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Generalized Frames on Super Hilbert Spaces

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Abstract. In this paper we study the relationship between g-frames for super Hilbert space $\mathscr{H}\oplus\mathscr{K}$ and g-frames for $\mathscr{H}$ and $\mathscr{K}$ and frames for $\mathscr{H}$, $\mathscr{K}$ and $\mathscr{H}\oplus\mathscr{K}$. Also the concepts of disjoint g-frames and strong complementary g-frame are introduced and studied.

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1. Introduction

The concept of frame was introduced by Duffin and Schaeffer [2] to study nonharmonic Fourier series in 1952. Frames are generalizations of orthonormal bases in Hilbert spaces. If $\mathscr{H}$ is a Hilbert space, $\mathscr{H}\oplus\mathscr{H}$ is called super Hilbert space in literatures [1, 3, 4] and have been widely studied recently. For example, Balan [1] introduced the concept of super frames and presented some density results for Weyl-Heisenberg super frames. In [4], Han and Larson derived necessary and sufficient conditions for the direct sum of two frames to be a super frame, and in [3], Gu and Han investigated the connection between decomposable Parseval wavelet frames and super wavelet frames, and gave some necessary and sufficient conditions for extendable Parseval wavelet frames. In [6], a generalization of the frame concept was introduced. Sun introduced a g-frame in a Hilbert space and showed that this includes more other cases of generalizations of frame concept and proved that many basic properties can be derived within this more general context. G-frames and g-Riesz bases in complex Hilbert spaces have some properties similar to those of frames, Riesz bases, but not all the properties are similar (see [6]). In [7], some properties of g-frames for super Hilbert space $\mathscr{H}\oplus\mathscr{H}$ with respect to $\mathbb{C}^2$ were studied. The authors showed that a g-frame associated with a frame for $\mathscr{H}\oplus\mathscr{H}$ remains a g-frame whenever any one of its elements is removed. Furthermore, they showed that the excess of such a g-frame is at least $\dim\mathscr{H}$. In this paper we study the relationship between g-frames for super Hilbert space $\mathscr{H}\oplus\mathscr{K}$ and g-frames for $\mathscr{H}$ and $\mathscr{K}$ and frames for $\mathscr{H}$, $\mathscr{K}$ and $\mathscr{H}\oplus\mathscr{K}$. Also we characterize g-frames operators for supper Hilbert space $\mathscr{H}\oplus\mathscr{K}$ via the g-frame operators for $\mathscr{H}$ and $\mathscr{K}$.

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Throughout this paper $\mathscr{H}$, $\mathscr{K}$ are separable Hilbert spaces and $\{\mathscr{H}_i\}_{i\in J}$ is a sequence of separable Hilbert spaces, where J is a subset of $\mathbb{Z}$, $\mathscr{L}(\mathscr{H},\mathscr{H}_i)$ is the collection of all bounded linear operators from $\mathscr{H}$ to $\mathscr{H}_i$. For each sequence $\{\mathscr{H}_i\}_{i\in J}$, we define the space $\bigoplus_{i\in J}\mathscr{H}_i$ by

$$\bigoplus_{i\in J}\mathscr{H}_i=\{\{f_i\}_{i\in J}: f_i\in\mathscr{H}_i,\ i\in J \text{ and } \sum_{i\in J}\|f_i\|^2<\infty\}.$$

With the inner product defined by

$$\langle\{f_i\},\{g_i\}\rangle=\sum_{i\in J}\langle f_i,g_i\rangle,$$

it is clear that $\bigoplus_{i\in J}\mathscr{H}_i$ is a Hilbert space.

A frame for a complex Hilbert space $\mathscr{H}$ is a family of vectors $\{f_i\}_{i\in J}$ so that there are two positive constants A and B satisfying

$$A\|f\|^2\le\sum_{i\in J}|\langle f,f_i\rangle|^2\le B\|f\|^2, f\in\mathscr{H}.$$

The constants A and B are called lower and upper frame bounds.

We say that the frame pairs $(\{f_i\},\{g_i\})$ and $(\{h_i\},\{k_i\})$ are similar if there are bounded invertible operators T_1 and T_2 such that $f_i=T_1h_i$ and $g_i=T_2k_i$ for all $i\in J$. A pair of frames $\{f_i: i\in J\}$ and $\{g_i: i\in J\}$ is called disjoint if $\{(f_i,g_i): i\in J\}$ is a frame for $\mathscr{H}\oplus\mathscr{K}$. A pair of normalized tight frames $\{f_i: i\in J\}$ and $\{g_i: i\in J\}$ is called strongly disjoint if $\{(f_i,g_i): i\in J\}$ is a normalized tight frame for $\mathscr{H}\oplus\mathscr{K}$, and a pair of general frames $\{f_i: i\in J\}$ and $\{g_i: i\in J\}$ is called strongly disjoint if it is similar to a strongly disjoint pair of normalized tight frames.

Let $\{f_i: i\in J\}$ be a normalized tight frame for $\mathscr{H}$. If there exists a Hilbert space $\mathscr{K}$ and a normalized tight frame $\{g_i: i\in J\}$ for $\mathscr{K}$ such that $\{(f_i,g_i): i\in J\}$ is an orthonormal basis for $\mathscr{H}\oplus\mathscr{K}$, we call $\{g_i: i\in J\}$ a strong complementary frame to $\{f_i: i\in J\}$, and we call $(\{f_i\},\{g_i\})$ a strong complementary pair. If $\{f_i: i\in J\}$ is a general frame, we will define a strong complement to $\{f_i: i\in J\}$ to be any frame $\{g_i: i\in J\}$ such that the pair $(\{f_i\},\{g_i\})$ is similar to a strong complementary pair of normalized tight frames.

We say that $\{f_i\}_{i\in J}$ is a Riesz basis for $\mathscr{H}$, if $\{f_i\}_{i\in J}$ is complete in $\mathscr{H}$ and there exist constants $0<A\le B<\infty$, such that for all sequences of scalars $c=\{c_i\}_{i\in J}$,

$$A\sum_{i\in J}|c_i|^2\le\|\sum_{i\in J}c_if_i\|^2\le B\sum_{i\in J}|c_i|^2.$$

Riesz bases are special cases of frames.

A sequence $\{\Lambda_i\in\mathscr{L}(\mathscr{H},\mathscr{H}_i): i\in J\}$ is called a generalized frame, or simply a g-frame, for $\mathscr{H}$ with respect to $\{\mathscr{H}_i\}_{i\in J}$ if there exist two positive constants A and B such that for all $f\in\mathscr{H}$,

$$A\|f\|^2\le\sum_{i\in J}\|\Lambda_if\|^2\le B\|f\|^2.$$

The constants A and B are called the lower and upper g-frame bounds, respectively. If $A=B$ we call this g-frame a tight g-frame and if $A=B=1$ it is called a normalized tight g-frame. We say simply a g-frame for $\mathscr{H}$ whenever the space sequence $\mathscr{H}_i$ is clear. If we only have the upper bound, we call $\{\Lambda_i\}_{i\in J}$ a g-Bessel sequence with bound B. We say that $\{\Lambda_i\}_{i\in J}$ is g-complete, if $\{f:\Lambda_if=0,\forall i\in J\}=\{0\}$ and is called g-orthonormal basis for $\mathscr{H}$, if

$$\langle\Lambda_i^\star g_i,\Lambda_j^\star g_j\rangle=\delta_{i,j}\langle g_i,g_j\rangle,\ i,j\in J,\ g_i\in\mathscr{H}_i,\ g_j\in\mathscr{H}_j,$$

and

$$\sum_{i\in J}\|\Lambda_i f\|^2 = \|f\|^2.$$

We say that $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$ is a g- Riesz basis for $\mathscr{H}$, if it is g-complete and there exist constants $0 < A \le B < \infty$, such that for any finite subset $I \subseteq J$ and $g_i \in \mathscr{H}_i$, $i \in I$,

$$A\sum_{i\in I}\|g_i\|^2 \le \|\sum_{i\in I}\Lambda_i^\star g_i\|^2 \le B\sum_{i\in I}\|g_i\|^2.$$

Theorem 1.1. [6] *Let $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$ be a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{H}_i\}_{i\in J}$. The operator*

$$S : \mathscr{H} \to \mathscr{H}, Sf = \sum_{i\in J}\Lambda_i^\star\Lambda_i f,$$

is a positive invertible operator and every $f \in \mathscr{H}$ has an expansion

$$f = \sum_{i\in J} S^{-1}\Lambda_i^\star\Lambda_i f = \sum_{i\in J}\Lambda_i^\star\Lambda_i S^{-1} f.$$

So $\{\widetilde{\Lambda}_i = \Lambda_i S^{-1} \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$ is a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{H}_i\}_{i\in J}$ and is called canonical dual g-frame of $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$. The operator S is called the g-frame operator of $\{\Lambda_i\}_{i\in J}$.

Definition 1.1. [5] *Let $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$ be a g-frame for $\mathscr{H}$. Then the synthesis operator for $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$ is the operator*

$$T : \bigoplus_{i\in J}\mathscr{H}_i \longrightarrow \mathscr{H},$$

defined by

$$T(\{f_i\}_{i\in J}) = \sum_{i\in J}\Lambda_i^\star(f_i).$$

We call the adjoint $T^\star$ of the synthesis operator the analysis operator. The analysis operator for $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$ is the operator

$$T^\star : \mathscr{H} \longrightarrow \bigoplus_{i\in J}\mathscr{H}_i,$$

defined by

$$T^\star(f) = \{\Lambda_i(f)\}_{i\in J}.$$

Proposition 1.1. [5] *Let $\{\Lambda_i\}_{i\in J}$ be a sequence in $\mathscr{L}(\mathscr{H},\mathscr{H}_i)$. Then the following are equivalent:*

(i) *$\{\Lambda_i\}_{i\in J}$ is a g-frame for $\mathscr{H}$;*
(ii) *The operator $T : (\{f_i\}_{i\in J}) \mapsto \sum_{i\in J}\Lambda_i^\star(f_i)$ is well-defined and bounded from $(\bigoplus_{i\in J}\mathscr{H}_i)_{l_2}$ onto $\mathscr{H}$;*
(iii) *The operator $S : f \mapsto \sum_{i\in J}\Lambda_i^\star\Lambda_i f$ is well-defined and bounded from $\mathscr{H}$ onto $\mathscr{H}$.*

In order to present the main results of this paper, we need the following Theorem which can be found in [6] and gives a characterization of g-frames.

Theorem 1.2. *Let $\{\Lambda_i\}_{i\in J}$ be a sequence in $\mathscr{L}(\mathscr{H},\mathscr{H}_i)$, $\{e_{i,k} : k \in K_i\}$ be an orthonormal basis for $\mathscr{H}_i$, $i \in J$ where K_i is a subset of $\mathbb{Z}$ and let $\psi_{i,k} = \Lambda_i^\star e_{i,k}$. Then we have the followings.*

(i) $\Lambda_i f = \sum_{k\in K_i}\langle f,\psi_{i,k}\rangle e_{i,k}$, *and* $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i\in J\}$ *is a g-frame (resp. g-Bessel sequence, tight g-frame, g-Riesz basis, g-orthonormal basis) for* $\mathscr{H}$ *if and only if* $\{\psi_{i,k} : i\in J, k\in K_i\}$ *is a frame (resp. Bessel sequence, tight frame, Riesz basis, orthonormal basis) for* $\mathscr{H}$.

(ii) *The g-frame operator for* $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i\in J\}$ *coincides with the frame operator for* $\{\psi_{i,k} : i\in J, k\in K_i\}$.

(iii) *Moreover,* $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i\in J\}$ *and* $\{\widetilde{\Lambda}_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i\in J\}$ *are a pair of (canonical) dual g-frames if and only if the induced sequences are a pair of (canonical) dual frames.*

We call $\{\psi_{i,k} : i\in J, k\in K_i\}$ the sequence induced by $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i\in J\}$ with respect to $\{e_{i,k} : k\in K_i\}$.

The paper is organized as follows. In section 2 we study the relationship between g-frames for super Hilbert spaces $\mathscr{H}\oplus\mathscr{K}$ with respect to $\mathscr{H}_i\oplus\mathscr{H}'_i$, $\mathscr{H}_i$ and g-frames for $\mathscr{H}$ and $\mathscr{K}$ with respect to $\mathscr{H}_i$ and $\mathscr{H}'_i$ and frames for $\mathscr{H}$, $\mathscr{K}$ and $\mathscr{H}\oplus\mathscr{K}$. Also we give generalized version of disjoint frames and strong complementary frame and the concepts of disjoint g-frames and strong complementary g-frame are introduced and studied. We show that strong complementary g-frames to a given g-frame are similar and strongly disjoint pairs of g-frames on the same Hilbert space have some useful additional structural properties. Also we give some other properties of g-frames.

2. Main results

The following Proposition is proved in [5, Theorem 2.20], we give another proof.

Proposition 2.1. *Let* $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i\in J\}$ *be a normalized tight g-frame for* $\mathscr{H}$ *with respect to* $\{\mathscr{H}_i\}_{i\in J}$. *Then there exists a Hilbert space* $\mathscr{K}\supseteq\mathscr{H}$ *and a g-orthonormal basis* $\{\Theta_i \in \mathscr{L}(\mathscr{K},\mathscr{H}_i) : i\in J\}$ *for* $\mathscr{K}$ *with respect to* $\{\mathscr{H}_i\}_{i\in J}$ *such that* $\Lambda_i = \Theta_i P$, *where* P *is the orthogonal projection from* $\mathscr{K}$ *onto* $\mathscr{H}$.

Proof. By using Theorem 1.2, $\Lambda_i f = \sum_{k\in K_i}\langle f,\psi_{i,k}\rangle e_{i,k}$ where $\{\psi_{i,k}\}$ is a normalized tight frame for $\mathscr{H}$, by [4, Proposition 1.1] there exists a Hilbert space $\mathscr{K}\supseteq\mathscr{H}$ and an orthonormal basis $\{\phi_{i,k}\}$ for $\mathscr{K}$ such that $\psi_{i,k} = P\phi_{i,k}$, where P is the orthogonal projection from $\mathscr{K}$ onto $\mathscr{H}$. Let $\Theta_i f = \sum_{k\in K_i}\langle f,\phi_{i,k}\rangle e_{i,k}$. By Theorem 1.2, $\{\Theta_i \in \mathscr{L}(\mathscr{K},\mathscr{H}_i) : i\in J\}$ is a g-orthonormal basis for $\mathscr{K}$ with respect to $\{\mathscr{H}_i\}_{i\in J}$, and

$$\Theta_i P f = \sum_{k\in K_i}\langle Pf,\phi_{i,k}\rangle e_{i,k} = \sum_{k\in K_i}\langle f,\psi_{i,k}\rangle e_{i,k} = \Lambda_i f. \qquad \blacksquare$$

Corollary 2.1.

(i) *Suppose that* $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i\in J\}$ *is a g-orthonormal basis for* $\mathscr{H}$, *and* V *is a partial isometry in* $\mathscr{L}(\mathscr{H})$. *Then* $\{\Lambda_i V\}$ *is a normalized tight g-frame for the range of* V.

(ii) *Let* $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i\in J\}$ *be a normalized tight g-frame for* $\mathscr{H}$ *with respect to* $\{\mathscr{H}_i\}_{i\in J}$ *and* $\{\Theta_i \in \mathscr{L}(\mathscr{K},\mathscr{H}_i) : i\in J\}$ *be a g-orthonormal basis for a Hilbert space* $\mathscr{K}$ *with respect to* $\{\mathscr{H}_i\}_{i\in J}$. *If* T *is the isometry defined by* $T(f) = \sum_{i\in J}\Theta_i^\star\Lambda_i f$, *then* $\Theta_i T = \Lambda_i$, *and* $\Lambda_i T^\star = \Theta_i P$ *for all* $i\in J$, *where* P *is the projection from* $\mathscr{K}$ *onto the range of* T. *More generally, if* $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i\in J\}$ *is a general frame for* $\mathscr{H}$, *then* T *defined above is a bounded linear operator and* $\Theta_i T = \Lambda_i$ *for all* $i\in J$.

(iii) *Suppose that* $\{\Lambda_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ *is a normalized tight g-frame for* $\mathscr{H}$ *with respect to* $\{\mathscr{H}_i\}_{i\in J}$ *and* $\{e_{i,k} : k \in K_i\}$ *is an orthonormal basis for* $\mathscr{H}_i$, $i \in J$ *where* K_i *is a subset of* $\mathbb{Z}$. *Then we have* $\sum_{i\in J}\sum_{k\in K_i} \|\Lambda_i^\star e_{i,k}\|^2$ *is equal to the dimension of* $\mathscr{H}$.

Proof. (i) Statement (i) follows from the definition.

(ii) By using Theorem 1.2, $\Lambda_i f = \sum_{k\in K_i} \langle f, \psi_{i,k}\rangle e_{i,k}$, where $\{\psi_{i,k}\}$ is a normalized tight frame for $\mathscr{H}$, and $\Theta_i^\star f_i = \sum_{k\in K_i} \langle f_i, e_{i,k}\rangle \phi_{i,k}$, where $\{\phi_{i,k}\}$ is an orthonormal basis for $\mathscr{K}$. Hence we have

$$T(f) = \sum_{i\in J} \Theta_i^\star \Lambda_i f = \sum_{i\in J}\sum_{k\in K_i} \langle f, \psi_{i,k}\rangle \phi_{i,k}.$$

Further, by [4, Corollary 1.2], $T^\star \phi_{i,k} = \psi_{i,k}$ and $T\psi_{i,k} = P\phi_{i,k}$, and it follows

$$\Theta_i T(f) = \sum_{k\in K_i} \langle Tf, \phi_{i,k}\rangle e_{i,k} = \sum_{k\in K_i} \langle f, \psi_{i,k}\rangle e_{i,k} = \Lambda_i f,$$

and

$$\Lambda_i T^\star(f) = \sum_{k\in K_i} \langle T^\star f, \psi_{i,k}\rangle e_{i,k} = \sum_{k\in K_i} \langle f, P\phi_{i,k}\rangle e_{i,k} = \Theta_i P f,$$

for all $i \in J$.

(iii) Since $\psi_{i,k} = \Lambda_i^\star e_{i,k}$, the conclusion follows from [4, Corollary 1.2]. ∎

Let $(\varphi_i, \psi_i) \in \mathscr{H} \oplus \mathscr{H}$ and $\Lambda_i f = (\langle f, \varphi_i\rangle, \langle f, \psi_i\rangle)^\perp$, $\forall f \in \mathscr{H}$, the authors in [7] have proved $\{\Lambda_i\}_{i\in J}$ is a g-frame for $\mathscr{H}$ with respect to $\mathbb{C}^2$ if and only if $\{\varphi_i\}_{i\in J} \cup \{\psi_i\}_{i\in J}$ is a frame for $\mathscr{H}$. The following Proposition is its extension to g-frames.

Proposition 2.2. *Let* $\{\Lambda_i\}_{i\in J}$ *and* $\{\Gamma_i\}_{i\in J}$ *be sequences in* $\mathscr{L}(\mathscr{H}, \mathscr{H}_i)$, $\{e_{i,k} : k \in K_i\}$ *be an orthonormal basis for* $\mathscr{H}_i$, $\Theta_i f = (\Lambda_i f, \Gamma_i f)$, $\psi_{i,k} = \Lambda_i^\star e_{i,k}$, *and* $\phi_{i,k} = \Gamma_i^\star e_{i,k}$. *Then* $\{\psi_{i,k} : i \in J, k \in K_i\} \cup \{\phi_{i,k} : i \in J, k \in K_i\}$ *is a frame (resp. Bessel sequence, tight frame, Riesz basis, orthonormal basis) for* $\mathscr{H}$ *if and only if* $\{\Theta_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i \oplus \mathscr{H}_i) : i \in J\}$ *is a g-frame (resp. g-Bessel sequence, tight g-frame, g-Riesz basis, g-orthonormal basis) for* $\mathscr{H}$.

Proof. By using Theorem 1.2 we have $\Lambda_i f = \sum_{k\in K_i} \langle f, \psi_{i,k}\rangle e_{i,k}$ and $\Gamma_i f = \sum_{k\in K_i} \langle f, \phi_{i,k}\rangle e_{i,k}$. Since

$$\begin{aligned}\sum_{i\in J} \|(\Lambda_i \oplus \Gamma_i)(f)\|^2 &= \sum_{i\in J} \|\Lambda_i(f)\|^2 + \sum_{i\in J} \|\Gamma_i(f)\|^2 \\ &= \sum_{i\in J}\sum_{k\in K_i} |\langle f, \psi_{i,k}\rangle|^2 + \sum_{i\in J}\sum_{k\in K_i} |\langle f, \phi_{i,k}\rangle|^2 \\ &= \sum_{i\in J}\sum_{k\in K_i} (|\langle f, \psi_{i,k}\rangle|^2 + |\langle f, \phi_{i,k}\rangle|^2),\end{aligned}$$

we conclude $\{\psi_{i,k} : i \in J, k \in K_i\} \cup \{\phi_{i,k} : i \in J, k \in K_i\}$ is a frame (resp. Bessel sequence, tight frame) for $\mathscr{H}$ if and only if $\{\Theta_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i \oplus \mathscr{H}_i) : i \in J\}$ is a g-frame (resp. g-Bessel sequence, tight g-frame) for $\mathscr{H}$.

Since $\{e_{i,k} : k \in K_i\}$ is an orthonormal basis for $\mathscr{H}_i$, every $f_i, g_i \in \mathscr{H}_i$ have expansions of the form $g_i = \sum_{k\in K_i} c_{i,k} e_{i,k}$ and $f_i = \sum_{k\in K_i} c'_{i,k} e_{i,k}$, where $\{c_{i,k}, c'_{i,k} : k \in K_i\} \in l^2(K_i)$. It follows that

$$\left\|\sum_{i\in I} \Theta_i^\star(g_i, f_i)\right\|^2 = \left\|\sum_{i\in I} \Lambda_i^\star g_i + \Gamma_i^\star f_i\right\|^2 = \left\|\sum_{i\in I}\sum_{k\in K_i} c_{i,k}\psi_{i,k} + c'_{i,k}\phi_{i,k}\right\|^2,$$

hence $\{\psi_{i,k} : i \in J, k \in K_i\} \cup \{\phi_{i,k} : i \in J, k \in K_i\}$ is a Riesz basis if and only if $\{\Theta_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i \oplus \mathscr{H}_i) : i \in J\}$ is a g-Riesz basis.

On the other hand if $g_{i,p}, g_{i,p'} \in \mathscr{H}_i$ and $g_{j,q}, g_{j,q'} \in \mathscr{H}_j$ we have $\Lambda_i^\star g_{i,p} = \sum_{k \in K_i} \langle g_{i,p}, e_{i,k}\rangle \psi_{i,k}$ and $\Gamma_i^\star g_{i,p'} = \sum_{k \in K_i} \langle g_{i,p'}, e_{i,k}\rangle \phi_{i,k}$,

$$\begin{aligned}
&\langle\langle \Theta_i^\star(g_{i,p}, g_{i,p'}), \Theta_j^\star(g_{j,q}, g_{j,q'})\rangle\rangle \\
&= \langle \Lambda_i^\star g_{i,p} + \Gamma_i^\star g_{i,p'}, \Lambda_j^\star g_{j,q} + \Gamma_j^\star g_{j,q'}\rangle \\
&= \langle \Lambda_i^\star g_{i,p}, \Lambda_j^\star g_{j,q}\rangle + \langle \Gamma_i^\star g_{i,p'}, \Gamma_j^\star g_{j,q'}\rangle + \langle \Lambda_i^\star g_{i,p}, \Gamma_j^\star g_{j,q'}\rangle + \langle \Gamma_i^\star g_{i,p'}, \Lambda_j^\star g_{j,q}\rangle \\
&= \sum_{k \in K_i} \sum_{l \in K_j} \langle g_{i,p}, e_{i,k}\rangle \langle e_{j,l}, g_{j,q}\rangle \langle \psi_{i,k}, \psi_{j,l}\rangle + \sum_{k \in K_i} \sum_{l \in K_j} \langle g_{i,p'}, e_{i,k}\rangle \langle e_{j,l}, g_{j,q'}\rangle \langle \phi_{i,k}, \phi_{j,l}\rangle \\
&\quad + \sum_{k \in K_i} \sum_{l \in K_j} \langle g_{i,p}, e_{i,k}\rangle \langle e_{j,l}, g_{j,q'}\rangle \langle \psi_{i,k}, \phi_{j,l}\rangle + \sum_{k \in K_i} \sum_{l \in K_j} \langle g_{i,p'}, e_{i,k}\rangle \langle e_{j,l}, g_{j,q}\rangle \langle \phi_{i,k}, \psi_{j,l}\rangle.
\end{aligned}$$

Then if $\{\psi_{i,k} : i \in J, k \in K_i\} \cup \{\phi_{i,k} : i \in J, k \in K_i\}$ is an orthonormal basis we conclude $\{\Theta_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i \oplus \mathscr{H}_i) : i \in J\}$ is a g-orthonormal basis.

If $\{\Theta_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i \oplus \mathscr{H}_i) : i \in J\}$ is a g-orthonormal basis, since $\langle \psi_{i,k}, \psi_{j,l}\rangle = \langle \Lambda_i^\star e_{i,k}, \Lambda_j^\star e_{j,l}\rangle$ and $\langle \psi_{i,k}, \phi_{j,l}\rangle = \langle \Lambda_i^\star e_{i,k}, \Gamma_j^\star e_{j,l}\rangle$ and

$$\begin{aligned}
\langle\langle \Theta_i^\star(g_{i,k}, g_{i,k'}), \Theta_j^\star(g_{j,l}, g_{j,l'})\rangle\rangle &= \langle \Lambda_i^\star g_{i,k} + \Gamma_i^\star g_{i,k'}, \Lambda_j^\star g_{j,l} + \Gamma_j^\star g_{j,l'}\rangle \\
&= \langle \Lambda_i^\star g_{i,k}, \Lambda_j^\star g_{j,l}\rangle + \langle \Gamma_i^\star g_{i,k}, \Gamma_j^\star g_{j,l'}\rangle \\
&\quad + \langle \Lambda_i^\star g_{i,k'}, \Gamma_j^\star g_{j,l}\rangle + \langle \Gamma_i^\star g_{i,k}, \Lambda_j^\star g_{j,l'}\rangle,
\end{aligned}$$

if let $g_{i,k} = g_{j,l} = 0$ and $g_{i,k'} = e_{i,k'}$, $g_{j,l'} = e_{j,l'}$ we have

$$\langle \Theta_i^\star(g_{i,k}, g_{i,k'}), \Theta_j^\star(g_{j,l}, g_{j,l'})\rangle = \langle \phi_{i,k}, \phi_{j,l}\rangle = \delta_{i,j}\delta_{k,l},$$

and if let $g_{i,k'} = g_{j,l'} = 0$ and $g_{i,k} = e_{i,k}$, $g_{j,l} = e_{j,l}$ we have

$$\langle \Theta_i^\star(g_{i,k}, g_{i,k'}), \Theta_j^\star(g_{j,l}, g_{j,l'})\rangle = \langle \psi_{i,k}, \psi_{j,l}\rangle = \delta_{i,j}\delta_{k,l},$$

also if let $g_{i,k'} = e_{i,k'}$, $g_{j,l'} = 0$ and $g_{i,k} = e_{i,k}$, $g_{j,l} = e_{j,l}$ we have

$$\delta_{i,j}\langle e_{i,k} e_{j,l}\rangle = \langle \Theta_i^\star(g_{i,k}, g_{i,k'}), \Theta_j^\star(g_{j,l}, g_{j,l'})\rangle = \langle \psi_{i,k}, \psi_{j,l}\rangle + \langle \psi_{i,k}, \phi_{j,l}\rangle.$$

Then $\{\psi_{i,k} : i \in J, k \in K_i\} \cup \{\phi_{i,k} : i \in J, k \in K_i\}$ is an orthonormal basis for $\mathscr{H}$. ∎

The following is an immediate consequence.

Corollary 2.2. *Let $\{\Lambda_i\}_{i \in J}$ and $\{\Gamma_i\}_{i \in J}$ be sequences in $\mathscr{L}(\mathscr{H}, \mathscr{H}_i)$ and $\Theta_i f = (\Lambda_i f, \Gamma_i f)$. Then $\{\Lambda_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\} \cup \{\Gamma_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ is a g-frame (resp. g-Bessel sequence, tight g-frame, g-Riesz basis, g-orthonormal basis) for $\mathscr{H}$ if and only if $\{\Theta_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i \oplus \mathscr{H}_i) : i \in J\}$ is a g-frame (resp. g-Bessel sequence, tight g-frame, g-Riesz basis, g-orthonormal basis) for $\mathscr{H}$.*

Putting Definition 1.2 and Corollary 2.2 together, we get

Proposition 2.3. *Let $\{\Theta_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i \oplus \mathscr{H}_i) : i \in J\}$, $\{\Lambda_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ and $\{\Gamma_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ be g-frames where $\Theta_i(f) = (\Lambda_i f, \Gamma_i f)$. Then the synthesis operator for $\{\Theta_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i \oplus \mathscr{H}_i) : i \in J\}$ is the operator*

$$T : \bigoplus_{i \in J} (\mathscr{H}_i \oplus \mathscr{H}_i) \longrightarrow \mathscr{H},$$

defined by

$$T(\{(f_i,g_i)\}_{i\in J})=\sum_{i\in J}(\Lambda_i^\star f_i+\Gamma_i^\star g_i).$$

The analysis operator for $\{\Theta_i\in\mathscr{L}(\mathscr{H},\mathscr{H}_i\oplus\mathscr{H}_i):i\in J\}$ *is the operator*

$$T^\star:\mathscr{H}\longrightarrow\bigoplus_{i\in J}(\mathscr{H}_i\oplus\mathscr{H}_i),$$

defined by

$$T^\star f=\{(\Lambda_i f,\Gamma_i f)\}_{i\in J}.$$

Also the g-frame operator for $\{\Theta_i\in\mathscr{L}(\mathscr{H},\mathscr{H}_i\oplus\mathscr{H}_i):i\in J\}$ *is the operator*

$$S_\Theta:\mathscr{H}\to\mathscr{H}$$

defined by

$$S_\Theta f=\sum_{i\in J}\Lambda_i^\star\Lambda_i f+\sum_{i\in J}\Gamma_i^\star\Gamma_i f=S_\Lambda f+S_\Gamma f.$$

Proposition 2.4. *Let* $\{\Lambda_i\}_{i\in J}$ *and* $\{\Gamma_i\}_{i\in J}$ *be sequences in* $\mathscr{L}(\mathscr{H},\mathscr{H}_i)$ *and* $\mathscr{L}(\mathscr{K},\mathscr{H'}_i)$ *respectively. Then* $\{\Lambda_i\in\mathscr{L}(\mathscr{H},\mathscr{H}_i):i\in J\}$ *and* $\{\Gamma_i\in\mathscr{L}(\mathscr{K},\mathscr{H'}_i):i\in J\}$ *are g-frames (resp. g-Bessel sequences, tight g-frames, g-Riesz bases, g-orthonormal bases) for* $\mathscr{H}$ *and* $\mathscr{K}$ *if and only if* $\{\Lambda_i\oplus\Gamma_i\in\mathscr{L}(\mathscr{H}\oplus\mathscr{K},\mathscr{H}_i\oplus\mathscr{H'}_i):i\in J\}$ *is a g-frame (resp. g-Bessel sequence, tight g-frame, g-Riesz basis, g-orthonormal basis) for* $\mathscr{H}\oplus\mathscr{K}$.

Proof. Since

$$\sum_{i\in J}\|(\Lambda_i\oplus\Gamma_i)(f,g)\|^2=\sum_{i\in J}\|\Lambda_i(f)\|^2+\sum_{i\in J}\|\Gamma_i(g)\|^2,$$

then $\{\Lambda_i\in\mathscr{L}(\mathscr{H},\mathscr{H}_i):i\in J\}$ and $\{\Gamma_i\in\mathscr{L}(\mathscr{K},\mathscr{H'}_i):i\in J\}$ are g-frames (resp. g-Bessel sequences, tight g-frames) for $\mathscr{H}$ and $\mathscr{K}$ if and only if $\{\Lambda_i\oplus\Gamma_i\in\mathscr{L}(\mathscr{H}\oplus\mathscr{K},\mathscr{H}_i\oplus\mathscr{H'}_i):i\in J\}$ is a g-frame (resp. g-Bessel sequence, tight g-frame) for $\mathscr{H}\oplus\mathscr{K}$. For any finite subset $I\subseteq J$, $f_i\in\mathscr{H}_i'$, $g_i\in\mathscr{H}_i$ we have

$$\begin{aligned}\left\|\sum_{i\in I}(\Lambda_i\oplus\Gamma_i)^\star(g_i,f_i)\right\|^2&=\left\|\sum_{i\in I}(\Lambda_i^\star g_i,\Gamma_i^\star f_i)\right\|^2=\left\|(\sum_{i\in I}\Lambda_i^\star g_i,\sum_{i\in I}\Gamma_i^\star f_i)\right\|^2\\&=\left\|\sum_{i\in I}\Lambda_i^\star g_i\right\|^2+\left\|\sum_{i\in I}\Gamma_i^\star f_i\right\|^2,\end{aligned}$$

and so $\{\Lambda_i\in\mathscr{L}(\mathscr{H},\mathscr{H}_i):i\in J\}$ and $\{\Gamma_i\in\mathscr{L}(\mathscr{K},\mathscr{H'}_i):i\in J\}$ are g-Riesz bases for $\mathscr{H}$ and $\mathscr{K}$ if and only if $\{\Lambda_i\oplus\Gamma_i\in\mathscr{L}(\mathscr{H}\oplus\mathscr{K},\mathscr{H}_i\oplus\mathscr{H'}_i):i\in J\}$ is a g-Riesz basis for $\mathscr{H}\oplus\mathscr{K}$.

On the other hand if $g_{i,l}\in\mathscr{H}_i$, $g_{i,k}\in\mathscr{H}_i'$ and $g_{j,l'}\in\mathscr{H}_j$, $g_{j,k'}\in\mathscr{H}_j'$ we have

$$\begin{aligned}\langle(\Lambda_i\oplus\Gamma_i)^\star(g_{i,l},g_{i,k}),(\Lambda_j\oplus\Gamma_j)^\star(g_{j,l'},g_{j,k'})\rangle&=\langle(\Lambda_i^\star g_{i,l},\Gamma_i^\star g_{i,k}),(\Lambda_j^\star g_{j,l'},\Gamma_j^\star g_{j,k'})\rangle\\&=\langle\Lambda_i^\star g_{i,l},\Lambda_j^\star g_{j,l'}\rangle+\langle\Gamma_i^\star g_{i,k},\Gamma_j^\star g_{j,k'}\rangle\end{aligned}$$

then $\{\Lambda_i\in\mathscr{L}(\mathscr{H},\mathscr{H}_i):i\in J\}$ and $\{\Gamma_i\in\mathscr{L}(\mathscr{K},\mathscr{H'}_i):i\in J\}$ are g-orthonormal bases for $\mathscr{H}$ and $\mathscr{K}$ if and only if $\{\Lambda_i\oplus\Gamma_i\in\mathscr{L}(\mathscr{H}\oplus\mathscr{K},\mathscr{H}_i\oplus\mathscr{H'}_i):i\in J\}$ is a g-orthonormal basis for $\mathscr{H}\oplus\mathscr{K}$. ∎

Corollary 2.3. *A set $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$ is a normalized tight g-frame for $\mathscr{H}$ with respect to $\{\mathscr{H}_i\}_{i\in J}$ if and only if there exists a Hilbert space $\mathscr{M}$ and a normalized tight g-frame $\{\Gamma_i \in \mathscr{L}(\mathscr{M},\mathscr{H}_i) : i \in J\}$ for $\mathscr{M}$ such that $\{\Lambda_i \oplus \Gamma_i \in \mathscr{L}(\mathscr{H}\oplus\mathscr{M},\mathscr{H}_i\oplus\mathscr{H}_i) : i \in J\}$ is a normalized tight g-frame for $\mathscr{H}\oplus\mathscr{M}$ with respect to $\{\mathscr{H}_i\oplus\mathscr{H}_i\}_{i\in J}$ and we have $\langle(\Lambda_i\oplus\Gamma_i)^\star(f_i,f_i),(\Lambda_j\oplus\Gamma_j)^\star(f_j,f_j)\rangle = \delta_{i,j}\langle f_i,f_j\rangle$.*

Proof. If $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$ is a normalized tight g-frame for $\mathscr{H}$, by using Proposition 2.1, there exists a Hilbert space $\mathscr{K}\supseteq\mathscr{H}$ and a g-orthonormal basis $\{\Theta_i \in \mathscr{L}(\mathscr{K},\mathscr{H}_i) : i \in J\}$ for $\mathscr{K}$ such that $\Lambda_i = \Theta_i P$, where P is the orthogonal projection from $\mathscr{K}$ onto $\mathscr{H}$. Let $\mathscr{M} = (I-P)\mathscr{K}$ and $\Gamma_i = \Theta_i(I-P)$. Then, by Proposition 2.4, $\{\Lambda_i\oplus\Gamma_i \in \mathscr{L}(\mathscr{H}\oplus\mathscr{M},\mathscr{H}_i\oplus\mathscr{H}_i) : i \in J\}$ is a normalized tight g-frame for $\mathscr{H}\oplus\mathscr{M}$. Also we have

$$\begin{aligned}\langle(\Lambda_i\oplus\Gamma_i)^\star(f_i,f_i),(\Lambda_j\oplus\Gamma_j)^\star(f_j,f_j)\rangle &= \langle\Lambda_i^\star f_i,\Lambda_j^\star f_j\rangle + \langle\Gamma_i^\star f_i,\Gamma_j^\star f_j\rangle\\ &= \langle P\Theta_i^\star f_i, P\Theta_j^\star f_j\rangle + \langle(I-P)\Theta_i^\star f_i,(I-P)\Theta_j^\star f_j\rangle\\ &= \langle\Theta_i^\star f_i,\Theta_j^\star f_j\rangle = \delta_{i,j}\langle f_i,f_j\rangle.\end{aligned}$$ ∎

Proposition 2.5. *Let $\{\Theta_i \in \mathscr{L}(\mathscr{H}\oplus\mathscr{K},\mathscr{H}_i\oplus\mathscr{H}'_i) : i \in J\}$, $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$ and $\{\Gamma_i \in \mathscr{L}(\mathscr{K},\mathscr{H}'_i) : i \in J\}_{i\in J}$ be g-frames where $\Theta_i(f,g) = (\Lambda_i f,\Gamma_i g)$. Then the synthesis operator for $\{\Theta_i \in \mathscr{L}(\mathscr{H}\oplus\mathscr{K},\mathscr{H}_i\oplus\mathscr{H}'_i) : i \in J\}$ is the operator*

$$T : \bigoplus_{i\in J}(\mathscr{H}_i\oplus\mathscr{H}'_i) \longrightarrow \mathscr{H}\oplus\mathscr{K},$$

defined by

$$T(\{(f_i,g_i)\}_{i\in J}) = (\sum_{i\in J}\Lambda_i^\star f_i, \sum_{i\in J}\Gamma_i^\star g_i).$$

The analysis operator for $\{\Theta_i \in \mathscr{L}(\mathscr{H}\oplus\mathscr{K},\mathscr{H}_i\oplus\mathscr{H}'_i) : i \in J\}$ is the operator

$$T^\star : \mathscr{H}\oplus\mathscr{K} \longrightarrow \bigoplus_{i\in J}(\mathscr{H}_i\oplus\mathscr{H}'_i),$$

defined by

$$T^\star(f,g) = \{(\Lambda_i f,\Gamma_i g)\}_{i\in J}.$$

Also the g-frame operator for $\{\Theta_i \in \mathscr{L}(\mathscr{H}\oplus\mathscr{K},\mathscr{H}_i\oplus\mathscr{H}'_i) : i \in J\}$ is the operator

$$S_\Theta : \mathscr{H}\oplus\mathscr{K} \to \mathscr{H}\oplus\mathscr{K},$$

defined by

$$S_\Theta(f,g) = (\sum_{i\in J}\Lambda_i^\star\Lambda_i f, \sum_{i\in J}\Gamma_i^\star\Gamma_i g) = (S_\Lambda f, S_\Gamma g).$$

Proposition 2.6. *Let $\{\Lambda_i\}_{i\in J}$ and $\{\Gamma_i\}_{i\in J}$ be sequences in $\mathscr{L}(\mathscr{H},\mathscr{H}_i)$ and $\mathscr{L}(\mathscr{K},\mathscr{H}_i)$ respectively, and $\{(e'_{i,k},e''_{i,k}) : k \in K_i\}$ be an orthonormal basis for $\mathscr{H}_i\oplus\mathscr{H}_i$, $i \in J$ where K_i is a subset of $\mathbb{Z}$ and let $\psi_{i,k} = \Lambda_i^\star e'_{i,k}$, $\phi_{i,k} = \Gamma_i^\star e''_{i,k}$. Then $\{(\psi_{i,k},\phi_{i,k}) : i \in J, k \in K_i\}$ is a frame (resp. Bessel sequence, tight frame, Riesz basis, orthonormal basis) for $\mathscr{H}\oplus\mathscr{K}$ if and only if $\{\Lambda_i\oplus\Gamma_i \in \mathscr{L}(\mathscr{H}\oplus\mathscr{K},\mathscr{H}_i\oplus\mathscr{H}_i) : i \in J\}$ is a g-frame (resp. g-Bessel sequence, tight g-frame, g-Riesz basis, g-orthonormal basis) for $\mathscr{H}\oplus\mathscr{K}$.*

Proof. Since $(\Lambda_i\oplus\Gamma_i)^\star = \Lambda_i^\star\oplus\Gamma_i^\star$ and $\{(e'_{i,k},e''_{i,k}) : k \in K_i\}$ is an orthonormal basis for $\mathscr{H}_i\oplus\mathscr{H}_i$, $i \in J$ we have $(\Lambda_i^\star\oplus\Gamma_i^\star)(e'_{i,k},e''_{i,k}) = (\psi_{i,k},\phi_{i,k})$, and hence $\{(\psi_{i,k},\phi_{i,k}) : i \in J, k \in K_i\}$ is the sequence induced by $\{\Lambda_i\oplus\Gamma_i \in \mathscr{L}(\mathscr{H}\oplus\mathscr{K},\mathscr{H}_i\oplus\mathscr{H}_i) : i \in J\}$ with respect to $\{(e'_{i,k},e''_{i,k}) : i \in J, k \in K_i\}$. So, by Theorem 1.2, $\{(\psi_{i,k},\phi_{i,k}) : i \in J, k \in K_i\}$ is a frame (resp.

Bessel sequence, tight frame, Riesz basis, orthonormal basis) for $\mathscr{H}\oplus\mathscr{K}$ if and only if $\{\Lambda_i\oplus\Gamma_i\in\mathscr{L}(\mathscr{H}\oplus\mathscr{K},\mathscr{H}_i\oplus\mathscr{H}_i):i\in J\}$ is a g-frame (resp. g-Bessel sequence, tight g-frame, g-Riesz basis, g-orthonormal basis) for $\mathscr{H}\oplus\mathscr{K}$. ∎

Proposition 2.7.

(i) *If T is a co-isometry and $\{\Lambda_i\in\mathscr{L}(\mathscr{H},\mathscr{H}_i):i\in J\}$ is a g-frame for $\mathscr{H}$, then $\{\Lambda_i T^\star\}_{i\in J}$ is a g-frame. Moreover, $\{\Lambda_i T^\star\}_{i\in J}$ is a normalized tight g-frame if $\{\Lambda_i\}_{i\in J}$ is.*

(ii) *Suppose that $\{\Lambda_i\in\mathscr{L}(\mathscr{H},\mathscr{H}_i):i\in J\}$ and $\{\Gamma_i\in\mathscr{L}(\mathscr{H},\mathscr{H}_i):i\in J\}$ are normalized tight g-frames, and suppose that T is a bounded linear operator which satisfies $\Lambda_i T^\star=\Gamma_i$ for all $i\in J$. Then T is a co-isometry. If T is invertible, then it is unitary.*

Proof. (i) Since T is a co-isometry, $T^\star$ is an isometry. Hence, for all $f\in\mathscr{H}$,

$$A\|f\|^2=A\|T^\star f\|^2\le\sum_{i\in J}\|\Lambda_i T^\star f\|^2\le B\|T^\star f\|^2=B\|f\|^2.$$

(ii) If $\{\Lambda_i\in\mathscr{L}(\mathscr{H},\mathscr{H}_i):i\in J\}$ and $\{\Gamma_i\in\mathscr{L}(\mathscr{H},\mathscr{H}_i):i\in J\}$ are normalized tight g-frames, the induced sequences $\{\psi_{i,k}:i\in J,k\in K_i\}$ and $\{\phi_{i,k}:i\in J,k\in K_i\}$ by $\{\Lambda_i\}_{i\in J}$ and $\{\Gamma_i\}_{i\in J}$ respectively, are normalized tight frames. Since $\Lambda_i T^\star=\Gamma_i$ we conclude $T\psi_{i,k}=\phi_{i,k}$, and so by Proposition 1.9(ii) of [4], T is a co-isometry and if T is invertible, then it is unitary. ∎

Proposition 2.8. *Let $\{\Lambda_i\}_{i\in J}$ and $\{\Gamma_i\}_{i\in J}$ be sequences in $\mathscr{L}(\mathscr{H},\mathscr{H}_i)$ and $\mathscr{L}(\mathscr{K},\mathscr{H}_i)$ respectively, and $\{e_{i,k}:k\in K_i\}$ be an orthonormal basis for $\mathscr{H}_i$, $i\in J$ where K_i is a subset of $\mathbb{Z}$ and let $\psi_{i,k}=\Lambda_i^\star e_{i,k}$, $\phi_{i,k}=\Gamma_i^\star e_{i,k}$ and $\Theta_i(f,g)=\Lambda_i f+\Gamma_i g$. Then $\{(\psi_{i,k},\phi_{i,k}):i\in J,k\in K_i\}$ is a frame (resp. Bessel sequence, tight frame, Riesz basis, orthonormal basis) for $\mathscr{H}\oplus\mathscr{K}$ if and only if $\{\Theta_i\in\mathscr{L}(\mathscr{H}\oplus\mathscr{K},\mathscr{H}_i):i\in J\}$ is a g-frame (resp. g-Bessel sequence, tight g-frame, g-Riesz basis, g-orthonormal basis) for $\mathscr{H}\oplus\mathscr{K}$ with respect to $\{\mathscr{H}_i\}_{i\in J}$.*

Proof. Since $\Theta_i^\star e_{i,k}=(\psi_{i,k},\phi_{i,k})$, by using Theorem 1.2, we have $\{(\psi_{i,k},\phi_{i,k}):i\in J,k\in K_i\}$ is a frame (resp. Bessel sequence, tight frame, Riesz basis, orthonormal basis) for $\mathscr{H}\oplus\mathscr{K}$ if and only if $\{\Theta_i\in\mathscr{L}(\mathscr{H}\oplus\mathscr{K},\mathscr{H}_i):i\in J\}$ is a g-frame (resp. g-Bessel sequence, tight g-frame, g-Riesz basis, g-orthonormal basis) for $\mathscr{H}\oplus\mathscr{K}$ with respect to $\{\mathscr{H}_i\}_{i\in J}$. ∎

Proposition 2.9. *Let $\{\Theta_i\in\mathscr{L}(\mathscr{H}\oplus\mathscr{K},\mathscr{H}_i):i\in J\}$, $\{\Lambda_i\in\mathscr{L}(\mathscr{H},\mathscr{H}_i):i\in J\}$ and $\{\Gamma_i\in\mathscr{L}(\mathscr{K},\mathscr{H}_i):i\in J\}_{i\in J}$ be g-frames where $\Theta_i(f,g)=\Lambda_i f+\Gamma_i g$. Then the synthesis operator for $\{\Theta_i\in\mathscr{L}(\mathscr{H}\oplus\mathscr{K},\mathscr{H}_i):i\in J\}$ is the operator*

$$T:\bigoplus_{i\in J}\mathscr{H}_i\longrightarrow\mathscr{H}\oplus\mathscr{K},$$

defined by

$$T(\{f_i\}_{i\in J})=(\sum_{i\in J}\Lambda_i^\star f_i,\sum_{i\in J}\Gamma_i^\star f_i).$$

The analysis operator for $\{\Theta_i\in\mathscr{L}(\mathscr{H}\oplus\mathscr{K},\mathscr{H}_i):i\in J\}$ is the operator

$$T^\star:\mathscr{H}\oplus\mathscr{K}\longrightarrow\bigoplus_{i\in J}\mathscr{H}_i,$$

defined by

$$T^\star(f,g)=\{(\Lambda_i f+\Gamma_i g)\}_{i\in J}.$$

Also the g-frame operator for $\{\Theta_i \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K}, \mathscr{H}_i) : i \in J\}$ *is the operator*

$$S_\Theta : \mathscr{H} \oplus \mathscr{K} \to \mathscr{H} \oplus \mathscr{K}$$

defined by

$$S_\Theta(f,g) = (\sum_{i\in J}(\Lambda_i^\star \Lambda_i f + \Lambda_i^\star \Gamma_i g), \sum_{i\in J}(\Gamma_i^\star \Lambda_i f + \Gamma_i^\star \Gamma_i g)).$$

We say that the g-frame pairs $(\{\Lambda_i\},\{\Gamma_i\})$ and $(\{\Upsilon_i\},\{\Theta_i\})$ are similar if there are bounded invertible operators T_1 and T_2 such that $\Lambda_i = \Upsilon_i T_1$ and $\Gamma_i = \Theta_i T_2$ for all $i \in J$. A pair of g-frames $\{\Lambda_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ and $\{\Gamma_i \in \mathscr{L}(\mathscr{K}, \mathscr{H}_i) : i \in J\}_{i\in J}$ is called disjoint if $\{\Theta_i \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K}, \mathscr{H}_i) : i \in J\}$ where $\Theta_i(f,g) = \Lambda_i f + \Gamma_i g$ is a g-frame for $\mathscr{H} \oplus \mathscr{K}$. A pair of normalized tight g-frames $\{\Lambda_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ and $\{\Gamma_i \in \mathscr{L}(\mathscr{K}, \mathscr{H}_i) : i \in J\}_{i\in J}$ is called strongly disjoint if $\{\Theta_i \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K}, \mathscr{H}_i) : i \in J\}$ is a normalized tight g-frame for $\mathscr{H} \oplus \mathscr{K}$, and a pair of general g-frames $\{\Lambda_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ and $\{\Gamma_i \in \mathscr{L}(\mathscr{K}, \mathscr{H}_i) : i \in J\}$ is called strongly disjoint if it is similar to a strongly disjoint pair of normalized tight g-frames.

Corollary 2.4. *A pair of g-frames* $\{\Lambda_i \in \mathscr{L}(\mathscr{K}, \mathscr{H}_i) : i \in J\}$ *and* $\{\Gamma_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}_{i\in J}$ *is disjoint (resp. strongly disjoint) if and only if* $\{\psi_{i,k} : i \in J, k \in K_i\}$ *and* $\{\phi_{i,k} : i \in J, k \in K_i\}$ *is a pair of disjoint (resp. strongly disjoint) frames where* $\psi_{i,k} = \Lambda_i^\star e_{i,k}$ *and* $\phi_{i,k} = \Gamma_i^\star e_{i,k}$.

Corollary 2.5. *If a pair of normalized tight frames* $\{\Lambda_i \in \mathscr{L}(\mathscr{K}, \mathscr{H}_i) : i \in J\}$ *and* $\{\Gamma_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}_{i\in J}$ *is strongly disjoint, then for all* $f \in \mathscr{H}$ *and* $g \in \mathscr{K}$ *we have*

$$\sum_{i\in J}\Lambda_i^\star \Gamma_i g = 0\ ,\ \sum_{i\in J}\Gamma_i^\star \Lambda_i f = 0.$$

Han and Larson in [4] have proved that $\{\varphi_i\}_{i\in J}$ is a normalized tight frame in a Hilbert space $\mathscr{H}$ if and only if there is a Hilbert space $\mathscr{K}$ and a normalized tight frame $\{\psi_i\}_{i\in J}$ in $\mathscr{K}$ such that $\{(\varphi_i, \psi_i)\}_{i\in J}$ is an orthonormal basis for $\mathscr{H} \oplus \mathscr{K}$. We extend this this result for g-frames.

Proposition 2.10. *A set* $\{\Lambda_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ *is a normalized tight g-frame for* $\mathscr{H}$ *with respect to* $\{\mathscr{H}_i\}_{i\in J}$ *if and only if there exists a Hilbert space* $\mathscr{K}$ *and a normalized tight g-frame* $\{\Gamma_i \in \mathscr{L}(\mathscr{K}, \mathscr{H}_i) : i \in J\}$ *for* $\mathscr{K}$ *such that* $\{\Theta_i \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K}, \mathscr{H}_i) : i \in J\}$*, where* $\Theta_i(f,g) = \Lambda_i f + \Gamma_i g$ *is a g-orthonormal basis for* $\mathscr{H} \oplus \mathscr{K}$ *with respect to* $\{\mathscr{H}_i\}_{i\in J}$.

Proof. If $\{\Lambda_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ is a normalized tight g-frame for $\mathscr{H}$ then by using Theorem 1.2, we have $\Lambda_i f = \sum_{k\in K_i}\langle f, \psi_{i,k}\rangle e_{i,k}$ where $\{\psi_{i,k}\}$ is a normalized tight frame for $\mathscr{H}$ and, by Corollary 1.3 of [4], there exists a Hilbert space $\mathscr{K}$ and a normalized tight frame $\{\phi_{i,k} : i \in J, k \in K_i\}$ for $\mathscr{K}$ such that $\{(\psi_{i,k}, \phi_{i,k}) : i \in J, k \in K_i\}$ is an orthonormal basis for $\mathscr{H} \oplus \mathscr{K}$. If $\Gamma_i f = \sum_{k\in K_i}\langle f, \phi_{i,k}\rangle e_{i,k}$, by Theorem 1.2, $\{\Gamma_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ is a normalized tight frame for $\mathscr{K}$ with respect to $\{\mathscr{H}_i\}_{i\in J}$. So, by Proposition 2.8, $\{\Theta_i \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K}, \mathscr{H}_i) : i \in J\}$ is a g-orthonormal basis for $\mathscr{H} \oplus \mathscr{K}$ with respect to $\{\mathscr{H}_i\}_{i\in J}$. ∎

Proposition 2.11. *The extension of a tight g-frame to a g-orthonormal basis described in the statement of Proposition 2.10 is unique up to unitary equivalence. That is if* $\mathscr{N}$ *is another Hilbert space and* $\{\Upsilon_i \in \mathscr{L}(\mathscr{N}, \mathscr{H}_i) : i \in J\}$ *is a tight g-frame for* $\mathscr{N}$ *such that* $\{\Lambda_i \oplus \Upsilon_i \in \mathscr{L}(\mathscr{H} \oplus \mathscr{N}, \mathscr{H}_i) : i \in J\}$ *is a g-orthonormal basis for* $\mathscr{H} \oplus \mathscr{N}$ *with respect to* $\{\mathscr{H}_i\}_{i\in J}$*, then there is a unitary transformation U mapping* $\mathscr{K}$ *onto* $\mathscr{N}$ *such that* $\Gamma_i U^\star = \Upsilon_i$ *for all* $i \in J$*. In particular,* $dim\mathscr{K} = dim\mathscr{N}$.

Proof. If $\{\Upsilon_i \in \mathscr{L}(\mathscr{N},\mathscr{H}_i) : i \in J\}$ is a normalized tight g-frame for $\mathscr{N}$ then, by using Theorem 1.2, we have $\Upsilon_i f = \sum_{k\in K_i}\langle f,\varphi_{i,k}\rangle e_{i,k}$ where $\{\varphi_{i,k}\}$ is a normalized tight frame for $\mathscr{N}$ and by Proposition 2.8 we know $\{(\psi_{i,k},\varphi_{i,k}) : i \in J, k \in K_i\}$ is an orthonormal basis for $\mathscr{H}\oplus\mathscr{N}$ so by [4, Corollary 1.4] there is a unitary transformation U mapping $\mathscr{K}$ onto $\mathscr{N}$ such that $U\phi_{i,k} = \varphi_{i,k}$ and $dim\mathscr{K} = dim\mathscr{N}$. Therefore

$$\Gamma_i U^\star f = \sum_{k\in K_i}\langle U^\star f,\phi_{i,k}\rangle e_{i,k} = \sum_{k\in K_i}\langle f,U\phi_{i,k}\rangle e_{i,k} = \sum_{k\in K_i}\langle f,\varphi_{i,k}\rangle e_{i,k} = \Upsilon_i f. \qquad \blacksquare$$

Proposition 2.12. *If a set $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$ is a g-frame then there exists a Hilbert space $\mathscr{K}$ and a normalized tight g-frame $\{\Gamma_i \in \mathscr{L}(\mathscr{K},\mathscr{H}_i) : i \in J\}$ for $\mathscr{K}$ such that $\{\Theta_i(f,g) \in \mathscr{L}(\mathscr{H}\oplus\mathscr{K},\mathscr{H}_i) : i \in J\}$, where $\Theta_i(f,g) = \Lambda_i f + \Gamma_i g$ is a g-Riesz basis for $\mathscr{H}\oplus\mathscr{K}$ with respect to $\{\mathscr{H}_i\}_{i\in J}$.*

Proof. If $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$ is a g-frame for $\mathscr{H}$ by using Theorem 1.2, we have $\Lambda_i f = \sum_{k\in K_i}\langle f,\psi_{i,k}\rangle e_{i,k}$, where $\{\psi_{i,k}\}$ is a frame for $\mathscr{H}$ and by [4, Corollary 1.6] there exists a Hilbert space $\mathscr{K}$ and a normalized tight frame $\{\phi_{i,k} : i \in J, k \in K_i\}$ for $\mathscr{K}$ such that $\{(\psi_{i,k},\phi_{i,k}) : i \in J, k \in K_i\}$ is a Riesz basis for $\mathscr{H}\oplus\mathscr{K}$. Now set $\Gamma_i f = \sum_{k\in K_i}\langle f,\phi_{i,k}\rangle e_{i,k}$. By Theorem 1.2, $\{\Gamma_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$ is a normalized tight g-frame for $\mathscr{K}$ with respect to $\{\mathscr{H}_i\}_{i\in J}$, and so by Proposition 2.8, we have $\{\Theta_i \in \mathscr{L}(\mathscr{H}\oplus\mathscr{K},\mathscr{H}_i) : i \in J\}$ is a g-Riesz basis for $\mathscr{H}\oplus\mathscr{K}$ with respect to $\{\mathscr{H}_i\}_{i\in J}$. $\blacksquare$

Proposition 2.13. *If $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$ and $\{\Gamma_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$ are normalized tight g-frames such that $\{\Theta_i \in \mathscr{L}(\mathscr{H}\oplus\mathscr{H},\mathscr{H}_i) : i \in J\}$, where $\Theta_i(f,g) = \Lambda_i f + \Gamma_i g$, is a normalized tight g-frame for $\mathscr{H}\oplus\mathscr{H}$ with respect to $\{\mathscr{H}_i\}_{i\in J}$ and if $\{\Upsilon_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$ is a normalized tight frame which is unitarily equivalent to $\{\Gamma_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$, then $\{\Omega_i \in \mathscr{L}(\mathscr{H}\oplus\mathscr{H},\mathscr{H}_i) : i \in J\}$, where $\Omega_i(f,g) = \Lambda_i f + \Upsilon_i g$, is also a normalized tight g-frame for $\mathscr{H}\oplus\mathscr{H}$ with respect to $\{\mathscr{H}_i\}_{i\in J}$.*

Proof. We know that the induced sequences $\{\psi_{i,k} : i \in J, k \in K_i\}$, $\{\phi_{i,k} : i \in J, k \in K_i\}$ and $\{\varphi_{i,k} : i \in J, k \in K_i\}$ by $\{\Lambda_i\}_{i\in J}$, $\{\Gamma_i\}_{i\in J}$ and $\{\Upsilon_i\}_{i\in J}$ respectively, are normalized tight frames and $\{\phi_{i,k} : i \in J, k \in K_i\}$ unitarily equivalent to $\{\varphi_{i,k} : i \in J, k \in K_i\}$. Since $\{\Theta_i \in \mathscr{L}(\mathscr{H}\oplus\mathscr{H},\mathscr{H}_i) : i \in J\}$ is a normalized tight g-frame for $\mathscr{H}\oplus\mathscr{H}$ with respect to $\{\mathscr{H}_i\}_{i\in J}$ by Proposition 2.8, $\{(\psi_{i,k},\phi_{i,k}) : i \in J, k \in K_i\}$ is normalized tight frame for $\mathscr{H}\oplus\mathscr{H}$ and so by Corollary 1.9 of [4] $\{(\psi_{i,k},\varphi_{i,k}) : i \in J, k \in K_i\}$ is normalized tight frame for $\mathscr{H}\oplus\mathscr{H}$ and therefore $\{\Omega_i \in \mathscr{L}(\mathscr{H}\oplus\mathscr{H},\mathscr{H}_i) : i \in J\}$ is also a normalized tight g-frame for $\mathscr{H}\oplus\mathscr{H}$ with respect to $\{\mathscr{H}_i\}_{i\in J}$.

Let $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$ be a normalized tight g-frame for $\mathscr{H}$ with respect to $\{\mathscr{H}_i\}_{i\in J}$. By Proposition 2.10 there exists a Hilbert space $\mathscr{K}$ and a normalized tight g-frame $\{\Gamma_i \in \mathscr{L}(\mathscr{K},\mathscr{H}_i) : i \in J\}$ for $\mathscr{K}$ such that $\{\Theta_i(f,g) \in \mathscr{L}(\mathscr{H}\oplus\mathscr{K},\mathscr{H}_i) : i \in J\}$ where $\Theta_i(f,g) = \Lambda_i f + \Gamma_i g$ is a g-orthonormal basis for $\mathscr{H}\oplus\mathscr{K}$ with respect to $\{\mathscr{H}_i\}_{i\in J}$. We will call $\{\Gamma_i \in \mathscr{L}(\mathscr{K},\mathscr{H}_i) : i \in J\}$ a strong complementary g-frame to $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$, and we will call $(\{\Lambda_i\},\{\Gamma_i\})$ a strong complementary pair. Proposition 2.11 says that the strong complement of a normalized tight g-frame is unique up to unitary equivalence. If $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$ is a g-frame, we will define a strong complement to $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i \in J\}$ be any g-frame $\{\Gamma_i \in \mathscr{L}(\mathscr{K},\mathscr{H}_i) : i \in J\}$ such that the pair $(\{\Lambda_i\},\{\Gamma_i\})$ is similar to a strong complementary pair of normalized tight g-frames. $\blacksquare$

Corollary 2.6. *A pair of g-frames* $\{\Lambda_i \in \mathscr{L}(\mathscr{K}, \mathscr{H}_i) : i \in J\}$ *and* $\{\Gamma_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}_{i\in J}$ *is strong complementary pair of g-frames if and only if* $\{\psi_{i,k} : i \in J, k \in K_i\}$ *and* $\{\phi_{i,k} : i \in J, k \in K_i\}$ *is complementary pair of frames.*

Proposition 2.14. *Let* $\{\Lambda_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ *be a g-frame for* $\mathscr{H}$ *and* $\{\Gamma_i \in \mathscr{L}(\mathscr{M}, \mathscr{H}_i) : i \in J\}$, $\{\Theta_i \in \mathscr{L}(\mathscr{N}, \mathscr{H}_i) : i \in J\}$ *be strong complementary g-frame to* $\{\Lambda_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ *in Hilbert spaces* $\mathscr{M}$ *and* $\mathscr{N}$*, respectively. Then there exists an invertible operator* $T \in \mathscr{L}(\mathscr{M}, \mathscr{N})$ *such that* $\Theta_i = \Gamma_i T^\star$.

Proof. By using Theorem 1.2, we have

$$\Lambda_i f = \sum_{k\in K_i} \langle f, \psi_{i,k}\rangle e_{i,k} \text{ and } \Gamma_i g = \sum_{k\in K_i} \langle g, \phi_{i,k}\rangle e_{i,k} \text{ and } \Theta_i h = \sum_{k\in K_i} \langle h, \varphi_{i,k}\rangle e_{i,k},$$

where $\{\psi_{i,k}\}$, $\{\phi_{i,k}\}$ and $\{\varphi_{i,k}\}$ are frames in $\mathscr{H}$, $\mathscr{M}$ and $\mathscr{N}$, respectively. So by Corollary 2.6 and [4, Proposition 2.1], there exists an invertible operator $T \in \mathscr{L}(\mathscr{M}, \mathscr{N})$ such that $T\phi_{i,k} = \varphi_{i,k}$ and hence $\Theta_i = \Gamma_i T^\star$. ∎

Proposition 2.15. *Suppose that* $\{\Lambda_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ *and* $\{\Gamma_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ *are strongly disjoint g-frames for the same Hilbert space* $\mathscr{H}$*. Then* $\{\Lambda_i + \Gamma_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ *is a g-frame for* $\mathscr{H}$*. In particular, if* $\{\Lambda_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ *and* $\{\Gamma_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ *are strongly disjoint proper normalized tight g-frames for* $\mathscr{H}$*, then* $\{\Lambda_i + \Gamma_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ *is a tight g-frame with g-frame bound 2.*

Proof. Conclusion follows from Theorem 1.2, Corollary 2.4 and [4, Proposition 2.19]. ∎

Proposition 2.16. *If* $\{\Lambda_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ *and* $\{\Gamma_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ *are disjoint g-frames for* $\mathscr{H}$*, then* $\{\Lambda_i + \Gamma_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ *is a g-frame for* $\mathscr{H}$.

Proof. The assertion follows from Theorem 1.2, Corollary 2.4 and [4, Proposition 2.20]. ∎

Proposition 2.17. *Suppose that* $\{\Lambda_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ *and* $\{\Gamma_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in J\}$ *are strongly disjoint normalized tight g-frames for* $\mathscr{H}$ *and* $A, B \in \mathscr{L}(\mathscr{H})$ *are operators such that* $AA^\star + BB^\star = I$*. Then* $\{\Lambda_i A^\star + \Gamma_i B^\star\}$ *is a normalized tight g-frame for* $\mathscr{H}$.

Proof. Conclusion follows from Theorem 1.2, Corollary 2.4 and [4, Proposition 2.21]. ∎

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