BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **35***(2) (2012), 315–323*

Numerical Methods for Sequential Fractional Differential Equations for Caputo Operator

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Abstract. To obtain the solution of nonlinear sequential fractional differential equations for Caputo operator two methods namely the Adomian decomposition method and Daftardar-Gejji and Jafari iterative method are applied in this paper. Finally some examples are presented to illustrate the efficiency of these methods.

2010 Mathematics Subject Classification: 65L05, 26A33

Keywords and phrases: Sequential fractional derivative, fractional integral, Adomian decomposition method, iterative method, Caputo fractional derivative.

1. Introduction

Nowadays there is increasing attention paid to fractional differential equations and their applications in different research areas. It is well known that these equations are concluded from many physical and chemical problems [19] such as the motion of a large thin plate in a Newtonian fluid, the process of cooling a semi-infinite body by radiation, the phenomena in electromagnetic acoustic viscoelasticity, electrochemistry and material science and so on. These equations are more adequate for modeling physical and chemical process than integer-order differential equations. So far there have been several fundamental works on the fractional derivative and fractional differential equations, written by Oldham and Spanier [17], Miller and Ross [12], Poldubny [17] and others.

The objective of the present paper is to extend the applications of the Adomian decomposition method (ADM) and the Daftardar-Gejji and Jafari iterative method (DJIM) to provide approximate solutions for the nonlinear sequential fractional differential equations:

$$\mathscr{D}_* y(x) = f(x, y),$$

where

$$\mathscr{D}_* \equiv D_*^{\alpha_n} D_*^{\alpha_{n-1}} \dots D_*^{\alpha_1},$$

Communicated by Dongwoo Sheen.
Received: October 13, 2009; *Revised:* September 26, 2010.

and $D_*^{\alpha_j}$ is the derivative of y of order α_j in the sense of Caputo and $m-1 < \alpha_j \le m$, subject to the initial conditions

$$y^0 = c_0, y^1 = c_1, \ldots, y^{(m-1)} = c_{m-1}.$$

Most nonlinear sequential fractional differential equations do not have exact analytic solutions, so approximation techniques (see [4, 7, 16]) must be used. The ADM [1, 2, 3, 7, 20] and the DJIM [8] are relatively new approaches to provide numerical approximations to nonlinear problems.

The application of these two methods are successfully extended to obtain a numerical approximate solutions to linear and nonlinear sequential differential equations of fractional order. A comparison between the DJIM and ADM for solving fractional differential equations is given in examples.

2. Preliminaries and notations

In order to proceed, we need the following definitions of fractional derivatives and integrals. We first introduce the Riemann-Liouville definition of fractional derivative operator $D_a^{-\alpha}$.

Definition 2.1. *Let $\alpha \in R^+$. The operator $D_a^{-\alpha}$, defined on the usual Lebesque space $L^1[a,b]$ by*

$$D_a^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \tag{2.1}$$

$$D_a^0 f(x) = f(x),$$

for $a \le x \le b$, is called the Riemann-Liouville fractional integral operator of order α .

Properties of the operator D_a^{α} can be found in [12]. For $f \in L^1[a,b], \alpha, \beta \ge 0$, and $\gamma > -1$, we mention only the following:

(1) $D_a^{-\alpha}$ exists for almost every $x \in [a,b]$.
(2) $D_a^{-\alpha} D_a^{-\beta} f(x) = D_a^{-(\alpha+\beta)} f(x)$,
(3) $D_a^{-\alpha} D_a^{-\beta} f(x) = D_a^{-\beta} D_a^{-\alpha} f(x)$,
(4) $D_a^{-\alpha} (x-a)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} (x-a)^{\alpha+\gamma}$.

Definition 2.2. *The fractional derivative of $f(x)$ in the Caputo sense is defined as*

$$D_*^{\alpha} f(x) = D^{-(m-\alpha)} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \tag{2.2}$$

for $m-1 < \alpha \le m, m \in N, x > 0$.

Also, we need here two of its basics properties. If $m-1 < \alpha \le m$, and $f \in L^1[a,b]$, then

$$D_*^{\alpha} D^{-\alpha} f(x) = f(x),$$

and,

$$D^{-\alpha} D_*^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}\left(0^+\right) \frac{x^k}{k!}, \quad x > 0.$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem.

In this paper, we are concerned with providing good quality algorithms for the solution of a nonlinear sequential fractional differential equations of the general form:

$$\frac{d^{\sigma}y(x)}{dx^{\sigma}} = f(x,y), \tag{2.3}$$

subject to the initial conditions

$$y^{(k)} = c_k, \quad k = 0,1,\ldots,m-1, \quad m-1 < \alpha \le m, \quad m \in N,$$

where c_k is a specified constant vector and $y(x)$ is the solution vector.

The fractional derivative in Equation (2.3) is considered in the Caputo sense. The reason for adopting the Caputo definition is as follows: when we interpret the fractional derivative in Equation (2.3) as Caputo fractional derivatives then, with suitable conditions on the forcing function $f(x,y)$ and with initial values $y^{(k)} = c_k, \quad k = 0,1,\ldots,m-1$ specified, the system has a unique solution. The initial conditions required by the Caputo definition coincide with identifiable physical states, and this leads to the preference, among modellers, for the Caputo definition [9].

3. Justification of the numerical methods

The Equation (2.3) can be written in terms of operator form as

$$\mathscr{D}_*^{\sigma} y(x) = f(x,y), \tag{3.1}$$

where

$$\mathscr{D}_* \equiv D_*^{\alpha_n} D_*^{\alpha_{n-1}} \ldots D_*^{\alpha_1} \quad \sigma = \sum_{j=1}^{n} \alpha_j,$$

$$m-1 < \alpha_j \le m, \quad (j = 1,2,\ldots,n),$$

and The fractional differential operator $D_*^{\alpha_j}$ is defined as in Equation (2.2), $f(x,y)$ is a nonlinear function of x,y and y is an unknown function to be determined later.

Applying the $D^{-\alpha_j} = D_0^{-\alpha_j}$, the inverse of the operator $D_*^{\alpha_j}$, to both of sides of Equation (3.1) yields

$$y(x) = \sum_{k=0}^{m-1} y^{(k)}(0^+)\frac{x^k}{k!} + D^{-\sigma}[f(x,y)]. \tag{3.2}$$

The ADM is widely used in approximate calculation. This method is introduced by the American mathematician, Adomian (1923–1996) [2, 3]. The Adomain decomposition method [2, 3] assumes a series solution for $y(x)$ given by

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \tag{3.3}$$

The nonlinear function in Equation (3.2) is decomposed into a series of the so-called Adomian polynomial as follows:

$$f(x,y) = \sum_{n=0}^{\infty} A_n(y_0,\cdots,y_n), \tag{3.4}$$

which the terms can be calculated recursively form

$$A_n = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\left[f\left(x, \sum_{k=0}^{n} y_k\lambda^k\right)\right]_{\lambda=0}. \tag{3.5}$$

Substituting Equations (3.3) and (3.4) into both sides of Equation (3.2) gives

$$\sum_{n=0}^{\infty} y_n(x) = \sum_{k=0}^{m-1} y^{(k)}(0^+)\frac{x^k}{k!} + D^{-\sigma}\left[\sum_{n=0}^{\infty} A_n\right]. \tag{3.6}$$

From this equation, the iterates are determined by the following recursive way:

$$y_0(x) = \sum_{k=0}^{m-1} \frac{c_k}{k!} x^k$$

$$y_{n+1}(x) = D^{-\sigma}[A_n], \quad n = 0, 1, 2, \ldots. \tag{3.7}$$

Define the N-term approximation solution as

$$\phi_N(x) = \sum_{n=0}^{N-1} y_n(x), \tag{3.8}$$

and the exact solution $y(x)$ is given by

$$\lim_{N\to\infty} \phi_N(x) = y(x). \tag{3.9}$$

However, in many cases the exact solution in a closed form may be obtained. Moreover, the decomposition series solutions generally converge very rapidly. The convergence of the decomposition series has been investigated by several authors. The theoretical treatment of convergence of the decomposition method has been considered in the literature [1, 7].

An iterative method for solving nonlinear functional equations, nonlinear Volterra integral equations, algebraic equations and systems of ordinary differential equation, nonlinear algebraic equations and fractional differential equations has been discussed. This method is introduced by the Daftardar-Gejji and Jafari [8]. We show how the method can be applied to a general sequential fractional differential equations. Consider the following general functional Equation (3.2)

$$y = f + N(y), \tag{3.10}$$

where N is a nonlinear operator from a Banach space $B \to B$ and f is a known function.

We are looking for solution y of Equation (3.10) having the series form:

$$y = \sum_{n=0}^{\infty} y_n. \tag{3.11}$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{n=0}^{\infty} y_n\right) = N(y_0) + \sum_{n=1}^{\infty}\left(N\left(\sum_{j=0}^{n} y_j\right) - N\left(\sum_{j=0}^{n-1} y_j\right)\right). \tag{3.12}$$

From Equations (3.11) and (3.12), Equation (3.10) is equivalent to

$$\sum_{n=0}^{\infty} y_n = f + N(y_0) + \sum_{n=1}^{\infty}\left(N\left(\sum_{j=0}^{n} y_j\right) - N\left(\sum_{j=0}^{n-1} y_j\right)\right).$$

Then we define the recurrence relation:

$$y_0 = f,$$
$$y_1 = N(y_0),$$

(3.13) $$y_{n+1} = N(y_0 + \ldots + y_n) - N(y_0 + \ldots + y_{n-1}), \quad n = 1, 2, \ldots.$$

In view of Equation (3.2) and Equation (3.13), we have:

$$y_0(x) = c_0,$$

$$y_1(x) = D^{-\sigma}[f(x, y_0)] + \sum_{k=1}^{m-1} \frac{c_k}{k!} x^k,$$

(3.14) $$y_{n+1}(x) = D^{-\sigma}[f(x, y_0 + \ldots + y_n) - f(x, y_0 + \ldots + y_{n-1})], \quad n = 1, 2, \ldots.$$

and

$$y = y(0) + \sum_{n=1}^{\infty} y_n.$$

If N is a contraction, i.e. $||N(x) - N(y)|| \leq ||x - y||, \quad 0 < k < 1,$then

$$||y_{n+1}|| = ||N(y_0 + \ldots + y_n) - N(y_0 + \ldots + y_{n-1})|| \leq k||y_n|| \leq k^{n+1}||y_0||, \quad n \geq 0,$$

and the series $\sum_{i=0}^{\infty} y_i$ absolutely and uniformly converges to a solution of Equation (3.10) [6, 8], which is unique, in view of the Banach fixed point theorem [11].

4. Implementation of these methods

In this section, for the sake of comparison, we have selected some examples.

Example 4.1. Consider the nonlinear sequential differential equation of the fractional order:

(4.1) $$D_*^{0.75}\left(D_*^{1.5} y(x)\right) = \frac{20}{7\Gamma(0.7)x^{0.7}} + x^4 - y^2(x)$$

with the initial conditions

$$y(0) = 0, \quad y'(0) = 0.$$

Solution via the ADM: In accordance with the Adomian method, we re-write Equation (4.1) as follows:

(4.2) $$y(x) = D^{-2.25}\left(\frac{20}{7\Gamma(0.7)x^{0.7}} + x^4\right) - D^{-2.25}(y^2(x)).$$

Therefore, we have the following recursive relation:

$$y_0(x) = D^{-2.25}\left(\frac{20}{7\Gamma(0.7)x^{0.7}} + x^4\right),$$

(4.3) $$y_{n+1}(x) = D^{-2.25}(A_n(x)), \quad n \geq 0.$$

Solution by using the DJIM: According to the releation Equation (3.14), we have the following iteration formula for solving Equation (4.1):

$$y_0(x) = D^{-2.25}\left(\frac{20}{7\Gamma(0.7)x^{0.7}} + x^4\right),$$

$$y_1(x) = -D^{-2.25}\left(y_0^2(x)\right),$$

(4.4) $$y_{n+1}(x) = -D^{-2.25}\left[(y_0 + \ldots + y_n)^2\right] + D^{-2.25}\left[(y_0 + \ldots + y_{n-1})^2\right], \quad n \geq 1$$

In Figure 1, the approximate solution $y \approx \sum_{n=0}^{3} y_n(x)$ of Equation (4.3) of the ADM and the approximation solution $y \approx \sum_{n=0}^{2} y_n(x)$ of Equation (4.4) of the iterative method have been plotted.

Example 4.2. Consider the following nonlinear sequential fractional differential equation:

$$D_*^{1.5}\left(D_*^{2.4}y(x)\right) - y^3(x) = \frac{10}{\Gamma(\frac{3}{5})}x^{\frac{3}{5}} - x^6 - 3x^7 - 3x^8 - x^9 \tag{4.5}$$

with the initial conditions

$$y(0) = y'(0) = 0, \quad y''(0) = 2.$$

Solution by the ADM: We, according to the ADM, have

$$y_0(x) = x^2 + D^{-3.9}\left(\frac{10}{\Gamma(\frac{3}{5})}x^{\frac{3}{5}} - x^6 - 3x^7 - 3x^8 - x^9\right),$$

$$y_{n+1}(x) = D^{-3.9}\left(A_n(x)\right), \quad n \geq 0. \tag{4.6}$$

Solution by the DJIM: In the light of the relation Equation (3.14) the iteration formulation can be written in the form:

$$y_0(0) = x^2,$$

$$y_1(x) = D^{-3.9}\left(\frac{10}{\Gamma\left(\frac{3}{5}\right)}x^{\frac{3}{5}} - x^6 - 3x^7 - 3x^8 - x^9\right) + D^{-3.9}\left(y_0^3(x)\right),$$

$$y_{n+1}(x) = -D^{-3.9}\left[(y_0 + \ldots + y_n)^3\right] + D^{-3.9}\left[(y_0 + \ldots + y_{n-1})^3\right], \quad n \geq 1 \tag{4.7}$$

In Figure 1, the approximate solution $y \approx \sum_{n=0}^{4} y_n(x)$ of Equation (4.6) and the approximate solution $y \approx \sum_{n=0}^{3} y_n(x)$ of Equation (4.7) have been plotted.

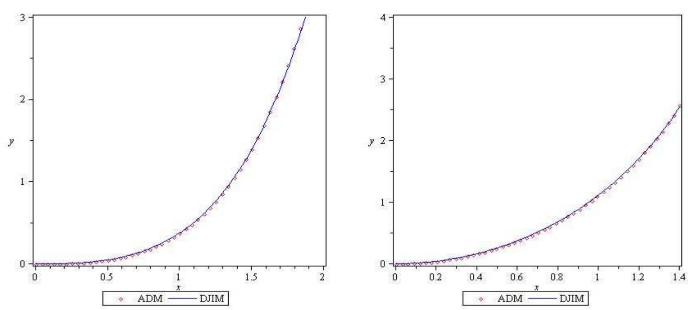

Figure 1. Approximate solution for Examples 4.1 and 4.2.

Example 4.3. Consider the following nonlinear sequential fractional differential equation with variable coefficients:

$$D_*^{1.3}\left(D_*^{0.2}\left(D_*^{0.4}y(x)\right)\right) + xy^2(x) = \frac{32}{21\Gamma(\frac{3}{4})}x^{\frac{7}{4}} + x^5 \tag{4.8}$$

with the initial condition

$$y(0) = y'(0) = 0.$$

Solution by using the ADM: In view of Equation (3.7), we have the following recursive relation:

$$y_0(x) = D^{-1.9}\left(\frac{32}{21\Gamma\left(\frac{3}{4}\right)}x^{\frac{7}{4}} + x^5\right),$$

$$y_{n+1}(x) = -D^{-1.9}(A_n(x)), n \geq 0, \tag{4.9}$$

Solution by using the DJIM: In light of Equation (3.14), the iteration formulae can be considered as

$$y_0(x) = D^{-1.9}\left(\frac{32}{21\Gamma\left(\frac{3}{4}\right)}x^{\frac{7}{4}} + x^5\right),$$

$$y_1(x) = -D^{-1.9}\left(xy_0^2(x)\right),$$

$$y_{n+1}(x) = -D^{-1.9}\left[x(y_0 + \ldots + y_n)^2\right] + D^{-1.9}\left[x(y_0 + \ldots + y_{n-1})^2\right], \quad n \geq 1 \tag{4.10}$$

The approximate solution $y \approx \sum_{n=0}^{3} y_n(x)$ of Equation (4.9) and the approximate solution $y \approx \sum_{n=0}^{2} y_n(x)$ of Equation (4.10) have been plotted in Figure 2.

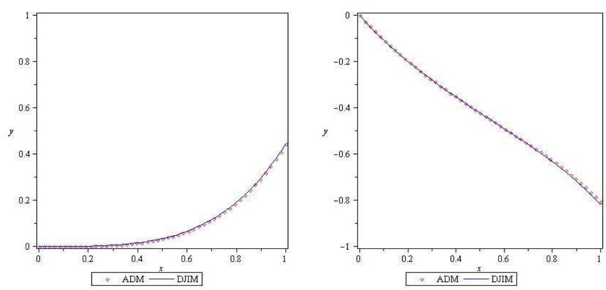

Figure 2. Approximate solution for Examples 4.3 and 4.4.

Example 4.4. Consider the following nonlinear sequential differential equation of the fractional order:

$$D_*^{0.75}\left(D_*^{0.2}y(x)\right) + e^{y(x)} = 0, \quad y(0) = 0 \tag{4.11}$$

Solution using the ADM: In accordance with Equation (3.7), we consider the following recursive relation:

$$y_0(x) = 0, \quad y_{n+1}(x) = -D^{-0.95}(A_n(x)), n \geq 0, \tag{4.12}$$

where

$$A_0 = e^{y_0},$$

$$A_1 = y_1 e^{y_0},$$

$$A_2 = \left(\frac{y_1^2}{2} + y_2\right)e^{y_0},$$

$$\vdots$$

Solution using the DJIM: In the light of Equation (3.14), we have the following variational iteration formulation:

$$y_0(x) = 0,$$
$$y_1(x) = e^{y_0},$$
$$y_{n+1}(x) = -D^{-0.95}\left[e^{(y_0+\ldots+y_n)}\right] + D^{-0.95}\left[e^{(y_0+\ldots+y_{n-1})}\right], \quad n \geq 1. \tag{4.13}$$

The approximate solution $y \approx \sum_{n=0}^{5} y_n(x)$ of Equation (4.12) and the approximate solution $y \approx \sum_{n=0}^{4} y_n(x)$ of Equation (4.13) have been plotted in Figure 2.

5. Conclusion

Analytical solution of the sequential fractional differential equations are usually difficult. In this paper the sequential fractional differential equations are solved by employing ADM and DJIM. The illustrative examples confirm the validity of these methods. A clear conclusion can be draw from the numerical results that the ADM and DJIM are highly accurate numerical techniques without spatial discretization for linear and nonlinear sequential fractional differential equations.

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