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Pseudo FQ-Injective Modules

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Abstract. A module M_R is called pseudo FQ-injective (or PFQ-injective for short) if every monomorphism from a finitely generated submodule of M to M extends to an endomorphism of M. Some characterizations and properties of this class of modules are investigated. In particular, finitely generated PFQ-injective modules are studied.

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1. Introduction and results

Throughout R is an associative ring with identity and all modules are unitary. Following [5], a right R-module M is called pseudo-injective if every monomorphism from a submodule of M to M extends to an endomorphism of M. And following [10], a right R-module M is called finitely quasi-injective (or FQ-injective for short) if every homomorphism from a finitely generated submodule of M to M extends to an endomorphism of M. In this paper, we generalize the concepts of pseudo-injective modules and FQ-injective modules to PFQ-injective modules and give some interesting results on these modules.

As usual, we denote the Jacobson radical of a ring R by $J(R)$ and denote the right singular ideal of R by $Z(R_R)$. Let M be a right R-module. Then we let $S = End(M_R)$, and we denote the injective hull of M by $E(M)$.

We start with the following definition.

Definition 1.1. *Let R be a ring. A right R-module M is called pseudo FQ-injective (or PFQ-injective for short) if every monomorphism from a finitely generated submodule of M to M extends to an endomorphism of M. A ring R is called right pseudo F-injective (or right PF-injective for short) if R_R is pseudo FQ-injective.*

Example 1.1. Let M be one of the following two examples of pseudo-injective modules which are not quasi-injective: either the Hallet's example or the Teply's example (see [5, p. 364]). Since M has only five submodules 0, M , N_1, N_2 and $N_1 \oplus N_2$, it is noetherian, so M is PFQ-injective but not FQ-injective.

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Theorem 1.1. *The following statements are equivalent for a module M_R with $S = End(M_R)$:*

(1) *M is PFQ-injective.*

(2) *$\mathbf{r}_{R_n}(x) = \mathbf{r}_{R_n}(y), x, y \in M^n, n \in Z^+$, implies that $Sx = Sy$.*

(3) *If $x_i \in M, i = 1, 2, \cdots, n$ and $f, g : \sum_{i=1}^n x_i R \to M$ are monic, then there exists $s \in S$ such that $f = sg$.*

Proof. (1) ⇒ (2). If $\mathbf{r}_{R_n}(x) = \mathbf{r}_{R_n}(y), x, y \in M^n, n \in Z^+$, write $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n)$, then the mapping $f : \sum_{i=1}^n x_i R \to M$; $\sum_{i=1}^n x_i r_i \mapsto \sum_{i=1}^n y_i r_i$ is a monomorphism. Since M is PFQ-injective, there exists $s \in S$ such that s extends f, then $y_i = f(x_i) = sx_i, i = 1, 2, \cdots, n$, so $y = sx$, and thus $Sy \subseteq Sx$. Similarly, $Sx \subseteq Sy$. Hence $Sx = Sy$.

(2) ⇒ (3). Since f, g are monic,

$$\mathbf{r}_{R_n}(f(x_1), f(x_2), \cdots, f(x_n)) = \mathbf{r}_{R_n}(g(x_1), g(x_2), \cdots, g(x_n)).$$

By (2), we have $S(f(x_1), f(x_2), \cdots, f(x_n)) = S(g(x_1), g(x_2), \cdots, g(x_n))$, which shows that there exists $s \in S$ such that $(f(x_1), f(x_2), \cdots, f(x_n)) = s(g(x_1), g(x_2), \cdots, g(x_n))$, and hence $f = sg$.

(3) ⇒ (1). Take $g : \sum_{i=1}^n x_i R \to M$ to be the inclusion mapping in (3). ∎

Corollary 1.1. *The following statements are equivalent for a ring R:*

(1) *R is right PF-injective.*

(2) *$\mathbf{r}_{R_n}(\alpha) = \mathbf{r}_{R_n}(\beta), \alpha, \beta \in R^n, n \in Z^+$, implies that $R\alpha = R\beta$.*

(3) *If $a_i \in R, i = 1, 2, \cdots, n$ and $f, g : \sum_{i=1}^n a_i R \to R$ are monic, then there exists $a \in R$ such that $f = ag$.*

Let M and N be two right R-modules. Then we call M finitely N-injective (or F-N-injective for short) if every homomorphism from a finitely generated submodule of N to M extends to a homomorphism of N to M; and we call M pseudo finitely N-injective (or PF-N-injective for short) if every monomorphism from a finitely generated submodule of N to M extends to a homomorphism of N to M. Clearly, M is FQ-injective if and only if M is F-M-injective, and M is PFQ-injective if and only if M is PF-M-injective.

Proposition 1.1. *Let M, N be two right R-modules and N' be a submodule of N. If M is PF-N-injective (resp., F-N-injective), then*

(1) *Every direct summand of M is PF-N-injective (resp., F-N-injective).*

(2) *M is PF-N'-injective (resp., F-N'-injective).*

Proof.

(1). Let $M = M_1 \oplus M_2$. Then for every finitely generated submodule K of N and every monomorphism (resp., homomorphism) f of K to M_1, since M is PF-N-injective (resp., F-N-injective), f extends to a homomorphism of N to M. It follows that f extends to a homomorphism of N to M_1 because M_1 is a direct summand of M.

(2). It is obvious. ∎

By Proposition 1.1, we have immediately the following corollary.

Corollary 1.2. *Every direct summand of a PFQ-injective module is PFQ-injective.*

Recall that a module M is called C_2 [8] if every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M; a module M is called GC_2 [17] if, every submodule of M that is isomorphic to M is itself a direct summand of M; a module

M is called C_3 [8] if, whenever N and K are direct summands of M with $N \cap K = 0$ then $N \oplus K$ is also a direct summand of M. We call a module M FC_2 if every finitely generated submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M; and we call a module M FC_3 if, whenever N and K are direct summands of M with $N \cap K = 0$ and K is finitely generated, then $N \oplus K$ is also a direct summand of M.

Theorem 1.2. *Every PFQ-injective module is FC_2 and FC_3.*

Proof. Let M_R be PFQ-injective with $S = End(M_R)$. If K is a finitely generated submodule of M and $K \cong sM$, where $s^2 = s \in S$, then sM is PF-M-injective by proposition 1.1 and hence K is also PF-M-injective, which implies that K is a direct summand of M because K is finitely generated. This proves FC_2. Now let N and K be direct summands of M with $N \cap K = 0$ and K finitely generated. Write $N = eM$ and $K = fM$, where e, f are idempotents in S, then $eM \oplus fM = eM \oplus (1-e)fM$. Since $(1-e)fM \cong fM$ is finitely generated, $(1-e)fM = hM$ for some $h^2 = h \in S$ by FC_2. Let $g = e + h - he$, then $g^2 = g$ and $eM \oplus fM = gM$, as required. ∎

Recall that a right R-module M is said to be weakly injective [4] if for every finitely generated submodule $N_R \subseteq E(M)$, we have $N \subseteq X_R \subseteq E(M)$ for some $X_R \cong M$.

Corollary 1.3. *Let M_R be a finitely generated module. Then M is injective if and only if it is weakly injective and PFQ-injective.*

Proof. We need only to prove the sufficiency. Let $x \in E(M)$, then there exists $X \subseteq E(M)$ such that $M + xR \subseteq X \cong M$. Since M is PFQ-injective, X is also PFQ-injective. By Theorem 1.2, X is FC_2 and hence M is a direct summand of X because M is a finitely generated submodule of X. But $M \subseteq^{ess} E(M)$, so $M \subseteq^{ess} X$. Thus $M = X$, and then $x \in M$. Therefore, $M = E(M)$ is injective. ∎

Recall that a module M_R is regular [16] if for every $m \in M$, mR is projective and is a direct summand of M, or equivalently, if every finitely generated submodule of M is projective and is a direct summand of M.

Definition 1.2. *A right R-module M is called pseudo regular if every finitely generated submodule of M is a direct summand of M.*

We note that pseudo regular modules are called strongly regular in [10]. Clearly, a ring R is von Neumann regular if and only if R_R is pseudo regular if and only if every free right R-module is pseudo regular if and only if every projective right R-module is pseudo regular.

The following Theorem 1.3 follows immediately from Theorem 1.2.

Theorem 1.3. *A right R-module M is pseudo regular if and only if M is PFQ-injective and every finitely generated submodule of M is isomorphic to a direct summand of M.*

About pseudo regular modules, we have the following results, which we state without proof.

Proposition 1.2. *Let M_R be a pseudo regular module. Then:*

(1) *Each submodule of M is also pseudo regular.*
(2) *If N is a finitely generated submodule of M, then M/N is pseudo regular.*
(3) *$Rad(M)=0$.*
(4) *M is a finitely generated semisimple module if and only if M is finitely cogenerated if and only if M is artinian if and only if M is noetherian.*

(5) *If I is an ideal such that $I \subseteq r_R(M)$, then M is also pseudo regular as an R/I-module.*

Theorem 1.4. *Let M_R be a finitely generated PFQ-injective module. Then*

(1) *M_R is a C_2 module.*
(2) *$J(S) = W(S)$, where $W(S) = \{s \in S \mid ker(s) \subseteq^{ess} M\}$.*
(3) *If M_R has finite Goldie dimension then S is semilocal.*
(4) *If M_R is uniform, then S is local.*

Proof.

(1). Since M_R is finitely generated, each direct summand of M_R is also finitely generated, so (1) follows because M_R is FC_2 by Theorem 1.2.

(2). Let $s \in J(S)$. If $s \notin W(S)$, then $ker(s) \cap K = 0$ for some $0 \neq K \leq M_R$. Take $k \in K$ such that $sk \neq 0$. Then $\mathbf{r}_R(k) = \mathbf{r}_R(sk)$, and so the mapping $f : skR \to kR; skr \mapsto kr$ is a monomorphism. Since M_R is PFQ-injective, there exists a $t \in S$ such that $f = t\cdot$. Hence $k = f(sk) = tsk$, i.e., $(1 - ts)k = 0$, and then $k = 0$, a contradiction. Thus, $J(S) \subseteq W(S)$. Since M_R is C_2 by (1), $W(S) \subseteq J(S)$ by [15, 41.22]. Therefore, $J(S) = W(S)$.

(3). Let s be any injective endomorphism of M. Then $s^k M \cong M$ for each positive integer k, and so $s^k M$ is a direct summand of M_R as M_R is a C_2 module by (1). Since M_R has finite Goldie dimension, it satisfies the descending conditions on direct summands. Hence $s^n M = s^{n+1} M$ for some positive integer n. This s is bijective. Therefore, S is semilocal by [3, Theorem 3].

(4). Let $s \in S$ and $S \neq S s$. Since M_R is C_2, it is GC_2. By [18, Theorem 4], $Ker(s) \neq 0$. So $Ker(s) \subseteq^{ess} M$ as M is uniform. Thus $s \in W(S) = J(S)$. This means that S is local. ∎

Corollary 1.4. *Let R be a right PF-injective ring. Then*

(1) *R is a right C_2 ring.*
(2) *$J(R) = Z(R_R)$.*
(3) *If R is right finite dimensional then R is semilocal.*
(4) *If R_R is uniform, then R is local.*

B. Stenström [14] defined and studied *FP-injective modules*. Following [14] , a right R-module M is said to be FP-injective if, for any projective right R-module P, every homomorphism from a finitely generated submodule of P to M can be extended to a homomorphism from P to M. FP-injective modules have been generalized by several authors. For example, principally injective modules, (m,n)-injective modules, (m,n)-small injective modules have been introduced and studied in [1, 7, 9], respectively. A right R-module M is called principally injective if every homomorphism from a principal right ideal of R to M can be extended to a homomorphism from R to M; a right R-module M is called (m,n)-injective if every homomorphism from an n-generated submodule of R^m to M can be extended to a homomorphism from R^m to M; a right R-module M is called (m,n)-small injective if every homomorphism from an n- generated submodule of J^m to M can be extended to a homomorphism from R^m to M, where $J = J(R)$ is the Jacobson radial of R. Clearly, a module M is FP-injective if and only if it is (m,n)-injective for each pair of positive integers m,n, a module M is principally injective if and only if it is $(1,1)$-injective, (m,n)-injective module is (m,n)- small injective. By [9, Theorem 2.12], if R is a semiregular ring, then an R-module is (m,n)-injective if and only if it is (m,n)-small injective. Motivated this notion of principally injectivity, in 1999, Sanh, Shum *et al.* [12] introduced the notion of M-pinjective module, following that, a right R-module N is M-p-injective if every homomorphism from an M-cyclic submodule of M to N can be extended to one from M to N; and a module M is called QP-injective [12] if M is M-p-injective. The detailed discussion of QP-injective modules can be found in [12, 13]. Continue this direction, Sanh *et al.*, introduced the notion of M-f-injectivity [6], following which, he replace an M-cylic submodule by a finitely M-generated submodule, this kind of submodules is of the form $\sum_{i=1}^{n} s_i(M)$ with all s_i are endomorphisms of M; and a module M is called quasi-f-injective [6] if M is M-f-injective. Recently, the generalizations of QP-injective modules have been studied by many authors also, for example, Sanh *et al.* studied the concepts of pseudo p-injectivity and quasi-rp-injectivity in [11] and [2] respectively. Next we generalize the concept of quasi-f-injective modules as following:

Definition 1.3. *Let R be a ring. A right R-module M is called pseudo quasi F-injective (or PQF-injective for short) if every monomorphism from a finitely M-generated submodule of M to M extends to an endomorphism of M.*

Proposition 1.3. *The following conditions are equivalent for a module M_R with $S = \text{End}(M_R)$.*

(1) *M is PQF-injective.*
(2) *$\mathbf{r}_{M_n}(\alpha) = \mathbf{r}_{M_n}(\beta), \alpha, \beta \in S^n, n \in Z^+$, implies that $S\alpha = S\beta$.*

Proof. (1) $\Rightarrow$ (2). Suppose that $\mathbf{r}_{M_n}(\alpha) = \mathbf{r}_{M_n}(\beta), \alpha, \beta \in S^n, n \in Z^+$. Write $\alpha = (s_1, s_2, \cdots, s_n)$, $\beta = (t_1, t_2, \cdots, t_n)$. Then the mapping $f : \sum_{i=1}^{n} s_i M \to M; \sum_{i=1}^{n} s_i m_i \mapsto \sum_{i=1}^{n} t_i m_i$ is a monomorphism. Since M is PQF-injective , there exists $s \in S$ such that s extends f, then $t_i m = f(s_i m) = s s_i m, i = 1, 2, \cdots, n$, for each $m \in M$. So $\beta = s\alpha$, and thus $S\beta \subseteq S\alpha$. Similarly, $S\alpha \subseteq S\beta$. Hence $S\alpha = S\beta$.

(2) $\Rightarrow$ (1). Assume (2). Let $f : \sum_{i=1}^{n} s_i M \to M$ be a monomorphism. Write $\alpha = (s_1, s_2, \cdots, s_n), \beta = (f s_1, f s_2, \cdots, f s_n)$, then $\mathbf{r}_{M_n}(\alpha) = \mathbf{r}_{M_n}(\beta)$. By (2), we have $\beta \in S\alpha$, so there exists $s \in S$ such that $\beta = s\alpha$, and hence s extends f. This proves (1). ∎

Proposition 1.4. *Let M be a right R-module with $S = End(M_R)$. Then*

(1) *If S is right PF-injective, then M_R is PQF-injective.*
(2) *If M_R is PQF-injective and M generates $\mathbf{r}_{M_n}(\alpha)$ for any positive integer n and $\alpha \in S^n$, then S is right PF-injective.*

Proof.

(1). Let $\mathbf{r}_{M_n}(\alpha) = \mathbf{r}_{M_n}(\beta), \alpha, \beta \in S^n, n \in Z^+$, then $\mathbf{r}_{S_n}(\alpha) = \mathbf{r}_{S_n}(\beta)$. Since S is right PF-injective, by Corollary 1.1, we have $S\alpha = S\beta$. Hence M is PQF-injective by Proposition 1.3.

(2). Let $\mathbf{r}_{S_n}(\alpha) = \mathbf{r}_{S_n}(\beta), \alpha, \beta \in S^n, n \in Z^+$. Then for any $x \in \mathbf{r}_{M_n}(\alpha)$, since M generates $\mathbf{r}_{M_n}(\alpha)$, we have $x = \sum_{i=1}^{k} \lambda_i m_i$ with $\lambda_i \in S_n$ and $\lambda_i M \subseteq \mathbf{r}_{M_n}(\alpha)$. Hence each $\lambda_i \in \mathbf{r}_{S_n}(\alpha)$. This implies that $\beta\lambda_i = 0, i = 1, 2, \cdots, k$, and thus $x \in \mathbf{r}_{M_n}(\beta)$. Hence, $\mathbf{r}_{M_n}(\alpha) \subseteq \mathbf{r}_{M_n}(\beta)$. Similarly, $\mathbf{r}_{M_n}(\beta) \subseteq \mathbf{r}_{M_n}(\alpha)$. And so $\mathbf{r}_{M_n}(\alpha) = \mathbf{r}_{M_n}(\beta)$. It follows that $S\alpha = S\beta$ since M_R is PQF-injective. Consequently, S is PF-injective by Corollary 1.1. ∎

Recall that a right R-module M is said to be a self-generator [15] if it generates all its submodules.

Lemma 1.1. *If M is a finitely generated right R-module which is a self-generator, then M is PFQ-injective if and only if M is PQF-injective.*

Proof. It is Obvious. ∎

Recall that a right R-module N is said to be subgenerated by a right R-module M, if N is isomorphic to a submodule of an M-generated module [15]. Following [15, p118], we denote by $\sigma[M]$ the full subcategory of $R-MOD$ whose objects are all R-modules subgenerated by M. By Proposition 1.4 and Lemma 1.1, we have immediately the following theorem.

Theorem 1.5. *Let M_R be a finitely generated module with $S = End(M_R)$. If M_R is a generator in $\sigma[M]$. Then the following conditions are equivalent:*

(1) *S is right PF-injective.*
(2) *M_R is PQF-injective.*
(3) *M_R is PFQ-injective.*

Corollary 1.5. *The following statements are equivalent for a ring R and a positive integer n:*

(1) *The free right R-module R^n is PFQ-injective.*
(2) *The full matrix ring $\mathbb{M}_n(R)$ is right PF-injective.*

Corollary 1.6. *The following statements are equivalent for a ring R:*

(1) *Every finitely generated free right R-module is PFQ-injective.*
(2) *Every finitely generated projective right R-module is PFQ-injective.*
(3) *The full matrix ring $\mathbb{M}_n(R)$ is right PF-injective for every positive integer n.*

We call two modules M, N mutually F-injective (resp., PF-injective) if M is F-N-injective (resp., PF-N-injective) and N is F-M-injective (resp., PF-M-injective).

Theorem 1.6. *If $M_1 \oplus M_2$ is PFQ-injective. Then M_1 and M_2 are mutually F-injective. In particular, if M is a right R-module such that $M \oplus M$ is PFQ-injective, then M is FQ-injective.*

Proof. Let $M_1 \oplus M_2$ be PFQ-injective. We show M_1 is F-M_2-injective. Let K be any finitely generated submodule of M_2 and $f : K \to M_1$ be an R-homomorphism. Define $g : K \to M_1 \oplus M_2$ by $g(x) = (f(x), x)$ for all $x \in K$, then g is a monomorphism. By Proposition 1.1, $M_1 \oplus M_2$ is PF-M_2-injective, whence g extends to a homomorphism $h : M_2 \to M_1 \oplus M_2$. If $\pi_1 : M_1 \oplus M_2 \to M_1$ is the natural projection, then $\pi_1 h : M_2 \to M_1$ is a homomorphism extending f. Consequently, M_1 is F-M_2-injective. ∎

Corollary 1.7. *If the full matrix ring $\mathbb{M}_2(R)$ is right PF-injective, then the ring R is right F-injective.*

Proof. It is by Corollary 1.5 and Theorem 1.6. ∎

Corollary 1.8. *If $\oplus_{i \in I} M_i$ is PFQ-injective, then M_j is F-M_k-injective for all distinct $j,k \in I$.*

Proof. It is by Theorem 1.6 and Proposition 1.1. ∎

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