

## Positive Solutions for Discrete Sturm-Liouville-Like Four-Point $p$ -Laplacian Boundary Value Problems

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**Abstract.** We consider the existence of positive solutions for a class of discrete second-order four-point boundary value problem with  $p$ -Laplacian. Using the well known Krasnosel'skii's fixed point theorem, some new existence criteria for positive solutions of the boundary value problem are presented.

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### 1. Introduction

Recently, the existence of positive solutions for Sturm-Liouville-like boundary value problems with  $p$ -Laplacian has received considerable attention [1–8]. The main tools to study this kind of problems are monotone iterative technique [1–3], fixed point theorem on a cone [4–8], Lerary-Schauder degree [8], upper and lower solution method [8], and so on.

Although, there are many authors considered discrete problems with  $p$ -Laplacian [9–12], there are no results referring the discrete Sturm-Liouville-like boundary value problems with  $p$ -Laplacian. The main difficulties are that when we define the operator of the problems in a common method, we can't guarantee that there exists integer  $t_0 \in [a, b] \cap \mathbb{N}$  such that  $\Delta u(t_0) = 0$ , if  $a, b$  are integers with  $a < b$ ,  $\Delta u(a) > 0$ ,  $\Delta u(b) < 0$ . So in [14], Zhang et al. try to study this kind of problems on time scales by imposing a special point  $\theta$ .

Motivated by the above works, we consider the following Sturm-Liouville-like four-point boundary value problem

$$(1.1) \quad \Delta(\varphi_p(\Delta u(n))) + h(n)f(u(n)) = 0, \quad n \in \mathcal{N}[0, N],$$

$$(1.2) \quad \alpha u(0) - \beta \Delta u(\xi) = 0, \quad \gamma u(N+2) + \delta \Delta u(\eta) = 0,$$

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where  $\Delta$  denotes the forward difference operator with stepsize 1,  $\varphi_p(u) = |u|^{p-2}u$ ,  $p > 1$ ,  $\alpha > 0$ ,  $\beta > 0$ ,  $\gamma > 0$ ,  $\delta > 0$ ,  $\mathcal{N}[0, N] = \{0, 1, \dots, N\}$ ,  $N$  is an integer with  $0 < \xi \leq \delta/\gamma < \max\{2\delta/\gamma, N + 2 - \beta/\alpha, N + 2/2 + \delta/2\gamma\} \leq \eta < N$ , and we denote  $(\varphi_p)^{-1} = \varphi_q$  with  $1/p + 1/q = 1$ .

By using a new method to express the operator of BVP (1.1), (1.2) and Krasnosel'skii's fixed point theorem, the special point doesn't need, and some new results for the existence of positive solution of BVP (1.1), (1.2) are obtained.

In this paper, we always assume that  $\sum_{i=s}^t a_i = 0$ , if  $t < s$ . And we list the following hypotheses:

(C<sub>1</sub>)  $f : [0, \infty) \rightarrow [0, \infty)$  is continuous.

(C<sub>2</sub>)  $h$  is a nonnegative value function defined on  $\mathcal{N}[0, N + 2]$ .

## 2. Preliminaries

In order to give our main results, first we give some conclusions with respect to the following boundary value problem

$$(2.1) \quad \Delta(\varphi_p(\Delta u(n))) + x(n) = 0, \quad n \in \mathcal{N}[0, N],$$

$$(2.2) \quad \alpha u(0) - \beta \Delta u(\xi) = 0, \quad \gamma u(N + 2) + \delta \Delta u(\eta) = 0,$$

where  $x(n) \in C([0, N + 2], [0, \infty))$ ,  $\alpha, \beta, \gamma, \delta, \xi, \eta$  are same to the coefficients in boundary condition (1.2)

### Lemma 2.1.

$$(2.3) \quad u(n) = \frac{\beta}{\alpha} \varphi_q \left( A_x - \sum_{j=0}^{\xi-1} x(j) \right) + \sum_{i=0}^{n-1} \varphi_q \left( A_x - \sum_{j=0}^{i-1} x(j) \right), \quad n \in \mathcal{N}[0, N + 2],$$

is a solution of BVP (2.1),(2.2), where  $A_x$  depends on  $x$  only is the unique solution of the following equation

$$(2.4) \quad \frac{\beta}{\alpha} \varphi_q \left( y - \sum_{j=0}^{\xi-1} x(j) \right) = \frac{\delta}{\gamma} \varphi_q \left( \sum_{j=0}^{\eta-1} x(j) - y \right) + \sum_{i=0}^{N+1} \varphi_q \left( \sum_{j=0}^{i-1} x(j) - y \right).$$

*Proof.* Let

$$F_x(y) = \frac{\beta}{\alpha} \varphi_q \left( y - \sum_{j=0}^{\xi-1} x(j) \right) - \frac{\delta}{\gamma} \varphi_q \left( \sum_{j=0}^{\eta-1} x(j) - y \right) - \sum_{i=0}^{N+1} \varphi_q \left( \sum_{j=0}^{i-1} x(j) - y \right).$$

It follows that

$$F_x(y) = \frac{\beta}{\alpha} \varphi_q \left( y - \sum_{j=0}^{\xi-1} x(j) \right) + \frac{\delta}{\gamma} \varphi_q \left( y - \sum_{j=0}^{\eta-1} x(j) \right) + \sum_{i=0}^{N+1} \varphi_q \left( y - \sum_{j=0}^{i-1} x(j) \right),$$

so  $F_x$  is increasing on  $(-\infty, \infty)$ . Since

$$F_x(0) = -\frac{\beta}{\alpha} \varphi_q \left( \sum_{j=0}^{\xi-1} x(j) \right) - \frac{\delta}{\gamma} \varphi_q \left( \sum_{j=0}^{\eta-1} x(j) \right) - \sum_{i=0}^{N+1} \varphi_q \left( \sum_{j=0}^{i-1} x(j) \right) < 0,$$

and

$$F_x \left( \sum_{i=0}^N x(j) \right) = \frac{\beta}{\alpha} \varphi_q \left( \sum_{j=\xi}^N x(j) \right) + \frac{\delta}{\gamma} \varphi_q \left( \sum_{j=\eta}^N x(j) \right) + \sum_{i=0}^{N+1} \varphi_q \left( \sum_{j=i}^{N+1} x(j) \right) > 0,$$

we get that there exists unique  $A_x$  such that (2.4) hold. It's easy to verify that (2.3) is a solution of BVP (2.1) and (2.2). ■

**Lemma 2.2.** *Suppose that  $A_x$  is given in Lemma 2.1. Then*

$$\sum_{j=0}^{\xi-1} x(j) \leq A_x \leq \sum_{j=0}^{\eta-1} x(j).$$

*Proof.* We only need to prove

$$F_x\left(\sum_{j=0}^{\xi-1} x(j)\right) \leq 0 \quad \text{and} \quad F_x\left(\sum_{j=0}^{\eta-1} x(j)\right) \geq 0.$$

$$\begin{aligned} F_x\left(\sum_{j=0}^{\xi-1} x(j)\right) &= -\frac{\delta}{\gamma} \varphi_q\left(\sum_{j=\xi}^{\eta-1} x(j)\right) + \sum_{i=0}^{\xi-1} \varphi_q\left(\sum_{j=i}^{\xi-1} x(j)\right) - \sum_{i=\xi}^{N+1} \varphi_q\left(\sum_{j=\xi}^{i-1} x(j)\right) \\ &\leq -\frac{\delta}{\gamma} \varphi_q\left(\sum_{j=\xi}^{\eta-1} x(j)\right) + \xi \varphi_q\left(\sum_{j=0}^{\xi-1} x(j)\right). \end{aligned}$$

Since  $0 < \xi \leq \delta/\gamma < \max\{2\delta/\gamma, N+2-\beta/\alpha, N+2/2+\delta/2\gamma\} \leq \eta < N$ , then  $\delta/\gamma \geq \xi$  and  $\xi \leq \eta/2$ . So,  $F_x\left(\sum_{j=0}^{\xi-1} x(j)\right) \leq 0$ . Similarly, we can also prove  $F_x\left(\sum_{j=0}^{\eta-1} x(j)\right) \geq 0$ . The proof is complete. ■

Lemma 2.2 implies  $u(0) \geq 0$ ,  $\Delta u(N+2) \geq 0$ . Let  $E = \{u : \mathcal{N}[0, N+2] \rightarrow \mathbb{R}^+\}$  be endowed with the ordering  $u_1 \leq u_2$ , if  $u_1(n) \leq u_2(n)$  for all  $n \in \mathcal{N}[0, N+2]$ , define the norm  $\|u\| = \max_{n \in \mathcal{N}[0, N+2]} |u(n)|$ . It's easy to see that  $E$  is a semi-ordered real Banach space. Choose cone  $P \subset E$

$$P = \left\{ u : \begin{array}{l} u \in E, u(n) \geq 0, n \in \mathcal{N}[0, N+2], \Delta^2 u(n) \leq 0, n \in \mathcal{N}[0, N], \text{ there exists} \\ n_0 \text{ such that } \Delta u(n) \geq 0, n \in \mathcal{N}[0, n_0], \Delta u(n) \leq 0, n \in \mathcal{N}[n_0+1, N+1] \end{array} \right\}.$$

Clearly,  $\|u\| = u(n_0+1)$  for  $u \in P$ . We define the operator  $A : P \rightarrow E$  by

$$Au(n) = \frac{\beta}{\alpha} \varphi_q\left(A_u - \sum_{j=0}^{\xi-1} h(j)f(u(j))\right) + \sum_{i=0}^{n-1} \varphi_q\left(A_u - \sum_{j=0}^{i-1} h(j)f(u(j))\right), n \in \mathcal{N}[0, N+2],$$

where  $A_u$  depends on  $u$  only is the unique solution of the following equation

$$\begin{aligned} \frac{\beta}{\alpha} \varphi_q\left(y - \sum_{j=0}^{\xi-1} h(j)f(u(j))\right) &= \frac{\delta}{\gamma} \varphi_q\left(\sum_{j=0}^{\eta-1} h(j)f(u(j)) - y\right) \\ &\quad + \sum_{i=0}^{N+1} \varphi_q\left(\sum_{j=0}^{i-1} h(j)f(u(j)) - y\right), \end{aligned}$$

From the definition of  $A$ , for each  $u \in P$ , we claim that  $Au \in P$ , and  $\|Au\| = Au(n_0+1)$ . In fact,  $\Delta Au(n) = \varphi_q\left(A_u - \sum_{j=0}^{n-1} h(j)f(u(j))\right)$  is decreasing on  $n \in \mathcal{N}[0, N+1]$ . If  $\Delta Au(n) > 0$  for all  $n \in \mathcal{N}[0, N+1]$ , then

$$Au(0) = \frac{\beta}{\alpha} \varphi_q\left(A_u - \sum_{j=0}^{\xi-1} h(j)f(u(j))\right) > 0,$$

and

$$Au(N+2) = -\frac{\delta}{\gamma} \varphi_q\left(A_u - \sum_{j=0}^{\eta-1} h(j)f(u(j))\right) < 0,$$

which contradicts  $\Delta Au(n) > 0$  for all  $n \in \mathcal{N}[0, N + 1]$ . Similarly, we can also prove  $\Delta Au(n) < 0$  does not hold for all  $n \in \mathcal{N}[0, N + 1]$ . Hence, there exists  $n_0 \in \mathcal{N}[0, N]$  such that  $\Delta Au(n_0) \geq 0$ , and  $\Delta Au(n_0 + 1) \leq 0$ . Similarly with Lemma 2.2, we know that  $Au(0) \geq 0$ ,  $Au(N + 2) \geq 0$ . So,  $Au(n) \geq 0$ ,  $n \in \mathcal{N}[0, N + 2]$ , and we have  $A : P \rightarrow P$ . Furthermore, it's easy to prove that  $A : P \rightarrow P$  is completely continuous.

**Lemma 2.3.** *If  $u \in P$ , then  $u(n) \geq n\|u\|/(n_0 + 1)$  for  $n \in \mathcal{N}[0, n_0 + 1]$ , and  $u(n) \geq (N + 2 - n)\|u\|/(N + 2 - n_0)$  for  $n \in \mathcal{N}[n_0 + 1, N + 2]$ .*

*Proof.* Since  $\Delta^2 u(n) \leq 0$ , it follows that  $\Delta u(n)$  is nonincreasing. Hence, for  $n \in \mathcal{N}[1, n_0 + 1]$ ,

$$u(n) - u(0) = \sum_{i=0}^{n-1} \Delta u(i) \geq n\Delta u(n-1) \geq n\Delta u(n),$$

and

$$u(n_0 + 1) - u(n) = \sum_{i=n}^{n_0} \Delta u(i) \leq (n_0 + 1 - n)\Delta u(n),$$

from which we have

$$u(n) \geq \frac{n}{n_0 + 1} \|u\|, \quad n \in \mathcal{N}[1, n_0 + 1].$$

For  $n \in \mathcal{N}[n_0 + 1, N + 1]$ ,

$$u(N + 2) - u(n) = \sum_{i=n}^{N+1} \Delta u(i) \leq (N + 2 - n)\Delta u(n),$$

and

$$u(n) - u(n_0 + 1) \geq u(n + 1) - u(n_0 + 1) = \sum_{i=n_0+1}^n \Delta u(i) \geq (n - n_0)\Delta u(n),$$

we know

$$u(n) \geq \frac{N + 2 - n}{N + 2 - n_0} \|u\|, \quad n \in \mathcal{N}[n_0 + 1, N + 2].$$

Clearly, when  $n = 0, N + 2$ , Lemma 2.3 also hold. The proof is complete. ■

**Lemma 2.4.** [3] *Let  $P$  be a cone in a Banach space  $E$ . Assume  $\Omega_1, \Omega_2$  are open subsets of  $E$  with  $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$ . If  $A : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow P$  is a completely continuous operator such that either*

- (i)  $\|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_1$  and  $\|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_2$ , or
- (ii)  $\|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_1$  and  $\|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_2$ .

*Then  $A$  has a fixed point in  $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$ .*

### 3. Main results

In this section, we present our main results with respect to BVP (1.1) and (1.2). For the notational convenience, we denote

$$L_1 = \frac{\beta}{\alpha} \varphi_q \left( \sum_{j=\xi}^{n_0} h(j) \right) + \sum_{i=0}^{n_0} \varphi_q \left( \sum_{j=i}^{n_0} h(j) \right),$$

$$L_2 = \frac{\beta}{\alpha} \varphi_q \left( \sum_{j=\xi}^{n_0-1} h(j) \right) + \sum_{i=\xi-1}^{n_0} \varphi_q \left( \sum_{j=i}^{n_0-1} h(j) \right),$$

where  $n_0 \in \mathcal{N}[0, N]$  and we define as [9]  $f_0 = \lim_{u \rightarrow 0^+} \frac{f(u)}{\varphi_p(u)}$ ,  $f_\infty = \lim_{u \rightarrow \infty} \frac{f(u)}{\varphi_p(u)}$ ,  $i_0 =$  number of zeros in the set  $\{f_0, f_\infty\}$ ,  $i_\infty =$  number of  $\infty$  in the set  $\{f_0, f_\infty\}$ . Clearly,  $i_0, i_\infty = 0, 1, \text{ or } 2$  and there are six possible cases:

- (i)  $i_0 = 0$  and  $i_\infty = 0$ ;
- (ii)  $i_0 = 0$  and  $i_\infty = 1$ ;
- (iii)  $i_0 = 0$  and  $i_\infty = 2$ ;
- (iv)  $i_0 = 1$  and  $i_\infty = 0$ ;
- (v)  $i_0 = 1$  and  $i_\infty = 1$ ;
- (vi)  $i_0 = 2$  and  $i_\infty = 0$ .

**Theorem 3.1.** *The BVP (1.1) and (1.2) has at least one positive solution in the case  $i_0 = 1$  and  $i_\infty = 1$ .*

*Proof.* First, we consider the case  $f_0 = 0$  and  $f_\infty = \infty$ . Since  $f_0 = 0$ , then there exists  $H_1 > 0$  such that  $f(u) \leq \varphi_p(\varepsilon)\varphi_p(u) = \varphi_p(\varepsilon u)$ ,  $0 < u \leq H_1$ , where  $\varepsilon$  satisfies  $\varepsilon L_1 \leq 1$ . If  $u \in P$  with  $\|u\| = H_1$ , then

$$\begin{aligned} \|Au\| &= Au(n_0 + 1) = \frac{\beta}{\alpha} \varphi_q \left( A_u - \sum_{j=0}^{\xi-1} h(j)f(u(j)) \right) \\ &\quad + \sum_{i=0}^{n_0} \varphi_q \left( A_u - \sum_{j=0}^{i-1} h(j)f(u(j)) \right) \\ &\leq \frac{\beta}{\alpha} \varphi_q \left( \sum_{j=0}^{n_0} h(j)f(u(j)) - \sum_{j=0}^{\xi-1} h(j)f(u(j)) \right) \\ &\quad + \sum_{i=0}^{n_0} \varphi_q \left( \sum_{j=0}^{n_0} h(j)f(u(j)) - \sum_{j=0}^{i-1} h(j)f(u(j)) \right) \\ &= \frac{\beta}{\alpha} \varphi_q \left( \sum_{j=\xi}^{n_0} h(j)f(u(j)) \right) + \sum_{i=0}^{n_0} \varphi_q \left( \sum_{j=i}^{n_0} h(j)f(u(j)) \right) \\ &\leq \|u\| \varepsilon L_1 \\ &\leq \|u\|. \end{aligned}$$

It follows that if  $\Omega_{H_1} = \{u \in E : \|u\| < H_1\}$ , then  $\|Au\| \leq \|u\|$  for  $u \in P \cap \partial\Omega_{H_1}$ . Since  $f_\infty = \infty$ , then there exists  $H'_2 > 0$  such that  $f(u) \geq \varphi_p(k)\varphi_p(u) = \varphi_p(ku)$ ,  $u \geq H'_2$ , where  $k > 0$  is chosen such that

$$k \frac{\xi - 1}{n_0 + 1} \frac{\xi - 1}{n_0 + 1} \geq 1.$$

Set  $H_2 = \max\{2H_1, n_0 + 1/\xi - 1H'_2\}$ , and  $\Omega_{H_2} = \{u \in E : \|u\| < H_2\}$ . If  $u \in P$  with  $\|u\| = H_2$ , then

$$\min_{n \in \mathcal{N}[\xi-1, n_0]} u(n) = u(\xi - 1) \geq \frac{\xi - 1}{n_0 + 1} \|u\| \geq H'_2.$$

So that

$$\|Au\| = Au(n_0 + 1) = \frac{\beta}{\alpha} \varphi_q \left( A_u - \sum_{j=0}^{\xi-1} h(j)f(u(j)) \right)$$

$$\begin{aligned}
 & + \sum_{i=0}^{n_0} \varphi_q \left( A_u - \sum_{j=0}^{i-1} h(j)f(u(j)) \right) \\
 \geq & \frac{\beta}{\alpha} \varphi_q \left( \sum_{j=0}^{n_0-1} h(j)f(u(j)) - \sum_{j=0}^{\xi-1} h(j)f(u(j)) \right) \\
 & + \sum_{i=0}^{n_0} \varphi_q \left( \sum_{j=0}^{n_0-1} h(j)f(u(j)) - \sum_{j=0}^{i-1} h(j)f(u(j)) \right) \\
 \geq & \frac{\beta}{\alpha} \varphi_q \left( \sum_{j=\xi}^{n_0-1} h(j)f(u(j)) \right) \\
 & + \sum_{i=\xi-1}^{n_0} \varphi_q \left( \sum_{j=i}^{n_0-1} h(j)f(u(j)) \right) \\
 = & \|u\| k \frac{\xi-1}{n_0+1} \|u\|_{L_2} \\
 \geq & \|u\|.
 \end{aligned}$$

In other words, if  $u \in P \cap \partial\Omega_{H_2}$ , then  $\|Au\| \geq \|u\|$ . Thus by (i) if Lemma 2.4, it follows that  $A$  has a fixed point in  $P \cap (\overline{\Omega}_{H_2} \setminus \Omega_{H_1})$  with  $H_1 \leq \|u\| \leq H_2$ .

Now we consider the case  $f_0 = \infty$  and  $f_\infty = 0$ . Since  $f_0 = \infty$ , there exists  $H_3 > 0$  such that  $f(u) \geq \varphi_p(m)\varphi_p(u) = \varphi_p(mu)$  for  $0 < u \leq H_3$  where  $m$  is such that  $mL_2 \frac{\xi-1}{n_0+1} \geq 1$ . If  $u \in P$  with  $\|u\| = H_3$ , then we have

$$\begin{aligned}
 \|Au\| = & Au(n_0+1) = \frac{\beta}{\alpha} \varphi_q \left( A_u - \sum_{j=0}^{\xi-1} h(j)f(u(j)) \right) \\
 & + \sum_{i=0}^{n_0} \varphi_q \left( A_u - \sum_{j=0}^{i-1} h(j)f(u(j)) \right) \\
 \leq & \frac{\beta}{\alpha} \varphi_q \left( \sum_{j=0}^{n_0-1} h(j)f(u(j)) - \sum_{j=0}^{\xi-1} h(j)f(u(j)) \right) \\
 & + \sum_{i=0}^{n_0} \varphi_q \left( \sum_{j=0}^{n_0-1} h(j)f(u(j)) - \sum_{j=0}^{i-1} h(j)f(u(j)) \right) \\
 \leq & \frac{\beta}{\alpha} \varphi_q \left( \sum_{j=\xi}^{n_0-1} h(j)f(u(j)) \right) \\
 & + \sum_{i=\xi-1}^{n_0} \varphi_q \left( \sum_{j=i}^{n_0-1} h(j)f(u(j)) \right) \\
 = & mL_2 \frac{\xi-1}{n_0+1} \|u\| \\
 \leq & \|u\|.
 \end{aligned}$$

Thus, we let  $\Omega_{H_3} = \{u \in E : \|u\| < H_3\}$ , so that  $\|Au\| \geq \|u\|$  for  $u \in P \cap \partial\Omega_{H_3}$ . Next consider  $f_\infty = 0$ . By definition, there exists  $H'_4 > 0$  such that  $f(u) \leq \varphi_p(\varepsilon)\varphi_p(u) = \varphi_p(\varepsilon u)$  for  $u \geq H'_4$  where  $\varepsilon > 0$  satisfies

$$(3.1) \quad \varepsilon L_1 \leq 1.$$

Suppose  $f$  is bounded. Then  $f(u) \leq \varphi_p(K)$  for all  $u \geq 0$ , pick

$$H_4 = \max\{2H_3, KL_1\}.$$

If  $u \in P$  with  $\|u\| = H_4$ , then

$$\begin{aligned} \|Au\| &= Au(n_0 + 1) = \frac{\beta}{\alpha} \varphi_q \left( A_u - \sum_{j=0}^{\xi-1} h(j)f(u(j)) \right) \\ &\quad + \sum_{i=0}^{n_0} \varphi_q \left( A_u - \sum_{j=0}^{i-1} h(j)f(u(j)) \right) \\ &\leq \frac{\beta}{\alpha} \varphi_q \left( \sum_{j=0}^{n_0} h(j)f(u(j)) - \sum_{j=0}^{\xi-1} h(j)f(u(j)) \right) \\ &\quad + \sum_{i=0}^{n_0} \varphi_q \left( \sum_{j=0}^{n_0} h(j)f(u(j)) - \sum_{j=0}^{i-1} h(j)f(u(j)) \right) \\ &= \frac{\beta}{\alpha} \varphi_q \left( \sum_{j=\xi}^{n_0} h(j)f(u(j)) \right) + \sum_{i=0}^{n_0} \varphi_q \left( \sum_{j=i}^{n_0} h(j)f(u(j)) \right) \\ &\leq KL_1 \\ &\leq H_4 = \|u\|. \end{aligned}$$

Now suppose  $f$  is unbounded. From condition  $(C_1)$ , it's easy to know that  $f(u) \leq f(H_4)$  for  $0 \leq u \leq H_4$ . If  $u \in P$  with  $\|u\| = H_4$ , then by using (3.1) we have

$$\begin{aligned} \|Au\| &= Au(n_0 + 1) = \frac{\beta}{\alpha} \varphi_q \left( A_u - \sum_{j=0}^{\xi-1} h(j)f(u(j)) \right) \\ &\quad + \sum_{i=0}^{n_0} \varphi_q \left( A_u - \sum_{j=0}^{i-1} h(j)f(u(j)) \right) \\ &\leq \frac{\beta}{\alpha} \varphi_q \left( \sum_{j=0}^{n_0} h(j)f(u(j)) - \sum_{j=0}^{\xi-1} h(j)f(u(j)) \right) \\ &\quad + \sum_{i=0}^{n_0} \varphi_q \left( \sum_{j=0}^{n_0} h(j)f(u(j)) - \sum_{j=0}^{i-1} h(j)f(u(j)) \right) \\ &= \frac{\beta}{\alpha} \varphi_q \left( \sum_{j=\xi}^{n_0} h(j)f(u(j)) \right) + \sum_{i=0}^{n_0} \varphi_q \left( \sum_{j=i}^{n_0} h(j)f(u(j)) \right) \\ &= H_4 \varepsilon L_1 \\ &\leq H_4 = \|u\|. \end{aligned}$$

Consequently, in either case we take

$$\Omega_{H_4} = \{u \in E : \|u\| < H_4\},$$

so that for  $u \in P \cap \partial\Omega_{H_4}$ , we have  $\|Au\| \geq \|u\|$ . Thus by (ii) of Lemma 2.4, it follows that  $A$  has a fixed point  $u$  in  $P \cap (\overline{\Omega_{H_4}} \setminus \Omega_{H_3})$  with  $H_3 \leq \|u\| \leq H_4$ .

The proof is complete. ■

**Theorem 3.2.** *Suppose the following conditions hold:*

- (C<sub>3</sub>) *there exist constant  $p' > 0$  such that  $f(u) \leq \varphi_p(p'A_1)$  for  $0 \leq u \leq p'$ , where  $A_1 = L_1^{-1}$ ,*
- (C<sub>4</sub>) *there exist constant  $q' > 0$  such that  $f(u) \geq \varphi_p(q'A_2)$  for  $\xi - 1/n_0 + 1q' \leq u \leq q'$ , where  $A_2 = L_2^{-1}$ , furthermore,  $p' \neq q'$ . Then BVP (1.1) and (1.2) has at least one positive solution  $u$  such that  $\|u\|$  lies between  $p'$  and  $q'$ .*

*Proof.* Without loss of generality, we may assume that  $p' < q'$ . Let  $\Omega_{p'} = \{u \in E : \|u\| < p'\}$ . Then for any  $u \in P \cap \partial\Omega_{p'}$ . In view of (C<sub>3</sub>) we have

$$\begin{aligned} \|Au\| &= Au(n_0 + 1) = \frac{\beta}{\alpha} \varphi_q \left( A_u - \sum_{j=0}^{\xi-1} h(j)f(u(j)) \right) \\ &\quad + \sum_{i=0}^{n_0} \varphi_q \left( A_u - \sum_{j=0}^{i-1} h(j)f(u(j)) \right) \\ &\leq \frac{\beta}{\alpha} \varphi_q \left( \sum_{j=0}^{n_0} h(j)f(u(j)) - \sum_{j=0}^{\xi-1} h(j)f(u(j)) \right) \\ &\quad + \sum_{i=0}^{n_0} \varphi_q \left( \sum_{j=0}^{n_0} h(j)f(u(j)) - \sum_{j=0}^{i-1} h(j)f(u(j)) \right) \\ &= \frac{\beta}{\alpha} \varphi_q \left( \sum_{j=\xi}^{n_0} h(j)f(u(j)) \right) + \sum_{i=0}^{n_0} \varphi_q \left( \sum_{j=i}^{n_0} h(j)f(u(j)) \right) \\ &= p'A_1L_1 \\ &\leq p', \end{aligned}$$

which yields

$$(3.2) \quad \|Au\| \leq \|u\| \text{ for } u \in P \cap \partial\Omega_{p'}.$$

Now set  $\Omega_{q'} = \{u \in E : \|u\| < q'\}$  for  $u \in P \cap \partial\Omega_{q'}$ , we have

$$\frac{\xi - 1}{n_0 + 1} q' \leq u(n) \leq q' \text{ for } n \in \mathcal{N}[\xi - 1, n_0].$$

Hence by condition (C<sub>4</sub>), we can get

$$\begin{aligned} \|Au\| &= Au(n_0 + 1) = \frac{\beta}{\alpha} \varphi_q \left( A_u - \sum_{j=0}^{\xi-1} h(j)f(u(j)) \right) \\ &\quad + \sum_{i=0}^{n_0} \varphi_q \left( A_u - \sum_{j=0}^{i-1} h(j)f(u(j)) \right) \end{aligned}$$



$$\begin{aligned}
 &\geq \frac{\beta}{\alpha} \varphi_q \left( \sum_{j=0}^{n_0-1} h(j)f(u(j)) - \sum_{j=0}^{\xi-1} h(j)f(u(j)) \right) \\
 &\quad + \sum_{i=0}^{n_0} \varphi_q \left( \sum_{j=0}^{n_0-1} h(j)f(u(j)) - \sum_{j=0}^{i-1} h(j)f(u(j)) \right) \\
 &\geq \frac{\beta}{\alpha} \varphi_q \left( \sum_{j=\xi}^{n_0-1} h(j)f(u(j)) \right) \\
 &\quad + \sum_{i=\xi-1}^{n_0} \varphi_q \left( \sum_{j=i}^{n_0-1} h(j)f(u(j)) \right) \\
 &= q'A_2L_2 \\
 &\geq q'.
 \end{aligned}$$

So if we take  $\Omega_{q'} = \{u \in E : \|u\| < q'\}$ , then

$$(3.3) \quad \|Au\| \geq \|u\|, \quad u \in P \cap \partial\Omega_{q'}.$$

Consequently, in view of  $p' < q'$ , (3.2) and (3.3), it follows from Lemma 2.3 that  $A$  has a fixed point  $u$  in  $P \cap (\overline{\Omega}_{q'} \setminus \Omega_{p'})$ . Moreover, it is a positive solution of (1.1) and (1.2) and  $p' < u < q'$ . The proof is complete. ■

**Theorem 3.3.** *If  $f_0 \in [0, \varphi_p(A_1))$  and  $f_\infty \in (\varphi_p(n_0 + 1/\xi - 1A_2), \infty)$ , then the BVP (1.1) and (1.2) has at least one positive solution.*

*Proof.* It is easy to see that under the assumptions, the conditions  $(C_3)$  and  $(C_4)$  in Theorem 3.2 are satisfied. So the proof is easy and we omit it here. ■

**Theorem 3.4.** *If  $f_0 \in (\varphi_p(n_0 + 1/\xi - 1A_2), \infty)$  and  $f_\infty \in [0, \varphi_p(A_1))$ , then the BVP (1.1) and (1.2) has at least one positive solution.*

*Proof.* In view of  $f_0 \in (\varphi_p(n_0 + 1/\xi - 1A_2), \infty)$ , for  $\varepsilon = f_0 - \varphi_p(n_0 + 1/\xi - 1A_2) \leq f_0 - \varphi_p(n_0 + 1/\xi - 1A_2)$ , there exists a sufficiently small  $q'$  such that

$$f(u)/\varphi_p(u) \geq f_0 - \varepsilon \geq \varphi_p \left( \frac{n_0 + 1}{\xi - 1} A_2 \right), \quad 0 < u \leq q'.$$

Thus, if  $\xi - 1/n_0 + 1q' < u \leq q'$ , then we have

$$f(u) \geq \varphi_p(u)\varphi_p(n_0 + 1/\xi - 1A_2) \geq \varphi_p(q'A_2).$$

So, for  $\xi - 1/n_0 + 1q' \leq u \leq q'$ , we have  $f(u) \geq \varphi_p(q'A_2)$ , which yields condition  $(C_4)$  in Theorem 3.2.

Next, by  $f_\infty \in [0, \varphi_p(A_1))$ , for  $\varepsilon = \varphi_p(A_1) - f_\infty$ , there exists a sufficiently large  $p'' (> q')$  such that

$$(3.4) \quad f(u)/\varphi_p(u) \leq f_\infty + \varepsilon = \varphi_p(A_1), \quad u \geq p''.$$

We consider two cases:

**Case (i).** Suppose that  $f$  is bounded, say

$$(3.5) \quad f(u) \leq \varphi_p(K), \quad u \geq 0.$$

In this case, take sufficiently large  $p'$  such that  $p' \geq \max\{K/A_1, p''\}$ , then from (3.5), we know  $f(u) \leq \varphi_p(K) \leq \varphi_p(A_1 p')$  for  $0 < u \leq p'$ , which yields condition  $(C_3)$  in Theorem 3.2.

**Case (ii).** Suppose that  $f$  is unbounded. It's easy to know that there is  $p' > p''$  such that

$$(3.6) \quad f(u) \leq f(p'), \quad 0 \leq u \leq p'.$$

Since  $p' > p''$  then from (3.4) and (3.6), we get

$$f(u) \leq f(p') \leq \varphi_p(p'A_1), \quad 0 \leq u \leq p'.$$

Thus, the condition  $(C_3)$  of Theorem 3.2 is satisfied. Hence, from Theorem 3.2, the BVP (1.1) and (1.2) has at least one positive solution. The proof is complete. ■

From Theorems 3.3 and 3.4, we have the following two results.

**Corollary 3.1.** *If  $f_0 = 0$  and the condition  $(C_4)$  in Theorem 3.2 hold, then the BVP (1.1) and (1.2) has at least one positive solution.*

**Corollary 3.2.** *If  $f_\infty = 0$  and the condition  $(C_4)$  in Theorem 3.2 hold, then the BVP (1.1) and (1.2) has at least one positive solution.*

**Theorem 3.5.** *If  $f_0 \in (0, \varphi_p(A_1))$  and  $f_\infty = \infty$  hold, then the BVP (1.1) and (1.2) has at least one positive solution.*

*Proof.* In view of  $f_\infty = \infty$ , similar to the first part of Theorem 3.1, we have

$$\|Au\| \geq \|u\|, \quad u \in P \cap \partial\Omega_{H_2}.$$

Since  $f_0 \in (0, \varphi_p(A_1))$ , for  $\varepsilon = \varphi_p(A_1) - f_0 > 0$ , there exists a sufficiently small  $p' \in (0, H_2)$  such that

$$f(u) \leq (f_0 + \varepsilon)\varphi_p(u) = \varphi_p(A_1 u) \leq \varphi_p(A_1 p'), \quad 0 \leq u \leq p'.$$

Similar to the proof of Theorem 3.2, we obtain

$$\|Au\| \leq \|u\|, \quad u \in P \cap \partial\Omega_{p'}.$$

The result is obtained and the proof is complete. ■

**Theorem 3.6.** *If  $f_\infty \in (0, \varphi_p(A_1))$  and  $f_0 = \infty$ , then the BVP (1.1) and (1.2) has at least one positive solution.*

*Proof.* Since  $f_0 = \infty$ , similar to the second part of Theorem 3.1 we have  $\|Au\| \geq \|u\|$  for  $u \in P \cap \partial\Omega_{H_3}$ .

By  $f_\infty \in (0, \varphi_p(A_1))$ , similar to the second part of proof of Theorem 3.4, we have  $\|Au\| \leq \|u\|$  for  $u \in P \cap \partial\Omega_{p'}$ , where  $p' > H_3$ . Thus the BVP (1.1) and (1.2) has at least one positive solution. The proof is complete. ■

From Theorem 3.5 and Theorem 3.6, we can get the following corollaries.

**Corollary 3.3.** *If  $f_\infty = \infty$  and the condition  $(C_3)$  in Theorem 3.2 hold, then the BVP (1.1) and (1.2) has at least one positive solution.*

**Corollary 3.4.** *If  $f_0 = \infty$  and the condition  $(C_3)$  in Theorem 3.2 hold, then the BVP (1.1) and (1.2) has at least one positive solution.*

**Theorem 3.7.** *If  $i_0 = 0, i_\infty = 2$ , and the condition  $(C_3)$  of Theorem 3.2 hold. Then the BVP (1.1) and (1.2) has at least two positive solutions  $u_1, u_2 \in P$  such that  $0 < \|u_1\| < p' < \|u_2\|$ .*

*Proof.* By using the method of proving Theorem 3.1 and Theorem 3.2, we can deduce the conclusion easily, so we omit it here.  $\blacksquare$

**Theorem 3.8.** *If  $i_0 = 2$ ,  $i_\infty = 0$ , and the condition  $(C_4)$  of Theorem 3.2 hold. Then the BVP (1.1) and (1.2) has at least two positive solutions  $u_1, u_2 \in P$  such that  $0 < \|u_1\| < q' < \|u_2\|$ .*

*Proof.* Combining the proof of Theorem 3.1 and Theorem 3.2, the conclusion is easy to see, and we omit it here.  $\blacksquare$

#### 4. Example

$$(4.1) \quad \Delta(\varphi_p(\Delta u(n))) + 4 - \arctan u = 0, \quad n \in \mathcal{N}[0, 8],$$

$$(4.2) \quad u(0) - 6\Delta u(1) = 0, \quad u(10) + 2\Delta u(6) = 0,$$

where  $p = 1.5$ ,  $q = 3$ ,  $h(n) \equiv 1$ ,  $f(u) = 4 - \arctan u$ ,  $\alpha = \gamma = \xi = 1$ ,  $\beta = 6$ ,  $\delta = 2$ ,  $\eta = 6$ .

$$f_0 = \lim_{u \rightarrow 0^+} \frac{f(u)}{\varphi_p(u)} = \infty, \text{ and } f_\infty = \lim_{u \rightarrow \infty} \frac{f(u)}{\varphi_p(u)} = 0.$$

Therefore, by Theorem 3.1, the boundary value problem (4.1), and (4.2) has at least a positive solution.

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