\left(-\frac{|\omega|^4}{1-|\omega|^8}(z')^2\right) \exp\left(-\frac{2|\omega|^8}{(1-|\omega|^8)(|\omega|^8+|\omega|^4+1)}\left((x')^2-(y')^2\right)\right) \\
&|z'|^{2\alpha+2} dx' dy', \qquad (4.8)
\end{aligned}$$

where $z' = x' + iy'$.

Proof. Let S^α be the adjoint operator of $\mathscr{X}_\alpha^* \mathscr{X}_\alpha$ from $\mathbf{H}_{1,c}^\alpha$ to $\mathbf{H}_{-c}^\alpha$. Then $\mathscr{X}_\alpha^* \mathscr{X}_\alpha$ is an isometry of $\mathbf{H}_{-c}^\alpha$ onto $\mathbf{H}_{1,c}^\alpha$, the operator S^α is the inverse of $\mathscr{X}_\alpha^* \mathscr{X}_\alpha$. Hence, $\mathscr{X}_\alpha^* f = \mathscr{X}_\alpha^* \mathscr{X}_\alpha S^\alpha \mathscr{X}_\alpha^* f$ and $f = \mathscr{X}_\alpha S^\alpha \mathscr{X}_\alpha^* f$, since

$$\begin{aligned}
\widehat{f}(z) &= [S^\alpha \mathscr{X}_\alpha^* f](z) \\
&= \langle S^\alpha \mathscr{X}_\alpha^* f, \mathscr{N}^\alpha(.,\overline{z},-c)\rangle_{-c,\alpha} \\
&= \langle \mathscr{X}_\alpha^* f, \mathscr{X}_\alpha^* \mathscr{X}_\alpha \mathscr{N}^\alpha(.,\overline{z},-c)\rangle_{\mathbf{H}_{1,c}^\alpha},
\end{aligned}$$

from formula (4.4), (4.6) and (4.7) we obtain the desired representation (4.8). ∎

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