**BULLETIN** *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

*Bull. Malays. Math. Sci. Soc. (2)* **36***(2) (2013), 363–372*

# Global Signed Domination in Graphs

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**Abstract.** A function $f : V(G) \to \{-1,1\}$ defined on the vertices of a graph $G$ is a signed dominating function (SDF) if the sum of its function values over any closed neighborhood is at least one. A SDF $f : V(G) \to \{-1,1\}$ is called a global signed dominating function (GSDF) if $f$ is also a SDF of the complement $\overline{G}$ of $G$. The global signed domination number $\gamma_{gs}(G)$ of $G$ is defined as $\gamma_{gs}(G) = \min\{\sum_{v\in V(G)} f(v) \mid f \text{ is a GSDF of } G\}$. In this paper we study this parameter and pose some open problems.

2010 Mathematics Subject Classification: 05C69, 05C05

Keywords and phrases: Signed dominating function, signed domination number, global signed dominating function, global signed domination number

## 1. Introduction

In the whole paper, $G$ is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly $V$ and $E$). For every vertex $v \in V$, the open neighborhood $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v\in S} N(v)$, and the *closed neighborhood* of $S$ is the set $N[S] = N(S) \cup S$. The *minimum* and *maximum* degrees of $G$ are respectively denoted by $\delta$ and $\Delta$. For a vertex $v$ in a rooted tree $T$, let $D(v)$ denote the set of descendants of $v$ and $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_v$. We use [12] for terminology and notation which are not defined here.

For a real-valued function $f : V \to \mathbb{R}$ the *weight* of $f$ is $\omega(f) = \sum_{v\in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v\in S} f(v)$, so $\omega(f) = f(V)$. For a vertex $v$ in $V$, we denote $f(N[v])$ by $f[v]$. Let $f : V \to \{-1,1\}$ be a function which assigns to each vertex of $G$ an element of the set $\{-1,1\}$. The function $f$ is said to be a *signed dominating function* (SDF) of $G$ (see [4]) if $f[v] \geq 1$ for every $v \in V$. The *signed domination number* of $G$, denoted by $\gamma_s(G)$, is the minimum weight of a signed dominating function on $G$. In the definition of the signed dominating function if we replace $\{-1,1\}$ with $\{0,1\}$, then the function is said to be a *dominating function*. The *domination number* of $G$, denoted by $\gamma(G)$, is the minimum

*Communicated by* Sanming Zhou.
*Received:* February 21, 2011; *Revised:* July 24, 2011.

weight of a dominating function on $G$. The domination and signed domination numbers have been studied by several authors (see for example [1–3, 5–9, 11, 13, 14]).

A signed dominating function $f : V(G) \to \{-1, 1\}$ is called a *global signed dominating function* (GSDF) if $f$ is also a SDF of its complement $\overline{G}$. This definition is parallel to the definition of a global dominating function of a graph defined in [10]. The global signed domination number of $G$, denoted by $\gamma_{gs}(G)$, is the minimum weight of a GSDF on $G$. A $\gamma_s(G)$-function is a SDF of $G$ with $\omega(f) = \gamma_s(G)$. For a (global) signed dominating function $f$ of $G$ we define $P = \{v \in V \mid f(v) = 1\}$ and $M = \{v \in V \mid f(v) = -1\}$. Since every GSDF of $G$ is a SDF on both $G$ and $\overline{G}$, we have

$$\gamma_{gs}(G) \geq \max\{\gamma_s(G), \gamma_s(\overline{G})\}. \tag{1.1}$$

Our purpose in this paper is to initiate the study of the global signed domination numbers in graphs. We first present two classes of graphs with equal signed domination number and global signed domination number, then we give bounds on global signed domination numbers. We make use of the following results.

**Theorem 1.1.** *[4] For every graph $G$ of order $n$, $\gamma_s(G) = n$ if and only if every non-isolated vertex is either a leaf or adjacent to a leaf.*

**Theorem 1.2.** *[4] For every graph $G$ of order $n \geq 3$ with $\Delta \leq 3$, $\gamma_s(G) \geq n/3$.*

**Theorem 1.3.** *[4] For every tree $T$ of order $n \geq 2$, $\gamma_s(T) \geq (n+4)/3$.*

**Theorem 1.4.** *[4] For $n \geq 2$, $\gamma_s(P_n) = n - 2\lfloor (n-2)/3 \rfloor$.*

**Theorem 1.5.** *[4] For $n \geq 3$, $\gamma_s(C_n) = n - 2\lfloor n/3 \rfloor$.*

**Theorem 1.6.** *[5] Every connected cubic graph of order $n$ different from the Petersen graph has signed domination number at most $3n/4$.*

**Theorem 1.7.** *[14] Let $K_{a,b}$ be a complete bipartite graph with $b \leq a$. Then*

$$\gamma_s(K_{a,b}) = \begin{cases} a+1 & \text{if } b = 1 \\ b & \text{if } 2 \leq b \leq 3 \text{ and } a \text{ is even} \\ b+1 & \text{if } 2 \leq b \leq 3 \text{ and } a \text{ is odd} \\ 4 & \text{if } b \geq 4 \text{ and } a, b \text{ are both even} \\ 6 & \text{if } b \geq 4 \text{ and } a, b \text{ are both odd} \\ 5 & \text{if } b \geq 4 \text{ and } a, b \text{ have different parity.} \end{cases}$$

We conclude this section with a proposition on $\gamma_{gs}(G)$.

**Proposition 1.1.** *For every graph $G$ of order $n \geq 2$, $\gamma_{gs}(G) \equiv n \pmod 2$.*

*Proof.* Let $f$ be a $\gamma_{gs}(G)$-function. Obviously, $n = |P| + |M|$ and $\gamma_{gs}(G) = |P| - |M|$. Therefore, $n - \gamma_{gs}(G) = 2|M|$ and the result follows. ∎

## 2. Some classes of graphs with $\gamma_{gs}(G) = \gamma_s(G)$

In this section we present two classes of graphs with equal signed domination number and global signed domination number. Recall that, for every pair $u, v$ of distinct vertices in $V$, the *distance* $\mathrm{dist}(u, v)$ is the minimum length of a ($u$-$v$)-*path*.

**Theorem 2.1.** *For every graph $G$ of order $n \geq 8$ with $\Delta \leq 3$, $\gamma_{gs}(G) = \gamma_s(G)$.*

*Proof.* Let $f$ be a $\gamma_s(G)$-function and $v \in V$. We prove that $f$ is also a SDF of $\overline{G}$. If $v$ is a leaf of $G$ and $uv \in E(G)$, then $f(u) = f(v) = 1$ and we have $f(N_{\overline{G}}[v]) \geq 2$ by Theorem 1.2. Assume $v$ is not a leaf. First let $\deg(v) = 2$. Since $f$ is a $\gamma_s(G)$-function, we have $f[v] \geq 1$, which implies that

$$f(N_{\overline{G}}[v]) = f(V) - f(N_G[v] \setminus \{v\}) \geq n/3 - 2.$$

Since $n \geq 8$ and $f(N_{\overline{G}}[v])$ is an integer we obtain $f(N_{\overline{G}}[v]) \geq 1$.

Finally, let $\deg(v) = 3$ and $N(v) = \{v_1, v_2, v_3\}$. Since $f[v] \geq 1$, we must have $f[v] = 2$ or $f[v] = 4$. If $n \geq 10$, then as above $f(N_{\overline{G}}[v]) = f(v) + f(V \setminus N_G[v]) \geq 1$.

Let $n = 8$. Since each vertex in $M$ must be adjacent to at least two vertices in $P$ and every two distinct vertices in $M$ have no common neighbors in $P$ (because $\Delta \leq 3$), $f$ assigns $-1$ to at most two vertices of $G$. Therefore $\gamma_s(G) \geq 4$ and the result follows as before.

Now let $n = 9$. First let $f[v] = 4$. Then $f(v) = 1$ and $v$ has no neighbor in $M$. As in case $n = 8$, it is easy to see that $f$ assigns $-1$ to at most two vertices of $G$. Therefore $\gamma_s(G) \geq 5$ and the result follows as before. Suppose now that $f[v] = 2$. If $f(v) = 1$, then we have $f(N_{\overline{G}}[v]) = f(v) + f(V \setminus N_G[v]) \geq 2$ by Theorem 1.2. Let $f(v) = -1$. Then $v_i$ $(i = 1, 2, 3)$ has no neighbors in $M$. Therefore $f$ assign $-1$ to at most one vertex in $V \setminus N_G[v]$ and so $f(V \setminus N_G[v]) \geq 3$. Hence, $f(N_{\overline{G}}[v]) = f(v) + f(V \setminus N_G[v]) \geq 2$. Thus, in all cases $f(N_{\overline{G}}[v]) \geq 1$ and $f$ is a signed dominating function on $\overline{G}$. Therefore $f$ is a global signed dominating function on $G$, and hence $\gamma_s(G) \geq \gamma_{gs}(G)$. Now the result follows by (1.1). ∎

$\gamma_s(G) = 2$ and $\gamma_{gs}(G) = 4$ $\qquad$ $\gamma_s(G) = 3$ and $\gamma_{gs}(G) = 5$

Figure 1. The assumption $n \geq 8$ in Theorem 2.1 is necessary

Figure 1 and the fact that $\gamma_s(K_3) = 1$, $\gamma_{gs}(K_3) = 3$, $\gamma_s(K_4) = 2$ and $\gamma_{gs}(K_4) = 4$ show that the assumption $n \geq 8$ in Theorem 2.1 is necessary. An immediate consequence of Theorems 1.5 and 2.1 now follows.

**Corollary 2.1.**

$$\gamma_{gs}(C_n) = \begin{cases} n & \text{if } n = 3, 4 \\ n - 2 & \text{if } n = 5, 6 \\ n - 2\left\lfloor \dfrac{n}{3} \right\rfloor & \text{if } n \geq 7. \end{cases}$$

**Theorem 2.2.** *For every tree $T$ of order $n \geq 3$, $\gamma_{gs}(T) = \gamma_s(T)$.*

*Proof.* If $T$ is a star, then by Theorem 1.7, $\gamma_s(G) = n$ and the theorem is true. Suppose $T$ is not a star and $f$ is a $\gamma_s(T)$-function. We show that $f$ is a global signed dominating function on $T$. Let $v \in V(T)$. If $v$ is a leaf and $uv \in E(T)$, then obviously $f(u) = f(v) = 1$. By Theorem 1.3 we have

$$\sum_{x \in N_{\overline{T}}[v]} f(x) = \gamma_s(G) - 1 \geq \frac{n+4}{3} - 1 = \frac{n+1}{3} \geq 1.$$

Suppose that $v$ is not a leaf. Let $T$ be rooted at $v$ and let $z$ be a vertex with $\mathrm{dist}(v,z) = 2$. We claim that

$$\sum_{x \in V(T_z)} f(x) \geq 1. \tag{2.1}$$

Assume that $P_z = \{x \in V(T_z) \mid f(x) = 1\}$ and $M_z = \{x \in V(T_z) \mid f(x) = -1\}$. If $M_z = \emptyset$, then we are done. Suppose that $M_z \neq \emptyset$. For each $x \in M_z$ we set

$$B_x = \{y \in T_x \mid f \text{ assings } 1 \text{ to all vertices in } (x,y)\text{-path in } T \text{ except } x\}.$$

Obviously, $|B_x| \geq 2$. Therefore $|P_z| \geq \sum_{x \in M_z} |B_x| \geq 2|M_z| \geq 2$. Thus

$$\sum_{x \in V(T_z)} f(x) = |P_z| - |M_z| \geq 2|M_z| - |M_z| = |M_z| \geq 1,$$

which proves our claim.

Let $z_1, \ldots, z_r$ be the vertices of $T$ such that $\mathrm{dist}(v,z_i) = 2$ for $i \in \{1,2,\ldots,r\}$. Then

$$\sum_{x \in N_{\overline{T}}[v]} f(x) = f(v) + \sum_{i=1}^{r} \sum_{x \in V(T_{z_i})} f(x). \tag{2.2}$$

If $v$ is a support vertex, then obviously $f(v) = 1$ and by (2.1) and (2.2), $\sum_{x \in N_{\overline{T}}[v]} f(x) \geq 1$. Assume $v$ is not a support vertex. Then $r \geq 2$ and by (2.1) and (2.2), we have $\sum_{x \in N_{\overline{T}}[v]} f(x) \geq 1$. This completes the proof. ∎

## 3. Bounds on the global signed domination numbers

In this section, we give some bounds on the global signed domination numbers of general graphs. Our first theorem shows that the global signed domination number of a graph is a positive integer.

**Theorem 3.1.** *Let $G$ be a graph of order $n \geq 3$. Then*

$$\gamma_{gs}(G) \geq \max\{3, \gamma_s(G), \gamma_s(\overline{G})\}. \tag{3.1}$$

*Furthermore, this bound is sharp.*

*Proof.* By (1.1) we have $\gamma_{gs}(G) \geq \max\{\gamma_s(G), \gamma_s(\overline{G})\}$. Thus, it suffices to prove $\gamma_{gs}(G) \geq 3$. Let $f$ be a $\gamma_{gs}(G)$-function. If $M = \emptyset$, then the result follows. Let $M \neq \emptyset$. Assume $x \in M$. Then

$$|N_G(x) \cap P| \geq |N_G(x) \cap M| + 2 \tag{3.2}$$

and

$$|N_{\overline{G}}(x) \cap P| \geq |N_{\overline{G}}(x) \cap M| + 2. \tag{3.3}$$

By (3.2) and (3.3) we have

$$|N_G(x) \cap P| + |N_{\overline{G}}(x) \cap P| \geq |N_G(x) \cap M| + |N_{\overline{G}}(x) \cap M| + 4.$$

It follows that $|P| \geq |M| + 3$ and so $\gamma_{gs}(G) = |P| - |M| \geq 3$.

To prove the sharpness, let $k \geq 3$ and let $G$ be the graph with vertex set $V(G) = \{u_i, v_i \mid 1 \leq i \leq k\} \cup \{z_1, z_2, z_3\}$ and edge set $E(G) = \{u_i u_{i+1}, u_i v_i, u_{i+1} v_i \mid 1 \leq i \leq k\}$, where $u_{k+1} = u_1$. Define $f : V(G) \to \{-1, 1\}$ by $f(v_i) = -1$ for $1 \leq i \leq k$ and $f(x) = 1$ otherwise. It is easy to see that $f$ is a GSDF of $G$ with $\omega(f) = 3$. Thus $\gamma_{gs}(G) = 3$ and the proof is complete. ∎

Now we prove that the difference $\gamma_{gs}(G) - \max\{\gamma_s(G), \gamma_s(\overline{G})\}$ can be arbitrarily large.

**Theorem 3.2.** *For every positive integer $k$, there exists a graph $G$ that both of $G$ and $\overline{G}$ are connected and*

$$\gamma_{gs}(G) - \max\{\gamma_s(G), \gamma_s(\overline{G})\} \geq 2k + 1.$$

*Proof.* Let $G$ be the graph with vertex set $V(G) = \{u_i, v_i \mid 0 \leq i \leq 4k - 1\}$ and edge set $E(G) = \{v_i v_j \mid 0 \leq i \neq j \leq 4k - 1\} \cup \{u_i v_i, u_i v_{i+1}, \ldots, u_i v_{i+2k-1} \mid 0 \leq i \leq 4k - 1\}$, where the sum is taken modulo $4k$. Obviously, $G \simeq \overline{G}$ and so $\gamma_s(G) = \gamma_s(\overline{G})$. Define $f : V(G) \to \{-1, 1\}$ by $f(v_i) = 1$ if $i \in \{0, 1, \ldots, 3k\}$ and $f(x) = -1$ otherwise. It is easy to see that $f$ is a SDF of $G$ which implies that $\gamma_s(G) \leq \omega(f) = 2 - 2k$. Therefore $\max\{\gamma_s(G), \gamma_s(\overline{G})\} \leq 2 - 2k$. By Theorem 3.1 we have $\gamma_{gs}(G) - \max\{\gamma_s(G), \gamma_s(\overline{G})\} \geq 2k + 1$ and the proof is complete. ∎

**Theorem 3.3.** *Let $G$ be a graph of order $n$ with $\delta(G) \geq 2$. Then $\gamma_{gs}(G) = n$ if and only if $\gamma_s(\overline{G}) = n$.*

*Proof.* Let $\gamma_{gs}(G) = n$. We claim that $\Delta(\overline{G}) \leq 1$. Let, to the contrary, $\Delta(\overline{G}) \geq 2$. Suppose that $x \in V(G)$ is a vertex with $\deg_{\overline{G}}(x) = \Delta(\overline{G})$. Define $f : V(G) \to \{-1, 1\}$ by $f(x) = -1$ and $f(v) = 1$ otherwise. Obviously, $f$ is a GSDF on $G$ and this contradicts the fact that $\gamma_{gs}(G) = n$. Thus $\Delta(\overline{G}) \leq 1$. Now the result follows by Theorem 1.1.

Conversely, if $\gamma_s(\overline{G}) = n$, then by Theorem 3.1, $\gamma_{gs}(G) = n$ and the proof is complete. ∎

We conclude this section with some upper bounds on the global signed domination number of a graph.

**Theorem 3.4.** *For every graph $G$ of order $n$,*

$$\gamma_{gs}(G) \leq n - 2\min\left\{ \left\lfloor \frac{\delta(G)}{2} \right\rfloor, \left\lfloor \frac{\delta(\overline{G})}{2} \right\rfloor \right\}.$$

*Proof.* Let, without loss of generality, $\theta = \lfloor \delta(G)/2 \rfloor = \min\{\lfloor \delta(G)/2 \rfloor, \lfloor \delta(\overline{G})/2 \rfloor\}$. Suppose that $v_1, \ldots, v_\theta$ are distinct vertices of $G$. Define $f : V(G) \to \{-1, 1\}$ by $f(v_i) = -1$ for $i = 1, \ldots, \theta$ and $f(x) = 1$ if $x \notin \{v_1, \ldots, v_\theta\}$. It is easy to see that $f$ is a GSDF of $G$ and $\omega(f) = n - 2\theta$. This completes the proof. ∎

The *diameter* of $G$, $\text{diam}(G)$, is defined by $\text{diam}(G) = \max\{\text{dist}(u, v) \mid u, v \in V(G)\}$. A path of length $\text{diam}(G)$ is called a *diametral* path.

**Theorem 3.5.** *Let $G$ be a $K_3$-free graph of order $n \geq 3$ with $\delta(G) \geq 2$. Then*

$$\gamma_{gs}(G) \leq n - 2\left\lfloor \frac{\text{diam}(G) - 1}{3} \right\rfloor.$$

*Proof.* If $\text{diam}(G) \leq 3$, then obviously the theorem is true. Let $\text{diam}(G) \geq 4$. Suppose that $Q = v_1 v_2 v_3 \ldots v_{\text{diam}(G)+1}$ is a diametral path in $G$. Define $f : V(G) \to \{-1, 1\}$ by $f(v_{3i}) = -1$ for $1 \leq i \leq \lfloor (\text{diam}(G) - 1)/3 \rfloor$ and $f(x) = 1$ otherwise. We prove that $f$ is a GSDF of $G$. If $u \in V(G)$, then obviously $|N_G[u] \cap M| \leq 1$. Since $\delta(G) \geq 2$, it follows that $f[u] \geq 1$ in

$G$ and so $f$ is a SDF on $G$. On the other hand, it is easy to see that, if $u \in V(\overline{G})$, then $|N_{\overline{G}}[u] \cap M| \leq \lfloor (\mathrm{diam}(G)-1)/3 \rfloor$ and since $G$ is a $K_3$-free graph, it follows that $|N_{\overline{G}}[u] \cap P \cap V(Q)| \geq \lfloor (\mathrm{diam}(G)-1)/3 \rfloor + 2$. Hence, $f_{\overline{G}}[v] \geq 2$ and so $f$ is a SDF on $\overline{G}$. Therefore, $f$ is a GSDF of $G$. We have $\omega(f) = n - 2\lfloor (\mathrm{diam}(G)-1)/3 \rfloor$ and the result follows. ∎

We note that the bound given in Theorem 3.5 is sharp for paths, complete graphs, stars and subdivided stars.

The next theorem presents an upper bound for the global signed domination number of a graph $G$, which contains cycles, in terms of the girth of $G$. Recall that the *girth* of $G$ (denoted $g(G)$) is the length of a smallest cycle in $G$.

**Theorem 3.6.** *Let $G \neq C_6$ be a connected graph of order $n$ with $\delta(G) \geq 2$ and $\gamma_s(\overline{G}) \neq n$. Then*

$$\gamma_{gs}(G) \leq n - 2\left\lfloor \frac{g(G)}{3} \right\rfloor.$$

*Proof.* It follows straightforwardly from $\gamma_s(\overline{G}) \neq n$ that $G$ is not isomorphic to $C_3$ or $C_4$. Since $\gamma_s(\overline{G}) \neq n$, we have $\gamma_{gs}(G) < n$ by Theorem 3.3. By Proposition 1.1, $\gamma_{gs}(G) \leq n-2$. Hence, we may assume $g(G) \geq 6$ for otherwise the result follows. Let $C = (v_1, v_2, \ldots, v_{g(G)})$ be a cycle with $g(G)$ edges. (Note that every finite graph with $\delta(G) \geq 2$ contains a cycle.) If $G = C$, then the result follows by Corollary 2.1. Suppose that $G \neq C$. It follows that $n \geq 7$. Then each vertex in $V(G) \setminus V(C)$ can be adjacent to at most one vertex in $V(C)$. Let $g$ be a $\gamma_s(C)$-function. Define $f : V(G) \to \{-1, 1\}$ by $f(x) = g(x)$ if $x \in V(C)$ and $f(x) = 1$ otherwise. It is easy to see that $f$ is a GSDF of $G$ with weight $\omega(f) = n - 2\lfloor g(G)/3 \rfloor$. Thus $\gamma_{gs}(G) \leq n - 2\lfloor g(G)/3 \rfloor$. ∎

We need the following Theorem to characterize the family of graphs with girth at least five which achieve the bound in Theorem 3.6.

**Theorem 3.7.** *Every connected graph $G$ of order $n$ with $\delta(G) = 2$ and $\Delta(G) = 3$ has signed domination number at most $3n/4$.*

*Proof.* Let $X = \{v \in V(G) \mid \deg(v) = 2\} = \{v_1, \ldots, v_k\}$. Obviously, $X \neq \emptyset$. Suppose that $G'$ is a copy of $G$ and $X' = \{v' \in V(G') \mid \deg(v') = 2\} = \{v'_1, \ldots, v'_k\}$. Let $H$ be the graph obtained from $G$ and $G'$ by joining $v_i$ to $v'_i$ for $1 \leq i \leq k$. Then $H$ is a cubic graph of order $2n$ different from the Petersen graph. By Theorem 1.6, $\gamma_s(H) \leq 6n/4$. Let $f$ be a $\gamma_s(H)$-function and let $f|_G$ and $f|_{G'}$ be the restrictions of $f$ on $G$ and $G'$, respectively. Obviously, $f|_G$ and $f|_{G'}$ are SDFs on $G$ and $G'$, respectively. Without loss of generality we may assume $w(f|_G) \leq w(f|_{G'})$. Now we have

$$2w(f|_G) = w(f|_G) + w(f|_{G'}) = \gamma_s(H) \leq 6n/4.$$

Thus $\gamma_s(G) \leq w(f|_G) \leq 3n/4$ and the proof is complete. ∎

**Theorem 3.8.** *Let $G$ be a connected graph of order $n$ with $\delta(G) \geq 2$ and $g(G) = 5$. Then $\gamma_{gs}(G) = n-2$ if and only if $G \simeq C_5$.*

*Proof.* If $G \simeq C_5$, then the result follows by Corollary 2.1. Now let $\gamma_{gs}(G) = n-2$. Let, to the contrary, $G \not\simeq C_5$. Assume that $C = (v_1, v_2, \ldots, v_5)$ is a cycle of $G$. First let $\Delta(G) \leq 3$. By Theorems 1.2 and 3.7, $n - 2 \leq 3n/4$ and so $6 \leq n \leq 8$. Since $\delta(G) \geq 2$ and each vertex in $V(G) \setminus V(C)$ is adjacent to at most one vertex in $V(C)$, $n \neq 6$. Assume that $n = 7$ and $x, y \in V(G) \setminus V(C)$. Since each vertex in $V(G) \setminus V(C)$ is adjacent to at most one vertex in $V(C)$, $xy \in E(G)$. Since $\delta(G) \geq 2$, we may assume $xv_1 \in E(G)$. Since $g(G) = 5$ and $\delta(G) \geq 2$, $y$ must be adjacent to only one vertex in $\{v_3, v_4\}$. Without loss of generality we may assume $yv_3 \in E(G)$. Then $G \simeq G_1$ (see Figure 2). Define $f : V(G) \to \{-1, 1\}$ by $f(y) = f(v_5) = -1$ and $f(x) = 1$ if $x \in V(G) \setminus \{y, v_5\}$. Obviously, $f$ is a GSDF of $G$ and so $\gamma_{gs}(G) \leq w(f) = n-4$, a contradiction. Finally, let $n = 8$. Suppose that $x, y, z \in V(G) \setminus V(C)$. Since each vertex in $V(G) \setminus V(C)$ is adjacent to at most one vertex in $V(C)$, we may assume $xy, yz \in V(G)$. Then $xz \notin E(G)$. Consider two cases.

**Case 1:** $N(y) \cap V(C) \neq \emptyset$.

Let, without loss of generality, $yv_1 \in E(G)$. Since $\delta(G) \geq 2$, $x$ and $z$ must be adjacent to only one vertex in $\{v_3, v_4\}$. Without loss of generality, we may assume $xv_3, zv_4 \in E(G)$. Then $G \simeq G_2$ (see Figure 2). Define $f : V(G) \to \{-1, 1\}$ by $f(z) = f(v_2) = -1$ and $f(x) = 1$ if $x \in V(G) \setminus \{z, v_2\}$. Then $f$ is a GSDF of $G$ and so $\gamma_{gs}(G) \leq w(f) = n-4$, a contradiction.

**Case 2:** $N(y) \cap V(C) = \emptyset$.

Then $\deg(y) = 2$. Since $\delta(G) \geq 2$, $N(x) \cap V(C) \neq \emptyset$ and $N(z) \cap V(C) \neq \emptyset$. Without loss of generality, we may assume $xv_1 \in E(G)$. Then $z$ must be adjacent to only one vertex in $\{v_3, v_4\}$ or $z$ is adjacent to some vertex in $\{v_2, v_5\}$. If $zv_3 \in E(G)$ or $zv_4 \in E(G)$, then $G \simeq G_3$ (see Figure 2). If $z$ is adjacent to some vertex in $\{v_2, v_5\}$, then $z$ is adjacent to at most one of them because $g(G) = 5$ and so $G \simeq G_4$ or $G \simeq G_5$ (see Figure 2). It is now easy to see that $\gamma_{gs}(G) \leq n-4$, a contradiction.

Now let $\Delta(G) \geq 4$. Suppose that $x \in V(G)$ is a vertex of degree $\Delta(G)$ and $x_1, x_2 \in N(x)$. Since $g(G) = 5$, $N[x_1] \cap N[x_2] = \{x\}$. Define $f : V(G) \to \{-1, 1\}$ by $f(x_1) = f(x_2) = -1$ and $f(v) = 1$ if $v \in V(G) \setminus \{x_1, x_2\}$. Obviously, $f$ is a GSDF of $G$ and so $\gamma_{gs}(G) \leq w(f) = n-4$, a contradiction. ∎

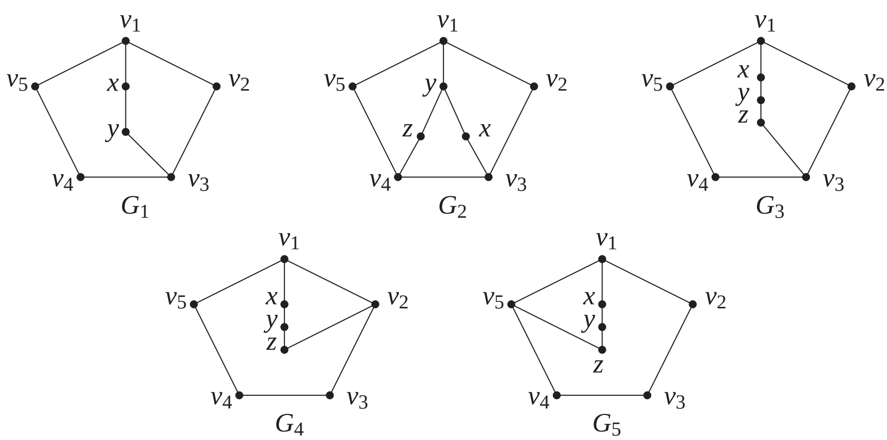


Figure 2. Connected graphs $G$ of order $n = 7, 8$ with $g(G) = 5$ and $\gamma_{gs} \leq n-4$

**Theorem 3.9.** *Let $G$ be a connected graph of order $n$ with $\delta(G) \geq 2$ and $g(G) \geq 7$. Then $\gamma_{gs}(G) = n - 2\lfloor g(G)/3 \rfloor$ if and only if $G \simeq C_n$.*

*Proof.* If $G \simeq C_n$, then the result follows by Corollary 2.1. Now let $\gamma_{gs}(G) = n - 2\lfloor g(G)/3 \rfloor$. Let, to the contrary, $G \not\simeq C_n$. Assume that $C = (v_1 v_2 \ldots v_{g(G)})$ is a cycle of $G$. Since $\delta(G) \geq 2$

and $g(G) \geq 7$, there exists a vertex $z \in V(G)$ such that $\text{dist}(z, V(C)) = 2$. Without loss of generality we may assume $N(v_1) \cap N(z) \neq \emptyset$. Let $x \in N(v_1) \cap N(z)$. If $g(G) = 7, 8$, then define $f : V(G) \to \{-1, 1\}$ by $f(z) = f(v_2) = f(v_{g(G)-2}) = -1$ and $f(x) = 1$ otherwise. If $g(G) \geq 9$, then define $f : V(G) \to \{-1, 1\}$ by $f(z) = f(v_{3i+2}) = -1$ for $0 \leq i \leq \lfloor g(G)/3 \rfloor - 1$ and $f(x) = 1$ otherwise. Obviously, $f$ is a GSDF of $G$ and so $\gamma_{gs}(G) \leq w(f) = n - 2(\lfloor g(G)/3 \rfloor + 1)$, a contradiction. This completes the proof. ∎

**Theorem 3.10.** *Let $G$ be a connected graph of order $n \geq 17$ with $\delta(G) \geq 2$ and $g(G) = 6$. Then $\gamma_{gs}(G) \leq n - 6$.*

*Proof.* Let, to the contrary, $\gamma_{gs}(G) \geq n - 4$. Note that by Proposition 1.1, $\gamma_{gs}(G) \neq n - 5$. By Theorems 3.3 and 3.6 we have $\gamma_{gs}(G) = n - 4$. First let $\Delta(G) \leq 3$. If $\delta(G) = \Delta(G) = 3$, then it follows from Theorem 1.6 that $n - 4 \leq (3n)/4$ and so $n \leq 16$, a contradiction. If $\delta(G) = \Delta(G) = 2$, then $G \simeq C_n$ and by Theorems 2.1 and 1.5, $n - 4 = n - 2\lfloor \frac{n}{3} \rfloor$ and so $n = 6, 7$ or $8$, a contradiction. If $\delta(G) = 2$ and $\Delta(G) = 3$, then it follows from Theorem 3.7 that $n - 4 \leq (3n)/4$ and so $n \leq 16$, a contradiction.

Now let $\Delta(G) \geq 4$ and let $x \in V(G)$ be a vertex of degree $\Delta(G)$. First assume that $x$ belongs to a minimal cycle in $G$, say $C = (v_0, v_1, \ldots, v_r)$ where $x = v_0$. Suppose that $x_1, x_2 \in N(x) \setminus V(C)$. Define $f : V(G) \to \{-1, 1\}$ by $f(x_1) = f(x_2) = f(v_2) = -1$ and $f(u) = 1$ for $u \in V(G) \setminus \{x_1, x_2, v_2\}$. Obviously, $f$ is a GSDF of $G$ and so $\gamma_{gs}(G) \leq w(f) = n - 6$, a contradiction. So we may assume $x$ does not belong to a cycle in $G$. Let $C = (v_1, v_2, \ldots, v_s)$ be a cycle in $G$ for which $\text{dist}(x, V(C))$ is minimum. If $\text{dist}(x, V(C)) = 1$, then without loss of generality we may assume $xv_1 \in E(G)$. Suppose that $x_1, x_2 \in N(x) \setminus \{v_1\}$. Then the function $f : V(G) \to \{-1, 1\}$ defined by $f(x_1) = f(x_2) = f(v_2) = -1$ and $f(u) = 1$ for $u \in V(G) \setminus \{x_1, x_2, v_2\}$, is a GSDF of $G$ which leads to a contradiction. Now let $\text{dist}(x, V(C)) \geq 2$. Suppose that $xz_1z_2 \ldots z_s$ is a shortest $(x, V(C))$-path and $x_1, x_2 \in N(x) \setminus \{z_1\}$. Define $f : V(G) \to \{-1, 1\}$ by $f(x_1) = f(x_2) = f(z_2) = -1$ and $f(u) = 1$ for $u \in V(G) \setminus \{x_1, x_2, z_2\}$. Obviously, $f$ is a GSDF of $G$ and so $\gamma_{gs}(G) \leq w(f) = n - 6$, a contradiction. This completes the proof. ∎

## 4. The global signed domination number of complete bipartite graphs

As the parameter $\gamma_{gs}(G)$ is new, it is important to determine its values for some familiar graphs. In this section we find the exact value of the global signed domination number for complete bipartite graphs.

**Theorem 4.1.** *Let $K_{a,b}$ be a complete bipartite graph with parts $A$, $B$ such that $|A| = a$, $|B| = b$, $b \leq a$. Then*

$$\gamma_{gs}(K_{a,b}) = \begin{cases} a+1 & \text{if } b = 1 \\ 4 & \text{if } a, b \text{ are both even} \\ 5 & \text{if } a \text{ is even and } b \geq 3 \text{ is odd} \\ 4 & \text{if } b = 3 \textit{ and a is odd} \\ 6 & \text{if } b \geq 5 \text{ and } a, b \text{ are both odd} \\ 3 & \text{if } b = 2 \text{ and } a \text{ is odd} \\ 5 & \text{if } b \geq 4 \text{ is even and } a \text{ is odd.} \end{cases}$$

*Proof.* If $b = 1$, then by Theorem 1.7, $\gamma_s(G) = a + 1$ and the result follows by Theorem 3.1. Let $b \geq 2$ and let $A = \{x_1, x_2, \ldots, x_a\}$ and $B = \{y_1, y_2, \ldots, y_b\}$. We consider four cases.

**Case 1:** $a$ and $b$ are both even.
Define $f : V(K_{a,b}) \to \{-1,1\}$ by $f(x_i) = 1$ for $1 \le i \le a/2+1$, $f(y_j) = 1$ if $1 \le j \le b/2+1$ and $f(x) = -1$ otherwise. Obviously, $f$ is a GSDF of $G$ and so $\gamma_{gs}(K_{a,b}) \le \omega(f) = 4$. Now the result follows by Proposition 1.1 and Theorem 3.1.

**Case 2:** $a$ is even and $b \ge 3$ is odd.
First note that by assumptions $a \ge 4$. Define $f : V(K_{a,b}) \to \{-1,1\}$ by $f(x_i) = 1$ for $1 \le i \le a/2+1$, $f(y_j) = 1$ for $1 \le j \le (b+3)/2$ and $f(x) = -1$ otherwise. It is easy to see that $f$ is a GSDF of $G$ with $\omega(f) = 5$. Now the result follows by Theorems 1.7 and 3.1.

**Case 3:** $a$ and $b$ are both odd.
First let $b = 3$. By Theorems 1.7 and 3.1, $\gamma_{gs}(K_{a,b}) \ge \gamma_s(K_{a,b}) = 4$. Define $f : V(K_{a,b}) \to \{-1,1\}$ by $f(x_i) = -1$ for $1 \le i \le \lfloor a/2 \rfloor$ and $f(x) = 1$, otherwise. Obviously, $f$ is a GSDF of $G$ with $\omega(f) = 4$. It follows that $\gamma_{gs}(K_{a,b}) = 4$.

Now suppose that $b \ge 5$. By Theorems 1.7 and 3.1, $\gamma_{gs}(K_{a,b}) \ge \max\{3,6,2\} = 6 = \gamma_s(K_{a,b})$. Define $f : V(K_{a,b}) \to \{-1,1\}$ by $f(x_i) = f(y_j) = 1$ for $1 \le i \le (a+3)/2$, $1 \le j \le (b+3)/2$ and $f(x) = -1$, otherwise. It is easy to verify that $f$ is a GSDF with $\omega(f) = 6$. This implies that $\gamma_{gs}(K_{a,b}) = 6$.

**Case 4:** $a$ is odd and $b$ is even.
First let $b = 2$. Define $f : V(K_{a,b}) \to \{-1,1\}$ by $f(x_i) = -1$ for $1 \le i \le \lfloor a/2 \rfloor$ and $f(x) = 1$ otherwise. Obviously, $f$ is a GSDF of $G$ with $\omega(f) = 3$. Therefore $\gamma_{gs}(K_{a,b}) \le 3$. Now the result follows by Theorems 1.7 and 3.1.

Let $b \ge 4$. Define $f : V(K_{a,b}) \to \{-1,1\}$ by $f(x_i) = f(y_j) = -1$ for $1 \le i \le \lfloor a/2 \rfloor - 1$, $1 \le j \le (b-2)/2$ and $f(x) = 1$ otherwise. Clearly, $f$ is a GSDF on $G$ with $\omega(f) = 5$. So $\gamma_{gs}(K_{a,b}) \le 5$. By Theorems 1.7 and 3.1, $\gamma_{gs}(K_{a,b}) \ge \max\{3,5,3\} = 5$, hence the result follows. This completes the proof. ∎

## 5. Some open problems

It is clear that for a graph $G$ of order $n$, $\gamma_{gs}(G) - |\gamma_s(G)| \le n-1$. A natural problem is the following.

**Problem 1.** Find a "good" lower bound for $\gamma_{gs}(G) - |\gamma_s(G)|$.

Define $g(n) = \max\{\gamma_{gs}(G) - \max\{\gamma_s(G), \gamma_s(\overline{G})\} \mid G \text{ is a graph of order } n\}$.
We know from the construction illustrated in the proof of Theorem 3.2, $g(n) \ge n/4+1$ holds when $n \equiv 0 \pmod 4$.

**Problem 2.** Find "good" lower and upper bounds for $g(n)$.

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