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Weak Annihilator Property of Malcev-Neumann Rings

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Abstract. Let R be an associative ring with identity, G an totally ordered group, σ a map from G into the group of automorphisms of R, and t a map from $G\times G$ to the group of invertible elements of R. The weak annihilator property of the Malcev-Neumann ring $R*((G))$ is investigated in this paper. Let $\mathrm{nil}(R)$ denote the set of all nilpotent elements of R, and for a nonempty subset X of a ring R, let $N_R(X)=\{a\in R \mid Xa\subseteq \mathrm{nil}(R)\}$ denote the weak annihilator of X in R. Under the conditions that R is an *NI* ring with $\mathrm{nil}(R)$ nilpotent and σ is compatible, we show that: (1) If the weak annihilator of each nonempty subset of R which is not contained in $\mathrm{nil}(R)$ is generated as a right ideal by a nilpotent element, then the weak annihilator of each nonempty subset of $R*((G))$ which is not contained in $\mathrm{nil}(R*((G)))$ is generated as a right ideal by a nilpotent element. (2) If the weak annihilator of each nonnilpotent element of R is generated as a right ideal by a nilpotent element, then the weak annihilator of each nonnilpotent element of $R*((G)))$ is generated as a right ideal by a nilpotent element. As a generalization of left APP-rings, we next introduce the notion of weak APP-rings and give a necessary and sufficient condition under which the ring $R*((G))$ over a weak APP-ring R is weak APP.

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1. Introduction

Throughout this paper R denotes an associative ring with identity, and $\mathrm{nil}(R)$ denotes the set of all nilpotent elements of R. For a nonempty subset X of R, $l_R(X)=\{a\in R \mid aX=0\}$ and $r_R(X)=\{a\in R \mid Xa=0\}$ stand for the left and right annihilator of X in R, respectively. Recall that a ring R is reduced if it has no nonzero nilpotent elements, and a ring R is semicommutative if for all $a, b\in R$, $ab=0$ implies $aRb=0$. Due to Marks [6], a ring R is called *NI* if $\mathrm{nil}(R)$ forms an ideal. Clearly, reduced rings and semicommutative rings are *NI* rings. An ideal I of R is said to be nilpotent if $I^k=0$ for some natural number k.

Let R be a ring, G a totally ordered group, and suppose that σ is a map from G into the group of automorphisms of R, $x\longrightarrow \sigma_x$, t is a map from $G\times G$ to $U(R)$, the group of invertible elements of R. Then we can form a Malcev-Neumann ring $R*((G))$: an element of $R*((G)$ is a infinite series $f=\sum_{x\in G} r_x x$ with $r_x\in R$ such that the set $\mathrm{supp}(f)=\{x\in G \mid$

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$r_x \neq 0\}$, called the support of f, is a well ordered subset of G, and the ring structure is given by componentwise addition and by a multiplication defined as follows:

$$\left(\sum_{x\in G} a_x x\right)\left(\sum_{y\in G} b_y y\right) = \sum_{z\in G}\left(\sum_{\{x,y|xy=z\}} a_x \sigma_x(b_y)t(x,y)\right) z.$$

In order to insure associativity, it is necessary to impose two additional conditions on σ and t, namely that for all x, y, $z \in G$,

$$\text{(i)}\ \ t(xy,z)\sigma_z(t(x,y)) = t(x,yz)t(y,z), \qquad \text{(ii)}\ \ \sigma_y\sigma_z = \sigma_{yz}\delta(y,z),$$

where $\delta(y,z)$ denotes the automorphism of R induced by the unit $t(y,z)$ (see [10, Lemma 1.1]). It is now routine to check that $R*((G))$ is a ring which we call the Malcev-Neumann ring.

Let U be a subset of R. We denote by $U*((G))$ the subset of $R*((G))$ consisting of those elements whose coefficients lie in U, that is, $U*((G)) = \{f = \sum_{x\in G} a_x x \in R*((G)) \mid a_x \in U, x \in \mathrm{supp}(f)\}$.

The Malcev-Neumann construction appeared for the first time in the latter part of 1940's (the Laurent series ring, a particular case of Malcev-Neumann rings, was used before by Hilbert). Using them, Malcev and Neumann independently showed (in 1948 and 1949 resp.) that the group ring of an ordered group over a division ring can be embedded in a division ring. Since then, the construction has appeared in many papers, mainly in the study of various properties of division rings and related topics. For instance, Makar-Limanov in [7] used a particular skew Laurent series division ring to prove that the skew field of fractions of the first Weyl-algebra contains a free noncommutative subalgebra. The study of Malcev-Neumann group rings over arbitrary rings was initiated in [5] by Lorenz while investigating properties of group algebras of nilpotent groups. Other results on Malcev-Neumann rings can be found in Musson and Stafford [8] and Sonin [11] and Zhao *et al.* [14]. In this paper, we investigate the relationship between the weak annihilator $N_R(X)$ of a nonempty subset X of R and the weak annihilator $N_{R*((G))}(V)$ of a nonempty subset V of $R*((G))$. Also as a generalization of left APP-rings, we introduce the notion of weak APP-rings, and study the conditions under which the ring $R*((G))$ is a weak APP-ring.

2. Weak annihilator property

As a generalization of annihilators, L. Ouyang and G. F. Birkenmeier in [9] introduced the concept of weak annihilators. For a nonempty subset X of a ring R, we define $N_R(X) = \{a \in R \mid Xa \subseteq \mathrm{nil}(R)\}$, which is called the weak annihilator of X in R. If X is a finite set, say $X = \{r_1, r_2, \cdots, r_n\}$, we use $N_R(r_1, r_2, \cdots, r_n)$ in place of $N_R(\{r_1, r_2, \cdots, r_n\})$. Obviously, for any nonempty subset X of a ring R, $N_R(X) = \{a \in R \mid Xa \subseteq \mathrm{nil}(R)\} = \{b \in R \mid bX \subseteq \mathrm{nil}(R)\}$, $r_R(X) \subseteq N_R(X)$ and $l_R(X) \subseteq N_R(X)$.

For example, Let $\mathbb{Z}$ be the ring of integers and $T_2(\mathbb{Z})$ the 2×2 upper triangular matrix ring over $\mathbb{Z}$. We consider the subset $X = \left\{\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)\right\}$. Then $r_{T_2(\mathbb{Z})}(X) = l_{T_2(\mathbb{Z})}(X) = 0$, but $N_{T_2(\mathbb{Z})}(X) = \left\{\left(\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right), |m \in \mathbb{Z}\right\}$. Thus $r_{T_2(\mathbb{Z})}(X) \subsetneq N_{T_2(\mathbb{Z})}(X)$ and $l_{T_2(\mathbb{Z})}(X) \subsetneq N_{T_2(\mathbb{Z})}(X)$.

If R is reduced, then $r_R(X) = N_R(X) = l_R(X)$ for any nonempty subset X of R. It is easy to see that for any nonempty subset $X \subseteq R$, $N_R(X)$ is an ideal of R in case $\mathrm{nil}(R)$ is an ideal. For more details and results of weak annihilators, see [9]. In this section, we mainly discuss the weak annihilator property of the ring $R*((G))$.

The next Lemma appears in [9].

Lemma 2.1. *Let X, Y be subsets of R. Then we have the following results:*

(1) $X \subseteq Y$ *implies* $N_R(X) \supseteq N_R(Y)$.
(2) $X \subseteq N_R(N_R(X))$.
(3) $N_R(X) = N_R(N_R(N_R(X)))$.

Lemma 2.2. *Let R be an NI ring. Then we have the following results:*

(1) $ab \in \mathrm{nil}(R)$ *implies* $RaRbR \subseteq \mathrm{nil}(R)$ *for any* $a, b \in R$.
(2) *Let $p \in R$ and let $p \cdot R$ denote the principal right ideal of R generated by p. Then* $N_R(p) = N_R(p \cdot R)$.
(3) *Let X be a subset of R and let I be the ideal of R generated by the subset X. Then* $N_R(X) = N_R(I)$.

Proof. (1) Since $\mathrm{nil}(R)$ of an *NI* ring is an ideal, we obtain $ab \in \mathrm{nil}(R) \Rightarrow abR \subseteq \mathrm{nil}(R) \Rightarrow bRa \subseteq \mathrm{nil}(R) \Rightarrow bRaR \subseteq \mathrm{nil}(R) \Rightarrow aRbR \subseteq \mathrm{nil}(R) \Rightarrow RaRbR \subseteq \mathrm{nil}(R)$.

(2) Since $p \in p \cdot R$, $N_R(p \cdot R) \subseteq N_R(p)$ is clear. Now we show that $N_R(p) \subseteq N_R(p \cdot R)$. If $x \in N_R(p)$, then $px \in \mathrm{nil}(R)$. By (1), we have $pRx \subseteq \mathrm{nil}(R)$, and so $x \in N_R(p \cdot R)$. Hence $N_R(p) \subseteq N_R(p \cdot R)$. Therefore $N_R(p) = N_R(p \cdot R)$.

(3) It suffices to show that $N_R(X) \subseteq N_R(I)$. Let $r \in N_R(X)$. Then $xr \in \mathrm{nil}(R)$ for all $x \in X$, and so by (1), we obtain $sxtr \in \mathrm{nil}(R)$ for any $s \in R$ and $t \in R$. Hence for any $\sum_{i=1}^n s_i x_i t_i \in I$, we have $\sum_{i=1}^n s_i x_i t_i r \in \mathrm{nil}(R)$, and so $r \in N_R(I)$. Thus $N_R(X) \subseteq N_R(I)$ is proved. ∎

Definition 2.1. *Let σ be a map from G into the group of automorphisms of R, $x \longrightarrow \sigma_x$. We say that σ is compatible if for each $a, b \in R$ and $x \in G$, $ab = 0 \Leftrightarrow a\sigma_x(b) = 0$.*

Lemma 2.3. *Let σ be a map from G into the group of automorphisms of R, $x \longrightarrow \sigma_x$. If σ is compatible, then for each $a, b \in R$, and each $x \in G$, we have the following results:*

(1) $ab \in \mathrm{nil}(R) \Leftrightarrow a\sigma_x(b) \in \mathrm{nil}(R)$.
(2) $ab \in \mathrm{nil}(R) \Leftrightarrow \sigma_x(a)b \in \mathrm{nil}(R)$.

Proof. (1) ($\Rightarrow$) Suppose $ab \in \mathrm{nil}(R)$. There exists some positive integer k such that $(ab)^k = 0$. Since σ is compatible, we have $0 = (ab)^k = abab \cdots ab \Rightarrow abab \cdots a\sigma_x(b) = 0 \Rightarrow abab \cdots aba\sigma_x(ba\sigma_x(b)) = abab \cdots aba\sigma_x(b)\sigma_x(a\sigma_x(b)) = 0 \Rightarrow abab \cdots aba\sigma_x(b)a\sigma_x(b) = 0 \Rightarrow \cdots \Rightarrow a\sigma_x(b) \in \mathrm{nil}(R)$.

($\Leftarrow$) Assume that $a\sigma_x(b) \in \mathrm{nil}(R)$. There exists some positive integer k such that $(a\sigma_x(b))^k = 0$. In the following computations, we use freely the condition that σ is compatible. $(a\sigma_x(b))^k = a\sigma_x(b)a\sigma_x(b) \cdots a\sigma_x(b) = 0 \Rightarrow a\sigma_x(b)a\sigma_x(b) \cdots a\sigma_x(b)ab = 0 \Rightarrow a\sigma_x(b)a\sigma_x(b) \cdots a\sigma_x(b)\sigma_x(ab) = 0 \Rightarrow a\sigma_x(b)a\sigma_x(b) \cdots a\sigma_x(b)a\sigma_x(bab) = 0 \Rightarrow a\sigma_x(b)a\sigma_x(b) \cdots a\sigma_x(b)abab = 0 \Rightarrow \cdots \Rightarrow ab \in \mathrm{nil}(R)$.

(2) $ab \in \mathrm{nil}(R) \Leftrightarrow ba \in \mathrm{nil}(R) \Leftrightarrow b\sigma_x(a) \in \mathrm{nil}(R) \Leftrightarrow \sigma_x(a)b \in \mathrm{nil}(R)$. ∎

Proposition 2.1. *Let R be an NI ring with $\mathrm{nil}(R)$ nilpotent, and let σ be compatible, and $f = \sum_{x \in G} a_x x \in R * ((G))$. Then $f \in \mathrm{nil}(R * ((G)))$ if and only if $a_x \in \mathrm{nil}(R)$ for every $x \in \mathrm{supp}(f)$.*

Proof. ($\Rightarrow$) Suppose that $f = \sum_{x \in G} a_x x \in \mathrm{nil}(R * ((G)))$. Then there exists some positive integer k such that

$$f^k = \left(\sum_{x \in G} a_x x \right)^k = 0. \tag{2.1}$$

We will use transfinite induction on the ordered group $(G,\leq)$ to show that $a_x \in \mathrm{nil}(R)$ for every $x \in \mathrm{supp}(f)$. Let x_0 be the minimal element of $\mathrm{supp}(f)$ on the $\leq$ order. If $v_1, v_2, \ldots, v_k \in \mathrm{supp}(f)$ are such that $v_1 v_2 \cdots v_k = x_0^k$, then $x_0 \leq v_i$ for all $1 \leq i \leq k$. If $x_0 < v_i$ for some $1 \leq i \leq k$, then $x_0^k < v_1 v_2 \cdots v_k = x_0^k$, a contradiction. Thus $x_0 = v_i$ for $1 \leq i \leq k$. Hence from Equation (2.1), it follows that

$$a_{x_0}\sigma_{x_0}(a_{x_0})t(x_0,x_0)\sigma_{x_0^2}(a_{x_0})t(x_0^2,x_0)\cdots\sigma_{x_0^{k-1}}(a_{x_0})t(x_0^{k-1},x_0) = 0.$$

Since σ is compatible and $t(x,y)$ is invertible for all $x, y \in G$, and $\mathrm{nil}(R)$ of an *NI* ring is an ideal, we have

$$\begin{aligned}
& a_{x_0}\sigma_{x_0}(a_{x_0})t(x_0,x_0)\sigma_{x_0^2}(a_{x_0})t(x_0^2,x_0)\cdots\sigma_{x_0^{k-1}}(a_{x_0})t(x_0^{k-1},x_0) = 0 \\
\Rightarrow\ & a_{x_0}\sigma_{x_0}(a_{x_0})t(x_0,x_0)\sigma_{x_0^2}(a_{x_0})t(x_0^2,x_0)\cdots\sigma_{x_0^{k-1}}(a_{x_0}) = 0 \\
\Rightarrow\ & a_{x_0}\sigma_{x_0}(a_{x_0})t(x_0,x_0)\sigma_{x_0^2}(a_{x_0})t(x_0^2,x_0)\cdots\sigma_{x_0^{k-2}}(a_{x_0})t(x_0^{k-2},x_0)a_{x_0} = 0 \\
\Rightarrow\ & a_{x_0}a_{x_0}\sigma_{x_0}(a_{x_0})t(x_0,x_0)\sigma_{x_0^2}(a_{x_0})t(x_0^2,x_0)\cdots\sigma_{x_0^{k-2}}(a_{x_0})t(x_0^{k-2},x_0) \in \mathrm{nil}(R) \\
\Rightarrow\ & a_{x_0}a_{x_0}\sigma_{x_0}(a_{x_0})t(x_0,x_0)\sigma_{x_0^2}(a_{x_0})t(x_0^2,x_0)\cdots\sigma_{x_0^{k-2}}(a_{x_0}) \in \mathrm{nil}(R) \\
\Rightarrow\ & a_{x_0}a_{x_0}\sigma_{x_0}(a_{x_0})t(x_0,x_0)\sigma_{x_0^2}(a_{x_0})t(x_0^2,x_0)\cdots\sigma_{x_0^{k-3}}(a_{x_0})t(x_0^{k-3},x_0)a_{x_0} \in \mathrm{nil}(R) \\
\Rightarrow\ & a_{x_0}a_{x_0}a_{x_0}\sigma_{x_0}(a_{x_0})t(x_0,x_0)\sigma_{x_0^2}(a_{x_0})t(x_0^2,x_0)\cdots\sigma_{x_0^{k-3}}(a_{x_0})t(x_0^{k-3},x_0) \in \mathrm{nil}(R) \\
\Rightarrow\ & \cdots \Rightarrow a_{x_0} \in \mathrm{nil}(R).
\end{aligned}$$

Now suppose that $w \in \mathrm{supp}(f)$ is such that for any $x \in \mathrm{supp}(f)$ with $x < w$, $a_x \in \mathrm{nil}(R)$. We will show that $a_w \in \mathrm{nil}(R)$ for $w \in \mathrm{supp}(f)$. For convenience, we write

$$\{(u_1,u_2,\cdots,u_k) \mid u_1u_2\cdots u_k = w^k, u_i \in \mathrm{supp}(f), i = 1,2,\ldots,k\}$$

as

$$\{(w,w,\cdots,w)\} \cup \{(u_{i1},u_{i2},\cdots,u_{ik}) \mid i = 2,3,\ldots,n\},$$

and for each

$$(u_{i1},u_{i2},\cdots,u_{ik}) \in \{(u_{i1},u_{i2},\cdots,u_{ik}) \mid i = 2,3,\ldots,n\},$$

there exists some $1 \leq l \leq k$ such that $u_{il} \neq w$. Now we show that for each

$$(u_{i1},u_{i2},\cdots,u_{ik}) \in \{(u_{i1},u_{i2},\cdots,u_{ik}) \mid i = 2,3,\ldots,n\},$$

there exists some $1 \leq p \leq k$ such that $u_{ip} < w$. If $u_{il} < w$, then we are done. So assume that $u_{il} > w$. If for all $1 \leq j \leq k$, $j \neq l$, $u_{ij} \geq w$, then $w^k < u_{i1}u_{i2}\cdots u_{ik} = w^k$, a contradiction. Thus for each

$$(u_{i1},u_{i2},\cdots,u_{ik}) \in \{(u_{i1},u_{i2},\cdots,u_{ik}) \mid i = 2,3,\ldots,n\},$$

there exists some $1 \leq p \leq k$ such that $u_{ip} < w$. Then by induction hypothesis, we obtain $a_{u_{ip}} \in \mathrm{nil}(R)$, and so by Lemma 2.3, $1 \cdot a_{u_{ip}} \in \mathrm{nil}(R)$ implies $1 \cdot \sigma_x(a_{u_{ip}}) = \sigma_x(a_{u_{ip}}) \in \mathrm{nil}(R)$ for every $x \in G$. Hence

$$a_{u_{i1}}\sigma_{u_{i1}}(a_{u_{i2}})t(u_{i1},u_{i2})\cdots\sigma_{(u_{i1}u_{i2}\cdots u_{i(k-1)})}(a_{u_{ik}})t(u_{i1}u_{i2}\cdots u_{i(k-1)},u_{ik}) \in \mathrm{nil}(R)$$

for all $2 \leq i \leq n$, because $\mathrm{nil}(R)$ of an *NI* ring is an ideal. Now from Equation (2.1), we have

$$a_w\sigma_w(a_w)t(w,w)\cdots\sigma_{w^{k-1}}(a_w)t(w^{k-1},w)$$

$$= -\sum_{i=2}^{n} a_{u_{i1}} \sigma_{u_{i1}}(a_{u_{i2}}) t(u_{i1}, u_{i2}) \cdots \sigma_{(u_{i1}u_{i2}\cdots u_{i(k-1)})}(a_{u_{ik}}) t(u_{i1}u_{i2}\cdots u_{i(k-1)}, u_{ik}) \in \mathrm{nil}(R).$$

Then

$$\begin{aligned}
& a_w \sigma_w(a_w) t(w,w) \cdots \sigma_{w^{k-1}}(a_w) t(w^{k-1}, w) \in \mathrm{nil}(R) \\
\Rightarrow\ & a_w \sigma_w(a_w) t(w,w) \cdots \sigma_{w^{k-1}}(a_w) \in \mathrm{nil}(R) \\
\Rightarrow\ & a_w \sigma_w(a_w) t(w,w) \cdots \sigma_{w^{k-2}}(a_w) t(w^{k-2}, w) a_w \in \mathrm{nil}(R) \\
\Rightarrow\ & a_w a_w \sigma_w(a_w) t(w,w) \cdots \sigma_{w^{k-2}}(a_w) t(w^{k-2}, w) \in \mathrm{nil}(R) \\
\Rightarrow\ & \cdots \Rightarrow a_w \in \mathrm{nil}(R).
\end{aligned}$$

Therefore by transfinite induction, $a_x \in \mathrm{nil}(R)$ for any $x \in \mathrm{supp}(f)$.

($\Leftarrow$) Assume that $a_x \in \mathrm{nil}(R)$ for every $x \in \mathrm{supp}(f)$. By Lemma 2.3, we have $\sigma_z(a_x) \in \mathrm{nil}(R)$ for each $z \in G$. Since $\mathrm{nil}(R)$ is nilpotent, there exists some positive integer k such that $(\mathrm{nil}(R))^k = 0$. Now we show that

$$f^k = \left(\sum_{x \in G} a_x x \right)^k = \sum_{y \in G} b_y y = 0.$$

For every $y \in \mathrm{supp}(f^k)$, we write

$$\{(u_1, u_2, \cdots, u_k) \mid u_1 u_2 \cdots u_k = y, u_i \in \mathrm{supp}(f), i = 1, 2, \ldots, k\}$$

as

$$\{(u_{i1}, u_{i2}, \cdots, u_{ik}) \mid i = 1, 2, \ldots, n\}.$$

Then from $f^k = (\sum_{x \in G} a_x x)^k = \sum_{y \in G} b_y y$, it follows that

$$b_y = \sum_{i=1}^{n} a_{u_{i1}} \sigma_{u_{i1}}(a_{u_{i2}}) t(u_{i1}, u_{i2}) \cdots \sigma_{(u_{i1}u_{i2}\cdots u_{i(k-1)})}(a_{u_{ik}}) t(u_{i1}u_{i2}\cdots u_{i(k-1)}, u_{ik}).$$

Since for each $1 \le i \le n$,

$$a_{u_{i1}} \sigma_{u_{i1}}(a_{u_{i2}}) t(u_{i1}, u_{i2}) \cdots \sigma_{u_{i1}u_{i2}\cdots u_{i(k-1)}}(a_{u_{ik}}) t(u_{i1}u_{i2}\cdots u_{i(k-1)}, u_{ik}) \in (\mathrm{nil}(R))^k = 0,$$

we have $b_y = 0$. Hence $f^k = 0$, and so $f \in \mathrm{nil}(R * ((G)))$. Then we finish our proof of Proposition 2.1. ∎

Remark 2.1. In the proof of the implication ($\Rightarrow$) in Proposition 2.1, the condition that $\mathrm{nil}(R)$ is nilpotent is not used. Hence if R is an *NI* ring, and σ is compatible, then $\mathrm{nil}(R * ((G))) \subseteq \mathrm{nil}(R) * ((G))$.

By Proposition 2.1 we have the following result.

Corollary 2.1. *Let R be an NI ring with* $\mathrm{nil}(R)$ *nilpotent, and let σ be compatible. Then*

(1) *$R * ((G))$ is an NI ring.*
(2) $\mathrm{nil}(R * ((G))) = \mathrm{nil}(R) * ((G))$.

Proposition 2.2. *Let R be an NI ring with* $\mathrm{nil}(R)$ *nilpotent, and let σ be compatible. If the weak annihilator of each nonempty subset of R which is not contained in* $\mathrm{nil}(R)$ *is generated as a right ideal by a nilpotent element, then the weak annihilator of each nonempty subset of $R * ((G))$ which is not contained in* $\mathrm{nil}(R * ((G)))$ *is generated as a right ideal by a nilpotent element.*

Proof. Let V be a nonempty subset of $R*((G))$ with $V \not\subseteq \mathrm{nil}(R*((G)))$. We show that $N_{R*((G))}(V)$ is generated as a right ideal by a nilpotent element. For any $f = \sum_{x\in G} a_x x \in R*((G))$, let C_f denote the set $\{a_x \mid x \in \mathrm{supp}(f)\}$, and for any subset $U \subseteq R*((G))$, let C_U denote the set $\bigcup_{f\in U} C_f$. Since $V \not\subseteq \mathrm{nil}(R*((G)))$, by Corollary 2.1, we have $C_V \not\subseteq \mathrm{nil}(R)$. So there exists an element $c \in \mathrm{nil}(R)$ such that $N_R(C_V) = c\cdot R$. Now we show that

$$N_{R*((G))}(V) = c\cdot(R*((G))).$$

Let $f = \sum_{x\in G} a_x x \in V$ and $g = \sum_{y\in G} b_y y \in R*((G))$. Then

$$f\cdot c\cdot g = \left(\sum_{x\in G} a_x x\right)\cdot c\cdot\left(\sum_{y\in G} b_y y\right) = \sum_{z\in G}\left(\sum_{\{x,y\mid xy=z\}} a_x\sigma_x(c)t(x,1)\sigma_x(b_y)t(x,y)\right)z.$$

Since $c \in \mathrm{nil}(R)$ and σ is compatible, for any $x \in \mathrm{supp}(f)$ and $y \in \mathrm{supp}(g)$, we have

$$\begin{aligned} c \in \mathrm{nil}(R) &\Rightarrow \sigma_x(c) \in \mathrm{nil}(R) \Rightarrow a_x\sigma_x(c)t(x,1)\sigma_x(b_y)t(x,y) \in \mathrm{nil}(R) \\ &\Rightarrow \sum_{\{x,y\mid xy=z\}} a_x\sigma_x(c)t(x,1)\sigma_x(b_y)t(x,y) \in \mathrm{nil}(R). \end{aligned}$$

Thus by Proposition 2.1, we obtain $f\cdot c\cdot g \in \mathrm{nil}(R*((G)))$. Hence $N_{R*((G))}(V) \supseteq c\cdot(R*((G)))$.

Conversely, let $g = \sum_{y\in G} b_y y \in N_{R*((G))}(V)$. Then $fg \in \mathrm{nil}(R*((G)))$ for any $f = \sum_{x\in G} a_x x \in V$. Let $fg = \left(\sum_{x\in G} a_x x\right)\left(\sum_{y\in G} b_y y\right) = \sum_{z\in G}\Delta_z z$. Then by Proposition 2.1, we have $\Delta_z \in \mathrm{nil}(R)$. Note that

$$\Delta_z = \sum_{\{x,y\mid xy=z\}} a_x\sigma_x(b_y)t(x,y). \tag{2.2}$$

We will use transfinite induction on the ordered group $(G,\leq)$ to show that $a_x b_y \in \mathrm{nil}(R)$ for every $x \in \mathrm{supp}(f)$ and $y \in \mathrm{supp}(g)$.

Let x_0 and y_0 be the minimal elements of $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ in the order $\leq$, respectively. If $x \in \mathrm{supp}(f)$ and $y \in \mathrm{supp}(g)$ are such that $xy = x_0y_0$, then $x_0 \leq x$, and $y_0 \leq y$. If $x_0 < x$, then $x_0y_0 < xy_0 \leq xy = x_0y_0$, a contradiction. Thus $x_0 = x$. Similarly, $y = y_0$. Then from Equation (2.2), we obtain $\Delta_{x_0y_0} = a_{x_0}\sigma_{x_0}(b_{y_0})t(x_0,y_0) \in \mathrm{nil}(R)$. Thus we have $a_{x_0}\sigma_{x_0}(b_{y_0})t(x_0,y_0) \in \mathrm{nil}(R) \Rightarrow a_{x_0}\sigma_{x_0}(b_{y_0})t(x_0,y_0)(t(x_0,y_0))^{-1} = a_{x_0}\sigma_{x_0}(b_{y_0}) \in \mathrm{nil}(R) \Rightarrow a_{x_0}b_{y_0} \in \mathrm{nil}(R)$.

Now suppose that $w \in G$ is such that for any $x \in \mathrm{supp}(f)$ and $y \in \mathrm{supp}(g)$ with $xy < w$, $a_x b_y \in \mathrm{nil}(R)$. We will show that $a_x b_y \in \mathrm{nil}(R)$ for any $x \in \mathrm{supp}(f)$ and $y \in \mathrm{supp}(g)$ with $xy = w$. For convenience, we write $\{(x,y) \mid xy = w, x \in \mathrm{supp}(f), y \in \mathrm{supp}(g)\}$ as $\{(x_i,y_i) \mid i = 1,2,\ldots,n, x_i \in \mathrm{supp}(f), y_i \in \mathrm{supp}(g)\}$ with $x_1 < x_2 < \cdots < x_n$ (Note that if $x_1 = x_2$, then from $x_1y_1 = x_2y_2$, it follows that $y_1 = y_2$, and thus $(x_1,y_1) = (x_2,y_2)$). Now from Equation (2.2), we have

$$\Delta_w = \sum_{\{x,y\mid xy=w\}} a_x\sigma_x(b_y)t(x,y) = \sum_{i=1}^{n} a_{x_i}\sigma_{x_i}(b_{y_i})t(x_i,y_i), \tag{2.3}$$

and $\Delta_w \in \mathrm{nil}(R)$. For any $1 \leq i \leq n-1$, $x_iy_n < x_ny_n = w$, and thus, by induction hypothesis, we have $a_{x_i}b_{y_n} \in \mathrm{nil}(R)$. Then by Lemma 2.2, $a_{x_i}\sigma_{x_i}(b_{y_i})t(x_i,y_i)b_{y_n} \in \mathrm{nil}(R)$. Hence multiplying Equation (2.3) on the right by b_{y_n}, we obtain

$$a_{x_n}\sigma_{x_n}(b_{y_n})t(x_n,y_n)b_{y_n} = \Delta_w b_{y_n} - \sum_{i=1}^{n-1} a_{x_i}\sigma_{x_i}(b_{y_i})t(x_i,y_i)b_{y_n}.$$

Then $a_{x_n}\sigma_{x_n}(b_{y_n})t(x_n,y_n)b_{y_n} \in \mathrm{nil}(R)$ because $\mathrm{nil}(R)$ of an *NI* ring is an ideal. Now

$$\begin{aligned} &a_{x_n}\sigma_{x_n}(b_{y_n})t(x_n,y_n)b_{y_n} \in \mathrm{nil}(R) \Rightarrow b_{y_n}a_{x_n}\sigma_{x_n}(b_{y_n})t(x_n,y_n) \in \mathrm{nil}(R) \\ &\Rightarrow b_{y_n}a_{x_n}\sigma_{x_n}(b_{y_n})t(x_n,y_n)(t(x_n,y_n))^{-1} = b_{y_n}a_{x_n}\sigma_{x_n}(b_{y_n}) \in \mathrm{nil}(R) \\ &\Rightarrow b_{y_n}a_{x_n}b_{y_n} \in \mathrm{nil}(R) \Rightarrow a_{x_n}b_{y_n} \in \mathrm{nil}(R). \end{aligned}$$

From Lemma 2.3, it follows that

$$a_{x_n}b_{y_n} \in \mathrm{nil}(R) \Rightarrow a_{x_n}\sigma_{x_n}(b_{y_n}) \in \mathrm{nil}(R) \Rightarrow a_{x_n}\sigma_{x_n}(b_{y_n})t(x_n,y_n) \in \mathrm{nil}(R).$$

Now Equation (2.3) becomes

$$\sum_{i=1}^{n-1} a_{x_i}\sigma_{x_i}(b_{y_i})t(x_i,y_i) = \Delta_w - a_{x_n}\sigma_{x_n}(b_{y_n})t(x_n,y_n) \in \mathrm{nil}(R). \tag{2.4}$$

Multiplying $b_{y_{n-1}}$ on Equation (2.4) from the right-hand side, we obtain $a_{x_{n-1}}b_{y_{n-1}} \in \mathrm{nil}(R)$ by the same way as above. Continuing this process, we can prove that $a_{x_i}b_{y_i} \in \mathrm{nil}(R)$ for $i = 1,2,\ldots,n$. Thus $a_x b_y \in \mathrm{nil}(R)$ for all $x \in \mathrm{supp}(f)$ and $y \in \mathrm{supp}(g)$ with $xy = w$.

Therefore, by transfinite induction, $a_x b_y \in \mathrm{nil}(R)$ for any $x \in \mathrm{supp}(f)$ and $y \in \mathrm{supp}(g)$. Thus for any $y \in \mathrm{supp}(g)$, $b_y \in N_R(C_V) = c \cdot R$. So for any $y \in \mathrm{supp}(g)$, there exists $r_y \in R$ such that $b_y = cr_y$. Hence $g = c \cdot h$ where $h = \sum_{y\in G} r_y y \in R*((G))$, and so $N_{R*((G))}(V) \subseteq c\cdot(R*((G)))$. Therefore $N_{R*((G))}(V) = c\cdot(R*((G)))$ where c is a nilpotent element. ∎

Corollary 2.2. *Let R be an NI ring with $\mathrm{nil}(R)$ nilpotent, and let σ be compatible. If the weak annihilator of each ideal of R which is not contained in $\mathrm{nil}(R)$ is generated as a right ideal by a nilpotent element, then the weak annihilator of each ideal of $R*((G))$ which is not contained in $nil(R*((G)))$ is generated as a right ideal by a nilpotent element.*

Proof. This is immediate from Lemma 2.2 and Proposition 2.2. ∎

Proposition 2.3. *Let R be an NI ring with $\mathrm{nil}(R)$ nilpotent, and let σ be compatible. If the weak annihilator of each nonnilpotent element of R is generated as a right ideal by a nilpotent element, then the weak annihilator of each nonnilpotent element of $R*((G))$ is generated as a right ideal by a nilpotent element.*

Proof. Let $f = \sum_{x\in G} a_x x$ be a nonnilpotent element of $R*((G))$. Then by Proposition 2.1, there exists some $u \in \mathrm{supp}(f)$ such that $a_u \notin \mathrm{nil}(R)$. Hence we can find $c \in \mathrm{nil}(R)$ such that $N_R(a_u) = c\cdot R$. Now we show that

$$N_{R*((G))}(f) = c\cdot(R*((G))).$$

For any $g = \sum_{y\in G} b_y y \in R*((G))$, we have

$$f\cdot c\cdot g = \left(\sum_{x\in G} a_x x\right)\cdot c\cdot\left(\sum_{y\in G} b_y y\right) = \sum_{z\in G}\left(\sum_{\{x,y|xy=z\}} a_x\sigma_x(c)t(x,1)\sigma_x(b_y)t(x,y)\right)z.$$

Since $c \in \mathrm{nil}(R)$ and σ is compatible, it is easy to see that

$$\sum_{\{x,y|xy=z\}} a_x\sigma_x(c)t(x,1)\sigma_x(b_y)t(x,y) \in \mathrm{nil}(R)$$

for any $z \in \mathrm{supp}(fcg)$. Then by Proposition 2.1, we obtain $fcg \in \mathrm{nil}(R*((G)))$, and so $c \cdot R*((G)) \subseteq N_{R*((G))}(f)$.

Conversely, let $g = \sum_{y \in G} b_y y \in N_{R*((G))}(f)$. Then $fg \in \mathrm{nil}(R*((G)))$. By analogy with the proof of Proposition 2.2, we obtain $a_x b_y \in \mathrm{nil}(R)$ for any $x \in \mathrm{supp}(f)$ and any $y \in \mathrm{supp}(g)$. Hence $b_y \in N_R(a_u)$ for any $y \in \mathrm{supp}(g)$. Thus for any $y \in \mathrm{supp}(g)$, there exists $r_y \in R$ such that $b_y = c \cdot r_y$. Then $g = ch$ where $h = \sum_{y \in G} r_y y \in R*((G))$, and so $N_{R*((G))}(f) \subseteq c \cdot (R*((G)))$. Therefore, $N_{R*((G))}(f) = c \cdot (R*((G)))$. ∎

Corollary 2.3. *Let R be an NI ring with* $\mathrm{nil}(R)$ *nilpotent, and let σ be compatible. If the weak annihilator of each principal right ideal of R which is not contained in* $\mathrm{nil}(R)$ *is generated as a right ideal by a nilpotent element, then the weak annihilator of each principal right ideal of $R*((G))$ which is not contained in* $\mathrm{nil}(R*((G)))$ *is generated as a right ideal by a nilpotent element.*

Proof. This is immediate from Lemma 2.2 and Proposition 2.3. ∎

Example 2.1. Let F be a field and let S denote the F-space on basis

$$\{1, c, c^2, \dots, c^n\},$$

where $c^{n+1} = 0$. Then $\mathrm{nil}(S) = \{a_1 c + a_2 c^2 + \cdots + a_n c^n \mid a_i \in F\}$ is an ideal of S. For any $m = b_0 + b_1 c + \cdots + b_n c^n \in S$, if $b_0 = 0$, then $m \in \mathrm{nil}(S)$. If $b_0 \neq 0$, then $m = b_0 + b_1 c + \cdots + b_n c^n$ is invertible. For any nonempty subset $V \not\subseteq \mathrm{nil}(S)$, now we show that $N_S(V)$ is generated as a right ideal by a nilpotent element. Let $\Omega = \{b_0 \mid b_0 + b_1 c + \cdots + b_n c^n \in V\}$. If $\Omega = \{0\}$, then $V \subseteq \mathrm{nil}(S)$. This is contrary to the fact that $V \not\subseteq \mathrm{nil}(S)$. Thus we have $\Omega \neq \{0\}$. In this case, we have $N_S(V) = \mathrm{nil}(S) = c \cdot S$, where $c \in \mathrm{nil}(S)$. Hence S is a ring such that for each nonempty subset $V \not\subseteq \mathrm{nil}(S)$, $N_S(V)$ is generated as a right ideal by a nilpotent element.

Let R be a field. Then the residue ring $R[x]/(x^{n+1})$ is an R-space on basis

$$\{\overline{1}, \overline{x}, \overline{x}^2, \dots, \overline{x}^n\},$$

where $\overline{x}^{n+1} = 0$. Hence $R[x]/(x^{n+1})$ is a ring such that for each nonempty subset $V \not\subseteq nil(R[x]/(x^{n+1}))$, $N_{R[x]/(x^{n+1})}(V)$ is generated as a right ideal by a nilpotent element.

Let R be a field and let

$$R_n = \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & a_1 & \cdots & a_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_1 \end{pmatrix} \mid a_i \in R \right\}$$

be the subring of $n \times n$ upper triangular matrix ring. Then $R_n \cong R[x]/(x^n)$. Thus R_n is also a ring that for each nonempty subset $V \not\subseteq \mathrm{nil}(R_n)$, $N_{R_n}(V)$ is generated as a right ideal by a nilpotent element.

Example 2.2. If p is a prime, the ring $\mathbb{Z}_{p^n}$ of integers modulo p^n is a commutative local ring and the Jacobson radical J of $\mathbb{Z}_{p^n}$ is $J = \mathrm{nil}(\mathbb{Z}_{p^n}) = \mathbb{Z}_{p^n} \cdot [p]$. Hence it is easy to see that for any nonempty subset $V \not\subseteq \mathrm{nil}(\mathbb{Z}_{p^n})$, $N_{\mathbb{Z}_{p^n}}(V)$ is generated as a right ideal by a nilpotent element.

3. Weak APP-rings

An ideal I of R is said to be right *s-unital* if $a \in aI$ for each $a \in I$. If I and J are right *s-unital* ideals, then so is $I \cap J$. It follows from [12, Theorem 1] that I is right *s-unital* if and only if for any finitely many elements $a_1, a_2, \ldots, a_n \in I$, there exists an element $x \in I$ such that $a_i = a_i x$, $i = 1, 2, \ldots, n$. A ring R is called a left APP-ring if the left annihilator $l_R(Ra)$ is right *s-unital* as an ideal of R for any element $a \in R$, right APP-rings may be defined analogously. A ring is biregular if every principal ideal is generated by some idempotent in the center of the ring, and a ring is quasi-Baer if the left annihilator of every left ideal is generated by an idempotent. Thus the class of left APP-rings includes all biregular rings and all quasi-Baer rings. It was shown in [4, Theorem 2] that if R is a ring satisfying descending chain condition on right annihilators, then the skew power series ring $R[[x;\alpha]]$ is left APP if and only if for any sequence $(b_0, b_1, \cdots)$ of elements of R, the ideal $l_R(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} R\alpha^k(b_j))$ is right *s-unital*, where α is an automorphism of R. It was also proved in [13, Theorem 3] that if $(S, \leq)$ is a strictly totally ordered monoid, $\omega : S \longrightarrow \mathrm{Aut}(R)$ a monoid homomorphism and R a ring satisfying descending chain condition on right annihilators, then the skew generalized power series ring $[[R^{S,\leq}, \omega]]$ is left APP if and only if for any S-indexed subset A of R, the ideal $l_R(\sum_{a\in A}\sum_{s\in S} R\omega_s(a))$ is right *s-unital*. For more details and properties of left APP-rings, see [2, 3, 4, 13].

As a generalization of left APP-rings, in this section, we introduce the notion of weak APP-rings and investigate its properties. We first briefly develop the definition of weak APP-rings. Also we provide several basic results. Next, we investigate the weak APP-property of Malcev-Neumann rings.

Definition 3.1. *Let R be an NI ring. An ideal I of R is said to be weak s-unital if, for each $a \in I$, there exists an element $x \in I$ such that $ax - a \in \mathrm{nil}(R)$.*

Obviously, for all a, $x \in R$, $ax - a = a(x-1) \in \mathrm{nil}(R) \Leftrightarrow (x-1)a = xa - a \in \mathrm{nil}(R)$. So all right s-unital ideals and all left s-unital ideals are weak s-unital. But the following example shows that the converse is not true in general.

Example 3.1. Let R be a domain and let

$$R_2 = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in R \right\}$$

be the subring of 2×2 upper triangular matrix ring. Consider the ideal

$$I = R_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R_2$$

generated by $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Then I is neither right s-unital nor left s-unital. But it is easy to see that I is weak s-unital.

Proposition 3.1. *Let R be an NI ring. Then the following conditions are equivalent:*

(1) *I is weak s-unital.*
(2) *For any finitely many elements $a_1, a_2, \ldots, a_n \in I$, there exists an element $x \in I$ such that $a_i x - a_i \in \mathrm{nil}(R)$, $i = 1, 2, \ldots, n$.*

Proof. $(1) \Longrightarrow (2)$ We prove it by induction on n with the case $n = 1$ clear. Now suppose that $n \geq 2$. From the condition that I is weak s-unital and the induction hypothesis, it follows that there exist e_1, $e_2 \in I$ such that $a_i e_1 - a_i \in \mathrm{nil}(R)$ for all $1 \leq i \leq n-1$, and

$a_ne_2 - a_n \in \mathrm{nil}(R)$. In the following computations, we use freely the condition that R is an *NI* ring. For each $1 \leq i \leq n-1$, $a_ie_1 - a_i = a_i(e_1-1) \in \mathrm{nil}(R) \Rightarrow (e_1-1)a_i \in \mathrm{nil}(R) \Rightarrow (e_1-1)a_i(e_2-1)(e_1-1) \in \mathrm{nil}(R) \Rightarrow a_i(e_2-1)(e_1-1)(e_1-1) = a_i(e_2e_1^2 - 2e_2e_1 + e_2 - e_1^2 + 2e_1 - 1) \in \mathrm{nil}(R) \Rightarrow a_i(e_2e_1^2 - 2e_2e_1 + e_2 - e_1^2 + 2e_1) - a_i \in \mathrm{nil}(R)$, and $a_ne_2 - a_n = a_n(e_2-1) \in \mathrm{nil}(R) \Rightarrow a_n(e_2-1)(e_1-1)(e_1-1) \in \mathrm{nil}(R) \Rightarrow a_n(e_2e_1^2 - 2e_2e_1 + e_2 - e_1^2 + 2e_1) - a_n \in \mathrm{nil}(R)$. Set $x = e_2e_1^2 - 2e_2e_1 + e_2 - e_1^2 + 2e_1$. Then we obtain $a_ix - a_i \in \mathrm{nil}(R)$ for all $1 \leq i \leq n$.

(2) $\Rightarrow$ (1) It is straightforward. ∎

Proposition 3.2. *Let R be an NI ring and I, J are weak s-unital ideals. Then $I \cap J$ and $I+J$ are weak s-unital.*

Proof. Let $a \in I \cap J$. Then there exist $x \in I$ and $y \in J$ such that $ax - a \in \mathrm{nil}(R)$ and $ay - a \in \mathrm{nil}(R)$. So we can find α, $\beta \in \mathrm{nil}(R)$ such that $ax = a + \alpha$ and $ay = a + \beta$. Thus $axy = (a+\alpha)y = ay + \alpha y = a + \beta + \alpha y$. Hence $axy - a \in \mathrm{nil}(R)$ with $xy \in IJ \subseteq I \cap J$. Therefore $I \cap J$ is weak s-unital.

Now we see that $I+J$ is weak s-unital. Let $a_1 + a_2 \in I+J$ with $a_1 \in I$ and $a_2 \in J$. Then there exist $e_1 \in I$ and $e_2 \in J$ such that $a_1e_1 - a_1 \in \mathrm{nil}(R)$ and $a_2e_2 - a_2 \in \mathrm{nil}(R)$. By analogy with the proof of Proposition 3.1, we can find $x = e_2e_1^2 - 2e_2e_1 + e_2 - e_1^2 + 2e_1 \in I+J$ such that $a_ix - a_i \in \mathrm{nil}(R)$, $i = 1,2$. Thus we have $(a_1+a_2)x - (a_1+a_2) \in \mathrm{nil}(R)$. This implies that $I+J$ is weak s-unital. ∎

Definition 3.2. *An NI ring R is called a weak APP-ring if the weak annihilator $N_R(a)$ is weak s-unital as an ideal of R for any element $a \in R$.*

Example 3.2. Here are some examples of weak APP-rings.

(1) Obviously, all domains and division rings are weak APP-rings. If a ring R is reduced, then for any $a \in R$, $N_R(a) = r_R(aR) = l_R(Ra)$. So reduced left (resp. right) APP-rings are weak APP-rings. Since reduced PP-rings and reduced *p.q.-Baer* rings are left (resp. right) APP-rings (see [3]), they are also weak APP-rings. Hence the class of weak APP-rings includes reduced left (resp. right) APP-rings. In particular, the class of weak APP-rings includes reduced PP-rings and reduced *p.q.-Baer* rings.

(2) Let R be an *NI* ring and let $T_n(R)$ be the $n \times n$ upper triangular matrix ring over R. Now we show that R is a weak APP-ring if and only if $T_n(R)$ is a weak APP-ring. Clearly, $T_n(R)$ is an *NI* ring. Suppose that R is a weak APP-ring. Let $A = (a_{ij}) \in T_n(R)$ and $B = (b_{ij}) \in N_{T_n(R)}(A)$. Then $BA \in \mathrm{nil}(T_n(R))$ and so $b_{ii}a_{ii} \in \mathrm{nil}(R)$ for all $1 \leq i \leq n$. Thus $b_{ii} \in N_R(a_{ii})$ for all $1 \leq i \leq n$. Because R is a weak APP-ring, there exists $c_{ii} \in N_R(a_{ii})$ such that $b_{ii}c_{ii} - b_{ii} \in \mathrm{nil}(R)$ for each $1 \leq i \leq n$. Now it is easy to see that

$$B\begin{pmatrix} c_{11} & 0 & \cdots & 0 \\ 0 & c_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & c_{nn} \end{pmatrix} - B \in \mathrm{nil}(T_n(R))$$

and

$$\begin{pmatrix} c_{11} & 0 & \cdots & 0 \\ 0 & c_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & c_{nn} \end{pmatrix} \in N_{T_n(R)}(A).$$

Conversely, assume that $T_n(R)$ is a weak APP-ring. Let a, $b \in R$ such that $b \in N_R(a)$. Set

$$A = \begin{pmatrix} c_{11} & 0 & \cdots & 0 \\ 0 & c_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & c_{nn} \end{pmatrix} \quad B = \begin{pmatrix} c_{11} & 0 & \cdots & 0 \\ 0 & c_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & c_{nn} \end{pmatrix}$$

Then $B \in N_{T_n(R)}(A)$. Since $T_n(R)$ is a weak APP-ring, there exists $C = (c_{ij}) \in N_{T_n(R)}(A)$ such that $BC - B \in \mathrm{nil}(T_n(R))$. Now it is easy to see that $bc_{11} - b \in \mathrm{nil}(R)$ and $c_{11} \in N_R(a)$. Thus R is a weak APP-ring. So if R is a domain, the $T_n(R)$ is a weak APP-ring.

(3) If an *NI* ring R satisfies the condition that for each element $p \notin \mathrm{nil}(R)$, $N_R(p)$ is generated as a right ideal by a nilpotent element, then we can show that R is a weak APP-ring. So the rings in Example 2.1 and Example 2.2 are all weak APP-rings, and by Proposition 2.2 and Proposition 2.3, we can construct more examples of weak APP-rings.

Proposition 3.3. *Let σ be compatible and* $\mathrm{nil}(R)$ *nilpotent, and let R be an NI ring satisfying the descending chain condition on weak annihilators. Then the following conditions are equivalent:*

(1) *R is a weak APP-ring.*
(2) *$R*((G))$ is a weak APP-ring.*

Proof. (1) $\Rightarrow$ (2) Suppose that $f = \sum_{x\in G} a_x x$, $g = \sum_{y\in G} b_y y \in R*((G))$ are such that $f \in N_{R*((G))}(g)$. Then $fg \in \mathrm{nil}(R*((G)))$. By analogy with the proof of Proposition 2.2, we obtain $a_x b_y \in \mathrm{nil}(R)$ for any $x \in \mathrm{supp}(f)$ and any $y \in \mathrm{supp}(g)$. Hence $a_x \in N_R(b_y)$ and $b_y \in N_R(a_x)$ for any $x \in \mathrm{supp}(f)$ and any $y \in \mathrm{supp}(g)$. For a set $Y \subseteq R$, $|Y| < \infty$ means the cardinal number of Y is finite. Let

$$\Omega = \{N_R(Y) \mid Y \subseteq \{a_x \mid x \in \mathrm{supp}(f)\}, |Y| < \infty\}.$$

The Ω is a nonempty set of weak annihilators. Since R satisfying descending chain condition on weak annihilators, Ω has a minimal element, say $N_R(Y_0)$. Assume that

$$Y_0 = \{a_{x_1}, a_{x_2}, \cdots, a_{x_n}\}.$$

Similarly, let

$$\Psi = \{N_R(X) \mid X \subseteq \{b_y \mid y \in \mathrm{supp}(g)\}, |X| < \infty\}.$$

Then Ψ has a minimal element, say $N_R(X_0)$. Also assume that

$$X_0 = \{b_{y_1}, b_{y_2}, \cdots, b_{y_m}\}.$$

Since a_{x_1}, a_{x_2}, $\ldots$, $a_{x_n} \in N_R(X_0) = \bigcap_{i=1}^m N_R(b_{y_i})$, by Proposition 3.1 and Proposition 3.2, there exists $c \in N_R(X_0) = \bigcap_{i=1}^m N_R(b_{y_i})$ such that for all $1 \le i \le n$, $a_{x_i}c - a_{x_i} \in \mathrm{nil}(R)$. Then $c - 1 \in N_R(Y_0)$. If $\mathrm{supp}(f) = \{x_1, x_2, \cdots, x_n\}$, Then for any $x \in \mathrm{supp}(f)$, $a_x c - a_x \in \mathrm{nil}(R)$. Now assume that $x \in \mathrm{supp}(f) - \{x_1, x_2, \cdots, x_n\}$. Then by the minimality of $N_R(Y_0)$, we have $N_R(a_{x_1}, a_{x_2}, \cdots, a_{x_n}) = N_R(a_{x_1}, a_{x_2}, \cdots, a_{x_n}, a_x)$. Thus

$$(c-1) \in N_R(a_{x_1}, a_{x_2}, \cdots, a_{x_n}, a_x),$$

and so $a_x c - a_x \in \mathrm{nil}(R)$. This implies that $a_x c - a_x = a_x(c-1) \in \mathrm{nil}(R)$ for any $x \in \mathrm{supp}(f)$. Since σ is compatible, for any $x \in \mathrm{supp}(f)$, we have $a_x(c-1) \in \mathrm{nil}(R) \Rightarrow a_x\sigma_x(c-1) \in \mathrm{nil}(R) \Rightarrow a_x\sigma_x(c-1)t(x,1) \in nil(R)$. Thus by Proposition 2.1, we obtain

$$fc - f = f\cdot(c-1) = \left(\sum_{x\in G} a_x x\right)(c-1) = \sum_{x\in G} a_x\sigma_x(c-1)t(x,1)x \in nil(R*((G))).$$

Now we show that $c \in N_{R*((G))}(g)$. If $\mathrm{supp}(g) = \{y_1, y_2, \cdots, y_m\}$, Then for any $y \in \mathrm{supp}(g)$, $b_y c \in \mathrm{nil}(R)$. Now assume that $y \in \mathrm{supp}(g) - \{y_1, y_2, \cdots, y_m\}$. Then by the minimality of $N_R(X_0)$, we have $N_R(b_{y_1}, b_{y_2}, \cdots, b_{y_m}) = N_R(b_{y_1}, b_{y_2}, \cdots, b_{y_m}, b_y)$. Thus $c \in N_R(b_{y_1}, b_{y_2}, \cdots, b_{y_m}, b_y)$, and so $b_y c \in \mathrm{nil}(R)$. Hence for any $y \in \mathrm{supp}(g)$, $b_y c \in \mathrm{nil}(R)$, and so $b_y \sigma_y(c) t(y,1) \in \mathrm{nil}(R)$ for any $y \in \mathrm{supp}(g)$. By Proposition 2.1, we obtain $gc = (\sum_{y \in G} b_y y)c = \sum_{y \in G} b_y \sigma_y(c) t(y,1) y \in \mathrm{nil}(R*((G)))$. Hence $c \in N_{R*((G))}(g)$. Therefore $R*((G))$ is a weak APP-ring.

(2) $\Rightarrow$ (1) Let a, $b \in R$ be such that $a \in N_R(b)$. Then $a \in N_{R*((G))}(b)$. Since $R*((G))$ is a weak APP-ring, there exists $f = \sum_{x \in G} a_x x \in N_{R*((G))}(b)$ such that $af - a \in \mathrm{nil}(R*((G)))$. By Proposition 2.1, we obtain $aa_1 - a \in nil(R)$ where 1 denotes the identity of G. Since $f \in N_{R*((G))}(b)$, by Proposition 2.1, we have $ba_1 \in \mathrm{nil}(R)$, and so $a_1 \in N_R(b)$. Therefore R is a weak APP-ring. ∎

Let α be an endomorphism of a ring R. According to Hashemi and Moussavi [1], the ring R is said to be α-compatible if for each a, $b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$.

Corollary 3.1. *Let $\alpha \in Aut(R)$ and $\mathrm{nil}(R)$ nilpotent, and let R be an α-compatible NI ring satisfying the descending chain condition on weak annihilators. Then the following conditions are equivalent:*

(1) *R is a weak APP-ring.*
(2) *$R[[x, x^{-1}, \alpha]]$ is a weak APP-ring.*

Proof. Take $G = \mathbb{Z}$ and $t(x,y) = 1$ for any x, $y \in \mathbb{Z}$. For any $x \in \mathbb{Z}$, let $\sigma_x = \alpha^x$. Now the result follows from Proposition 3.3. ∎

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