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Existence of Solutions for Weighted $p(t)$-Laplacian Impulsive Integro-Differential System with Integral Boundary Value Conditions

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Abstract. This paper investigates the existence of solutions for weighted $p(t)$-Laplacian impulsive integro-differential system with integral boundary value conditions via Leray-Schauder's degree, the sufficient conditions for the existence of solutions be given. Moreover, we get the existence of nonnegative solutions.

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1. Introduction

In this paper, we consider the existence of solutions for the weighted $p(t)$-Laplacian integro-differential system

$$-\triangle_{p(t)} u + f\left(t, u, (w(t))^{\frac{1}{p(t)-1}} u', S(u), T(u)\right) = 0, \quad t \in (0,1),\ t \neq t_i, \tag{1.1}$$

where $u : [0,1] \to \mathbb{R}^N,\ f(\cdot,\cdot,\cdot,\cdot,\cdot) : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N,\ t_i \in (0,1),\ i = 1, \cdots, k$, with the following impulsive boundary value conditions

$$\lim_{t\to t_i^+} u(t) - \lim_{t\to t_i^-} u(t) = A_i\left(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-} (w(t))^{\frac{1}{p(t)-1}} u'(t)\right), \quad i = 1, \cdots, k, \tag{1.2}$$

$$\begin{aligned} &\lim_{t\to t_i^+} w(t) \left|u'\right|^{p(t)-2} u'(t) - \lim_{t\to t_i^-} w(t) \left|u'\right|^{p(t)-2} u'(t) \\ &\qquad = B_i\left(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-} (w(t))^{\frac{1}{p(t)-1}} u'(t)\right), \quad i = 1, \cdots, k, \end{aligned} \tag{1.3}$$

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$$u(0)=0, \quad u(1)=\int_0^1 g(t)u(t)dt, \tag{1.4}$$

where $p\in C([0,1],\mathbb{R})$ and $p(t)>1$, $-\triangle_{p(t)}u=-\left(w(t)\,|u'|^{p(t)-2}u'\right)'$ is called the weighted $p(t)$-Laplacian; $0<t_1<t_2<\cdots<t_k<1$; $g\in L^1[0,1]$ is nonnegative, $\int_0^1 g(t)dt=\sigma$ and $\sigma\in[0,1]$; $A_i,B_i\in C(\mathbb{R}^N\times\mathbb{R}^N,\mathbb{R}^N)$; T and S are linear operators defined by $(Tu)(t)=\int_0^t k_*(t,s)u(s)ds$, $(Su)(t)=\int_0^1 h_*(t,s)u(s)ds$, $t\in[0,1]$, where k_* , $h_*\in C([0,1]\times[0,1],\mathbb{R})$.

Throughout the paper, $o(1)$ means function which uniformly convergent to 0 (as $n\to+\infty$); for any $v\in\mathbb{R}^N$, v^j will denote the j-th component of v; the inner product in $\mathbb{R}^N$ will be denoted by $\langle\cdot,\cdot\rangle$, $|\cdot|$ will denote the absolute value and the Euclidean norm on $\mathbb{R}^N$. Denote $J=[0,1]$, $J'=(0,1)\backslash\{t_1,\cdots,t_k\}$, $J_0=[t_0,t_1]$, $J_i=(t_i,t_{i+1}]$, $i=1,\cdots,k$, where $t_0=0$, $t_{k+1}=1$. Denote J_i^o the interior of J_i, $i=0,1,\cdots,k$. Let

$$PC(J,\mathbb{R}^N)=\left\{x:J\to\mathbb{R}^N\ \middle|\ \begin{array}{c} x\in C(J_i,\mathbb{R}^N), i=0,1,\cdots,k\\ \text{and}\lim\limits_{t\to t_i^+}x(t)\text{ exists for } i=1,\cdots,k\end{array}\right\},$$

$w\in PC(J,\mathbb{R})$ satisfies $0<w(t)$, $\forall\, t\in J'$, and $(w(t))^{-1/(p(t)-1)}\in L^1(0,1)$;

$$PC^1(J,\mathbb{R}^N)=\left\{x\in PC(J,\mathbb{R}^N)\ \middle|\ \begin{array}{c} x'\in C(J_i^o,\mathbb{R}^N),\ \lim\limits_{t\to t_i^+}(w(t))^{\frac{1}{p(t)-1}}x'(t)\\ \text{and}\ \lim\limits_{t\to t_{i+1}^-}(w(t))^{\frac{1}{p(t)-1}}x'(t)\text{ exists for } i=0,1,\cdots,k\end{array}\right\}.$$

For any $x=(x^1,\cdots,x^N)\in PC(J,\mathbb{R}^N)$, denote $|x^i|_0=\sup\{|x^i(t)|\mid t\in J'\}$. Obviously, $PC(J,\mathbb{R}^N)$ is a Banach space with the norm $\|x\|_0=(\sum_{i=1}^N|x^i|_0^2)^{1/2}$, $PC^1(J,\mathbb{R}^N)$ is a Banach space with the norm $\|x\|_1=\|x\|_0+\|(w(t))^{1/(p(t)-1)}x'\|_0$. Let $L^1=L^1(J,\mathbb{R}^N)$ with the norm $\|x\|_{L^1}=(\sum_{i=1}^N|x^i|_{L^1}^2)^{1/2}$, $\forall x\in L^1$, where $|x^i|_{L^1}=\int_0^1|x^i(t)|dt$. In the following, $PC(J,\mathbb{R}^N)$ and $PC^1(J,\mathbb{R}^N)$ will be simply denoted by PC and PC^1, respectively. We denote

$$u(t_i^+)=\lim_{t\to t_i^+}u(t),\quad u(t_i^-)=\lim_{t\to t_i^-}u(t),$$

$$w(0)\left|u'\right|^{p(0)-2}u'(0)=\lim_{t\to 0^+}w(t)\left|u'\right|^{p(t)-2}u'(t),$$

$$w(1)\left|u'\right|^{p(1)-2}u'(1)=\lim_{t\to 1^-}w(t)\left|u'\right|^{p(t)-2}u'(t),$$

$$A_i=A_i\left(\lim_{t\to t_i^-}u(t),\lim_{t\to t_i^-}(w(t))^{\frac{1}{p(t)-1}}u'(t)\right),\quad i=1,\cdots,k,$$

$$B_i=B_i\left(\lim_{t\to t_i^-}u(t),\lim_{t\to t_i^-}(w(t))^{\frac{1}{p(t)-1}}u'(t)\right),\quad i=1,\cdots,k.$$

The study of differential equations and variational problems with nonstandard $p(t)$-growth conditions attracted more and more interesting in recent years. The applied background of this kind of problems we refer to [1,4,22,31]. Many results have been obtained on this kinds of problems, for example [5,7–9,11,18,19,23,30]. If $p(t)\equiv p$ (a constant), (1.1)–(1.4) is the well known p-Laplacian problem. If $p(t)$ is a general function, one can see easily $-\triangle_{p(t)} cu \neq c^{p(t)}(-\triangle_{p(t)} u)$ in general, but $-\triangle_p cu = c^p(-\triangle_p u)$, so $-\triangle_{p(t)}$ represents a non-homogeneity and possesses more nonlinearity, thus $-\triangle_{p(t)}$ is more complicated than $-\triangle_p$. For example

(a) If $\Omega \subset \mathbb{R}^N$ is a bounded domain, the Rayleigh quotient

$$\lambda_{p(x)} = \inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx}{\int_\Omega \frac{1}{p(x)} |u|^{p(x)}\, dx}$$

is zero in general, and only under some special conditions $\lambda_{p(x)} > 0$ (see [9]), when $\Omega \subset \mathbb{R}$ ($N = 1$) is an interval, the results show that $\lambda_{p(x)} > 0$ if and only if $p(x)$ is monotone. But the property $\lambda_p > 0$ is very important in the study of p-Laplacian problems, for example, in [13], the authors use this property to deal with the existence of solutions.

(b) If $w(t) \equiv 1$ and $p(t) \equiv p$ (a constant) and $-\triangle_p u > 0$, then u is concave, this property is used extensively in the study of one dimensional p-Laplacian problems (see [2]), but it is invalid for $-\triangle_{p(t)}$. It is another difference between $-\triangle_p$ and $-\triangle_{p(t)}$.

In recent years, there are many papers studying the existence of solutions for the Laplacian ($p(t) \equiv 2$) impulsive differential equation boundary value problems, for examples [6, 10, 12, 14, 16, 20, 21, 24, 26]. There are many methods to deal with this problems, for example sub-super-solution method, fixed point theorem, monotone iterative method, coincidence degree, etc. Because of the nonlinear property of $-\triangle_p$, results on the existence of solutions for p-Laplacian impulsive differential equation boundary value problems are rare (see [3]). In [27, 28], through the coincidence degree method, the present author investigate the existence of solutions for $p(r)$-Laplacian impulsive differential equation with periodic-like and multi-point boundary value conditions, respectively. On the differential equations with integral boundary value problems, we refer to [15, 17, 25, 29].

In this paper, when $p(t)$ is a general function, we investigate the existence of solutions and nonnegative solutions for the weighted $p(t)$-Laplacian impulsive integro-differential system with integral boundary value conditions. Our method is based upon Leray-Schauder's degree. The homotopy transformation used in [27,28] is unsuitable for this paper. Moreover, this paper will consider the existence of (1.1) with (1.2), (1.4) and the following impulsive condition

$$\begin{aligned} &\lim_{t\to t_i^+} (w(t))^{\frac{1}{p(t)-1}} u'(t) - \lim_{t\to t_i^-} (w(t))^{\frac{1}{p(t)-1}} u'(t) \\ &\qquad = D_i\Big(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-} (w(t))^{\frac{1}{p(t)-1}} u'(t)\Big), \quad i = 1, \cdots, k, \end{aligned} \tag{1.5}$$

where $D_i \in C(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$, the impulsive condition (1.5) is called linear impulsive condition (LI for short), and (1.3) is called nonlinear impulsive condition (NLI for short). Generally speaking, p-Laplacian impulsive problems have two kinds of impulsive conditions, including LI and NLI; but Laplacian impulsive problems only have LI. It is another difference of p-Laplacian impulsive problems and Laplacian impulsive problems.

Let $N \geq 1$, the function $f : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be Caratheodory, by this we mean:

(i) For almost every $t \in J$ the function $f(t, \cdot, \cdot, \cdot, \cdot)$ is continuous;

(ii) For each $(x, y, s, z) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ the function $f(\cdot, x, y, s, z)$ is measurable on J;

(iii) For each $R > 0$ there is a $\alpha_R \in L^1(J,\mathbb{R})$ such that, for almost every $t \in J$ and every $(x,y,s,z) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|x| \le R$, $|y| \le R$, $|s| \le R$, $|z| \le R$, one has

$$|f(t,x,y,s,z)| \le \alpha_R(t).$$

We say a function $u : J \to \mathbb{R}^N$ is a solution of (1.1) if $u \in PC^1$ with $w(t)\,|u'|^{p(t)-2}\,u'$ absolutely continuous on J_i^o, $i = 0,1,\cdots,k$, which satisfies (1.1) *a.e.* on J.

In this paper, we always use C_i to denote positive constants, if it can not lead to confusion. Denote

$$z^- = \min_{t\in J} z(t), \quad z^+ = \max_{t\in J} z(t), \quad \text{for any} \quad z \in PC(J,\mathbb{R}).$$

We say f satisfies sub-($p^- - 1$) growth condition, if f satisfies

$$\lim_{|u|+|v|+|s|+|z|\to+\infty} \frac{f(t,u,v,s,z)}{(|u|+|v|+|s|+|z|)^{q(t)-1}} = 0, \quad \text{for } t \in J \text{ uniformly},$$

where $q(t) \in PC(J,\mathbb{R})$, and $1 < q^- \le q^+ < p^-$. We say f satisfies general growth condition, if f does not satisfy sub-($p^- - 1$) growth condition.

This paper is organized as three sections. In Section 2, we present some preliminary. In Section 3, we give the existence of solutions for system (1.1)–(1.4) or (1.1) with (1.2), (1.5) and (1.4). Moreover, we give the existence of nonnegative solutions for system (1.1)–(1.4).

2. Preliminary

For any $(t,x) \in J \times \mathbb{R}^N$, denote $\varphi(t,x) = |x|^{p(t)-2}x$. Obviously, φ has the following properties.

Lemma 2.1. [28] *φ is a continuous function and satisfies*

(i) *For any $t \in [0,1], \varphi(t,\cdot)$ is strictly monotone, i.e.*

$$\langle \varphi(t,x_1) - \varphi(t,x_2), x_1 - x_2 \rangle > 0, \quad \textit{for any} \quad x_1,x_2 \in \mathbb{R}^N,\ x_1 \ne x_2.$$

(ii) *There exists a function $\alpha : [0,+\infty) \to [0,+\infty)$, $\alpha(s) \to +\infty$ as $s \to +\infty$, such that*

$$\langle \varphi(t,x), x \rangle \ge \alpha(|x|)\,|x|, \quad \textit{for all} \quad x \in \mathbb{R}^N.$$

It is well known that $\varphi(t,\cdot)$ is a homeomorphism from $\mathbb{R}^N$ to $\mathbb{R}^N$ for any fixed $t \in J$. Denote

$$\varphi^{-1}(t,x) = |x|^{\frac{2-p(t)}{p(t)-1}}\,x, \quad \text{for } x \in \mathbb{R}^N\backslash\{0\}, \qquad \varphi^{-1}(t,0) = 0, \quad \forall\, t \in J.$$

It is clear that $\varphi^{-1}(t,\cdot)$ is continuous and sends bounded sets to bounded sets. Now consider the following simple impulsive problem with boundary value condition (1.4)

$$\text{(2.1)} \qquad \begin{cases} (w(t)\varphi(t,u'(t)))' = f(t), & t \in (0,1), t \ne t_i, \\ \lim\limits_{t\to t_i^+} u(t) - \lim\limits_{t\to t_i^-} u(t) = a_i, & i = 1,\cdots,k, \\ \lim\limits_{t\to t_i^+} w(t)\,|u'|^{p(t)-2}\,u'(t) - \lim\limits_{t\to t_i^-} w(t)\,|u'|^{p(t)-2}\,u'(t) = b_i, & i = 1,\cdots,k, \end{cases}$$

where $a_i, b_i \in \mathbb{R}^N$; $f \in L^1$.

If u is a solution of (2.1), by integrating (2.1) from 0 to t, we find that

$$\text{(2.2)} \qquad w(t)\varphi(t,u'(t)) = w(0)\varphi(0,u'(0)) + \sum_{t_i<t} b_i + \int_0^t f(s)ds, \quad \forall\, t \in J'.$$

Denote $a=(a_1,\cdots,a_k)\in\mathbb{R}^{kN}$, $b=(b_1,\cdots,b_k)\in\mathbb{R}^{kN}$, $\rho_1=w(0)\varphi(0,u'(0))$. It is easy to see that ρ_1 is dependent on a,b and $f(\cdot)$. Define operator $F:L^1\longrightarrow PC$ as

$$F(f)(t)=\int_0^t f(s)ds,\quad \forall\, t\in J,\ \forall\, f\in L^1.$$

Note that $u(0)=0$. By solving for u' in (2.2) and integrating, we find

$$u(t)=\sum_{t_i<t}a_i+F\left\{\varphi^{-1}\left[t,(w(t))^{-1}\left(\rho_1+\sum_{t_i<t}b_i+F(f)(t)\right)\right]\right\}(t),\quad \forall\, t\in J.$$

From $u(1)=\int_0^1 g(t)u(t)dt$, we obtain

$$\begin{aligned}&\sum_{i=1}^k a_i+\int_0^1\varphi^{-1}\left[t,(w(t))^{-1}\left(\rho_1+\sum_{t_i<t}b_i+F(f)(t)\right)\right]dt\\ &=\int_0^1 g(t)\left(F\left\{\varphi^{-1}\left[t,(w(t))^{-1}\left(\rho_1+\sum_{t_i<t}b_i+F(f)(t)\right)\right]\right\}(t)+\sum_{t_i<t}a_i\right)dt.\end{aligned}$$

Denote $W=\mathbb{R}^{2kN}\times L^1$ with the norm $\|\omega\|=\sum_{i=1}^k|a_i|+\sum_{i=1}^k|b_i|+\|h\|_{L^1}$, $\forall\,\omega=(a,b,h)\in W$; then W is a Banach space. For any $\omega\in W$, we denote

$$\begin{aligned}\Lambda_\omega(\rho_1)=&\int_0^1\varphi^{-1}\left[t,(w(t))^{-1}\left(\rho_1+\sum_{t_i<t}b_i+F(h)(t)\right)\right]dt+\sum_{i=1}^k a_i\\ &-\int_0^1 g(t)\left(F\left\{\varphi^{-1}\left[t,(w(t))^{-1}\left(\rho_1+\sum_{t_i<t}b_i+F(h)(t)\right)\right]\right\}(t)+\sum_{t_i<t}a_i\right)dt.\end{aligned}$$

It is easy to see that

$$\begin{aligned}\Lambda_\omega(\rho_1)=&\int_0^1(1-\sigma)\varphi^{-1}\left[t,(w(t))^{-1}\left(\rho_1+\sum_{t_i<t}b_i+F(h)(t)\right)\right]dt\\ &+\int_0^1 g(s)\left\{\int_s^1\varphi^{-1}\left[t,(w(t))^{-1}\left(\rho_1+\sum_{t_i<t}b_i+F(h)(t)\right)\right]dt\right\}ds\\ &+\sum_{i=1}^k a_i-\int_0^1 g(t)\sum_{t_i<t}a_idt,\quad \forall\,\rho_1\in\mathbb{R}^N.\end{aligned}$$

Throughout the paper, we denote $E=\int_0^1(w(t))^{-1/(p(t)-1)}dt$.

Lemma 2.2. *The function $\Lambda_\omega(\cdot)$ has the following properties*

(i) *For any fixed $\omega\in W$, the equation*

$$\Lambda_\omega(\rho_1)=0 \tag{2.3}$$

has a unique solution $\widetilde{\rho}_1(\omega)\in\mathbb{R}^N$.

(ii) *The function $\widetilde{\rho}_1:W\to\mathbb{R}^N$, defined in (i), is continuous and sends bounded sets to bounded sets. Moreover, for any $\omega=(a,b,h)\in W$, we have*

$$|\widetilde{\rho}_1(\omega)|\le 3N\left[(2N)^{p^+}\left(2\sum_{i=1}^k|a_i|\right)^{p^\#-1}+\sum_{i=1}^k|b_i|+\|h\|_{L^1}\right],$$

the notation $M^{p^{\#}-1}$ means

$$M^{p^{\#}-1}=\begin{cases}M^{p^{+}-1} & M>1\\ M^{p^{-}-1} & M\le 1.\end{cases}$$

Proof. (i) From Lemma 2.1, it is immediate that

$$\langle \Lambda_{\omega}(x_1)-\Lambda_{\omega}(x_2),x_1-x_2\rangle>0,\quad \text{for } x_1\neq x_2,\ \forall\, x_1,x_2\in\mathbb{R}^N,$$

and hence, if (2.3) has a solution, then it is unique. Set

$$R_0=3N\left[(2N)^{p^{+}}\left(2\sum_{i=1}^{k}|a_i|\right)^{p^{\#}-1}+\sum_{i=1}^{k}|b_i|+\|\,h\,\|_{L^1}\right].$$

Suppose $|\rho_1|>R_0$. It is easy to see that there exists some $j_0\in\{1,\cdots,N\}$ such that the absolute value of the j_0-th component $\rho_1^{j_0}$ of ρ_1 satisfing

$$\left|\rho_1^{j_0}\right|\ge\frac{|\rho|}{N}>3\left[(2N)^{p^{+}}\left(2\sum_{i=1}^{k}|a_i|\right)^{p^{\#}-1}+\sum_{i=1}^{k}|b_i|+\|\,h\,\|_{L^1}\right].$$

Obviously, $\left|\sum_{i=1}^{k}a_i-\int_0^1 g(t)\sum_{t_i<t}a_i dt\right|\le(1+\sigma)\sum_{i=1}^{k}|a_i|\le 2\sum_{i=1}^{k}|a_i|$.

Thus the j_0-th component of $\rho_1+\sum_{t_i<t}b_i+F(h)(t)$ keeps sign on J, and we have

$$\left|\left(\rho_1^{j_0}+\sum_{t_i<t}b_i^{j_0}+F(h)^{j_0}(t)\right)\right|\ge\frac{2|\rho|}{3N}>2\left[(2N)^{p^{+}}\left(2\sum_{i=1}^{k}|a_i|\right)^{p^{\#}-1}+\sum_{i=1}^{k}|b_i|+\|\,h\,\|_{L^1}\right],$$

then it is easy to see that the j_0-th component of $\Lambda_{\omega}(\rho_1)$ keeps the same sign of $\rho_1^{j_0}$. Thus,

$$\Lambda_{\omega}(\rho_1)\neq 0.$$

Consider the equation

$$\lambda\Lambda_{\omega}(\rho_1)+(1-\lambda)\rho_1=0,\quad \lambda\in[0,1]. \tag{2.4}$$

According to the preceding discussion, all the solutions of (2.4) belong to $b(R_0+1)=\{x\in\mathbb{R}^N\mid |x|<R_0+1\}$. Therefore

$$d_B[\Lambda_{\omega}(\rho_1),b(R_0+1),0]=d_B[I,b(R_0+1),0]\neq 0,$$

which means the existence of solutions of $\Lambda_{\omega}(\rho_1)=0$.

In this way, we define a function $\widetilde{\rho_1}(\omega):W\to\mathbb{R}^N$, which satisfies

$$\Lambda_{\omega}(\widetilde{\rho_1}(\omega))=0.$$

(ii) By the proof of (i), we also obtain $\widetilde{\rho_1}$ sends bounded sets to bounded sets, and

$$|\widetilde{\rho_1}(\omega)|\le 3N\left[(2N)^{p^{+}}\left(2\sum_{i=1}^{k}|a_i|\right)^{p^{\#}-1}+\sum_{i=1}^{k}|b_i|+\|\,h\,\|_{L^1}\right].$$

It only remains to prove the continuity of $\widetilde{\rho_1}$. Let $\{\omega_n\}$ be a convergent sequence in W and $\omega_n\to\omega$, as $n\to+\infty$. Since $\{\widetilde{\rho_1}(\omega_n)\}$ is a bounded sequence, it contains a convergent subsequence $\{\widetilde{\rho_1}(\omega_{n_j})\}$. Suppose $\widetilde{\rho_1}(\omega_{n_j})\to\rho_1^0$ as $j\to+\infty$. Since $\Lambda_{\omega_{n_j}}(\widetilde{\rho_1}(\omega_{n_j}))=0$, letting $j \to +\infty$, we have $\Lambda_\omega(\rho_1^0) = 0$, which together with (i) implies that $\rho_1^0 = \widetilde{\rho_1}(\omega)$. It means $\widetilde{\rho_1}$ is continuous. This completes the proof. ∎

Now we denote $N_f(u) : PC^1 \to L^1$ the Nemytskii operator associated to f defined by

$$N_f(u)(t) = f(t, u(t), (w(t))^{\frac{1}{p(t)-1}} u'(t), S(u), T(u)), \quad a.e. \quad \text{on } J. \tag{2.5}$$

We define $\rho_1 : PC^1 \to \mathbb{R}^N$ as

$$\rho_1(u) = \widetilde{\rho_1}(A, B, N_f)(u), \tag{2.6}$$

where $A = (A_1, \cdots, A_k)$, $B = (B_1, \cdots, B_k)$. It is clear that $\rho_1(\cdot)$ is continuous and sends bounded sets of PC^1 to bounded sets of $\mathbb{R}^N$, and hence it is compact continuous.If u is a solution of (2.1) with (1.4), we have

$$u(t) = \sum_{t_i < t} a_i + F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(\widetilde{\rho_1}(\omega) + \sum_{t_i < t} b_i + F(f)(t)\right)\right]\right\}(t), \quad \forall\, t \in [0,1].$$

For fixed $a, b \in \mathbb{R}^{kN}$, we denote $K_{(a,b)} : L^1 \to PC^1$ as

$$K_{(a,b)}(h)(t) = F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(\widetilde{\rho_1}(a,b,h) + \sum_{t_i < t} b_i + F(h)(t)\right)\right]\right\}(t), \quad \forall\, t \in J.$$

Define $K_1 : PC^1 \to PC^1$ as

$$K_1(u)(t) = F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(\rho_1(u) + \sum_{t_i < t} B_i + F(N_f(u))(t)\right)\right]\right\}(t), \quad \forall\, t \in J.$$

Similar to the proof of [30, Lemma 2.3], we have

Lemma 2.3.

(i) *The operator $K_{(a,b)}$ is continuous and sends equi-integrable sets in L^1 to relatively compact sets in PC^1.*

(ii) *The operator K_1 is continuous and sends bounded sets in PC^1 to relatively compact sets in PC^1.*

It is not hard to check that

Lemma 2.4. *u is a solution of* (1.1)–(1.4) *if and only if u is a solution of the following abstract operator equation*

$$u = \sum_{t_i < t} A_i + K_1(u).$$

3. Main results and proofs

In this section, we will apply Leray-Schauder's degree to deal with the existence of solutions in the following three cases:

- Case (i) Existence of solutions for system (1.1)–(1.4);
- Case (ii) Existence of solutions for system (1.1) with (1.2), (1.5) and (1.4);
- Case (iii) Existence of nonnegative solutions.

3.1. Case (i)

In this subsection, we will deal with the existence of solutions for system (1.1)–(1.4). When f satisfies sub-$(p^- - 1)$ growth condition, we have

Theorem 3.1. *If f satisfies sub-$(p^- - 1)$ growth condition, we also assume that*

$$\sum_{i=1}^{k} |A_i(u,v)| \le C_1(1+|u|+|v|)^{\frac{q^+-1}{p^+-1}},$$
$$\sum_{i=1}^{k} |B_i(u,v)| \le C_2(1+|u|+|v|)^{q^+-1}, \quad \forall\ (u,v) \in \mathbb{R}^N \times \mathbb{R}^N,$$

then problem (1.1)–(1.4) *has at least a solution.*

Proof. First we consider the following problem

$$\begin{cases} -\triangle_{p(t)} u + \lambda N_f(u)(t) = 0, & t \in (0,1), t \neq t_i, \\ \lim\limits_{t\to t_i^+} u(t) - \lim\limits_{t\to t_i^-} u(t) = \lambda A_i\left(\lim\limits_{t\to t_i^-} u(t), \lim\limits_{t\to t_i^-} (w(t))^{\frac{1}{p(t)-1}} u'(t)\right), & i = 1,\cdots,k, \\ \lim\limits_{t\to t_i^+} w(t)\left|u'\right|^{p(t)-2} u'(t) - \lim\limits_{t\to t_i^-} w(t)\left|u'\right|^{p(t)-2} u'(t) & \\ \qquad = \lambda B_i\left(\lim\limits_{t\to t_i^-} u(t), \lim\limits_{t\to t_i^-} (w(t))^{\frac{1}{p(t)-1}} u'(t)\right), & i = 1,\cdots,k, \\ u(0) = 0,\ u(1) = \int_0^1 g(t)u(t)dt. \end{cases} \tag{3.1}$$

Denote

$$\rho_{1,\lambda}(u) = \widetilde{\rho_1}(\lambda A, \lambda B, \lambda N_f)(u),$$
$$K_{1,\lambda}(u) = F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(\rho_{1,\lambda}(u) + \lambda \sum_{t_i<t} B_i + F(\lambda N_f(u))(t)\right)\right]\right\},$$
$$\Psi_f(u,\lambda) = \lambda \sum_{t_i<t} A_i + K_{1,\lambda}(u),$$

where $N_f(u)$ is defined in (2.5).

We know that (3.1) has the same solution of the following operator equation when $\lambda = 1$

$$u = \Psi_f(u,\lambda). \tag{3.2}$$

It is easy to see that operator $\rho_{1,\lambda}$ is compact continuous for any $\lambda \in [0,1]$. It follows from Lemma 2.2 and Lemma 2.3 that $\Psi_f(\cdot,\cdot)$ is compact continuous from $PC^1 \times [0,1]$ to PC^1.

We claim that all the solutions of (3.2) are uniformly bounded for $\lambda \in [0,1]$. In fact, if it is false, we can find a sequence of solutions $\{(u_n,\lambda_n)\}$ for (3.2) such that $\|u_n\|_1 \to +\infty$ as $n \to +\infty$, and $\|u_n\|_1 > 1$ for any $n = 1,2,\cdots$. From Lemma 2.2, we have

$$\left|\rho_{1,\lambda}(u)\right| \le C_3\left[(2N)^{p^+}\left(2\sum_{i=1}^{k}|A_i|\right)^{p^\#-1} + \sum_{i=1}^{k}|B_i| + \|\, N_f(u)\,\|_{L^1}\right] \le C_4(1+\|u\|_1^{q^+-1}).$$

Thus

$$\left|\rho_{1,\lambda}(u)+\sum_{t_i<t}\lambda B_i+F(\lambda N_f)\right|\le\left|\rho_{1,\lambda}(u)\right|+\left|\sum_{t_i<t}B_i\right|+\left|F(N_f)\right|\le C_5\left(1+\|u\|_1^{q^+-1}\right).$$

From (3.1), we have

$$w(t)\left|u_n'(t)\right|^{p(t)-2}u_n'(t)=\rho_{1,\lambda}(u_n)+\sum_{t_i<t}\lambda B_i+\int_0^t\lambda N_f(u_n)(s)ds,\quad\forall\, t\in J'.$$

It follows from (2.6) and Lemma 2.2 that

$$w(t)\left|u_n'(t)\right|^{p(t)-1}\le\left|\rho_{1,\lambda}(u_n)\right|+\sum_{i=1}^k|B_i|+\int_0^1\left|N_f(u_n)(s)\right|ds\le C_6+C_7\|u_n\|_1^{q^+-1},\quad\forall\, t\in J'.$$

Denote $\alpha=(q^+-1)/(p^--1)$, then the above inequality tells us that

$$\left\|(w(t))^{\frac{1}{p(t)-1}}u_n'(t)\right\|_0\le C_8\|u_n\|_1^{\alpha},\quad n=1,2,\cdots.\tag{3.3}$$

For any $j=1,\cdots,N$, we have

$$\begin{aligned}\left|u_n^j(t)\right|&=\left|\sum_{t_i<t}A_i+\int_0^t(u_n^j)'(s)ds\right|\le\left|\sum_{t_i<t}A_i\right|+\left|\int_0^t(w(s))^{\frac{-1}{p(s)-1}}\sup_{t\in(0,1)}\left|(w(t))^{\frac{1}{p(t)-1}}(u_n^j)'(t)\right|ds\right|\\&\le C_9E\|u_n\|_1^{\alpha}+\left|\sum_{t_i<t}A_i\right|\le C_{10}\|u_n\|_1^{\alpha},\quad\forall\, t\in J,n=1,2,\cdots,\end{aligned}$$

which implies

$$\left|u_n^j\right|_0\le C_{11}\|u_n\|_1^{\alpha},\quad j=1,\cdots,N;\ n=1,2,\cdots.$$

Thus

$$\|u_n\|_0\le NC_{11}\|u_n\|_1^{\alpha},\quad n=1,2,\cdots.\tag{3.4}$$

It follows from (3.3) and (3.4) that $\{\|u_n\|_1\}$ is uniformly bounded.

Thus, we can choose a large enough $R_1>0$ such that all the solutions of (3.2) belong to $B(R_1)=\left\{u\in C^1\mid\|u\|_1<R_1\right\}$. Therefore the Leray-Schauder degree $d_{LS}[I-\Psi_f(\cdot,\lambda),B(R_1),0]$ is well defined for $\lambda\in[0,1]$, and

$$d_{LS}[I-\Psi_f(\cdot,1),B(R_1),0]=d_{LS}[I-\Psi_f(\cdot,0),B(R_1),0].$$

It is easy to see that u is a solution of $u=\Psi_f(u,0)$ if and only if u is a solution of the following usual differential equation

$$\begin{cases}-\triangle_{p(t)}u=0, & t\in(0,1),\\ u(0)=0,\ u(1)=\int_0^1g(t)u(t)dt.\end{cases}\tag{3.5}$$

Obviously, system (3.5) possesses a unique solution $u_0\equiv0$. Since $u_0\in B(R_1)$, we have

$$d_{LS}[I-\Psi_f(\cdot,1),B(R_1),0]=d_{LS}[I-\Psi_f(\cdot,0),B(R_1),0]\ne0,$$

which implies that (1.1)–(1.4) has at least one solution. This completes the proof. ∎

When f satisfies general growth condition, we consider

$$-\triangle_{p(t)}u+f\left(t,u,(w(t))^{\frac{1}{p(t)-1}}u',S(u),T(u),\delta\right)=0,\quad t\in(0,1),\ t\neq t_i, \tag{3.6}$$

where δ is a parameter, and

$$\begin{aligned}&f\left(t,u,(w(t))^{\frac{1}{p(t)-1}}u',S(u),T(u),\delta\right)\\&=\phi\left(t,u,(w(t))^{\frac{1}{p(t)-1}}u',S(u),T(u)\right)+\delta h\left(t,u,(w(t))^{\frac{1}{p(t)-1}}u',S(u),T(u)\right),\end{aligned}$$

where $\phi,h:J\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ are Caratheodory. We have

Theorem 3.2. *If ϕ satisfies sub-$(p^- -1)$ growth condition, and we also assume*

$$\begin{aligned}&\sum_{i=1}^{k}|A_i(u,v)|\le C_1(1+|u|+|v|)^{\frac{q^+-1}{p^+-1}},\\&\sum_{i=1}^{k}|B_i(u,v)|\le C_2(1+|u|+|v|)^{q^+-1},\quad\forall\,(u,v)\in\mathbb{R}^N\times\mathbb{R}^N,\end{aligned}$$

then problem (3.6) *with* (1.2)–(1.4) *has at least one solution when parameter δ is small enough.*

Proof. Denote

$$\begin{aligned}&f_\lambda\left(t,u,(w(t))^{\frac{1}{p(t)-1}}u',S(u),T(u),\delta\right)\\&=\phi\left(t,u,(w(t))^{\frac{1}{p(t)-1}}u',S(u),T(u)\right)+\lambda\delta h\left(t,u,(w(t))^{\frac{1}{p(t)-1}}u',S(u),T(u)\right).\end{aligned}$$

We consider the existence of solutions of the following equation with (1.2)–(1.4)

$$-\triangle_{p(t)}u+f_\lambda\left(t,u,(w(t))^{\frac{1}{p(t)-1}}u',S(u),T(u),\delta\right)=0,\quad t\in(0,1),\ t\neq t_i. \tag{3.7}$$

Denote

$$\begin{aligned}&\rho^{\#}_{1,\lambda}(u,\delta)=\widetilde{\rho}_1(A,B,N_{f_\lambda})(u),\\&K^{\#}_{1,\lambda}(u,\delta)=F\{\varphi^{-1}[t,(w(t))^{-1}(\rho^{\#}_{1,\lambda}(u,\delta)+\sum_{t_i<t}B_i+F(N_{f_\lambda}(u))(t))]\},\\&\Phi_\delta(u,\lambda)=\sum_{t_i<t}A_i+K^{\#}_{1,\lambda}(u,\delta),\end{aligned}$$

where $N_{f_\lambda}(u)$ is defined in (2.5). We know that (3.7) with (1.2)–(1.4) has the same solution of

$$u=\Phi_\delta(u,\lambda).$$

Obviously, $f_0=\phi$. So $\Phi_\delta(u,0)=\Psi_\phi(u,1)$. As in the proof of Theorem 3.1, we know that all the solutions of $u=\Phi_\delta(u,0)$ are uniformly bounded, then there exists a large enough $R_*>0$ such that all the solutions of $u=\Phi_\delta(u,0)$ belong to $B(R_*)=\left\{u\in PC^1\mid\|u\|_1<R_*\right\}$. Since $\Phi_\delta(\cdot,0)$ is compact continuous from PC^1 to PC^1, we have

$$\inf_{u\in\partial B(R_*)}\|u-\Phi_\delta(u,0)\|_1>0.$$

Since ϕ and h are Caratheodory, we have

$$\begin{aligned}\left\|F(N_{f_\lambda}(u))-F(N_{f_0}(u))\right\|_0 &\to 0 \text{ for } (u,\lambda)\in\overline{B(R_*)}\times[0,1] \text{ uniformly, as } \delta\to 0,\\ \left|\rho^{\#}_{1,\lambda}(u,\delta)-\rho^{\#}_{1,0}(u,\delta)\right| &\to 0 \text{ for } (u,\lambda)\in\overline{B(R_*)}\times[0,1] \text{ uniformly, as } \delta\to 0,\\ \left\|K^{\#}_{1,\lambda}(u,\delta)-K^{\#}_{1,0}(u,\delta)\right\|_1 &\to 0 \text{ for } (u,\lambda)\in\overline{B(R_*)}\times[0,1] \text{ uniformly, as } \delta\to 0.\end{aligned}$$

Thus

$$\|\Phi_\delta(u,\lambda)-\Phi_0(u,\lambda)\|_1 \to 0 \text{ for } (u,\lambda)\in\overline{B(R_*)}\times[0,1] \text{ uniformly, as } \delta\to 0.$$

Obviously, $\Phi_0(u,\lambda)=\Phi_\delta(u,0)=\Phi_0(u,0)$. We obtain

$$\|\Phi_\delta(u,\lambda)-\Phi_\delta(u,0)\|_1 \to 0 \text{ for } (u,\lambda)\in\overline{B(R_*)}\times[0,1] \text{ uniformly, as } \delta\to 0.$$

Thus, when δ is small enough, we can conclude that

$$\begin{aligned}&\inf_{(u,\lambda)\in\partial B(R_*)\times[0,1]}\|u-\Phi_\delta(u,\lambda)\|_1\\ &\ge \inf_{u\in\partial B(R_*)}\|u-\Phi_\delta(u,0)\|_1-\sup_{(u,\lambda)\in\overline{B(R_*)}\times[0,1]}\|\Phi_\delta(u,0)-\Phi_\delta(u,\lambda)\|_1>0.\end{aligned}$$

Thus $u=\Phi_\delta(u,\lambda)$ has no solution on $\partial B(R_*)$ for any $\lambda\in[0,1]$, when δ is small enough. It means that the Leray-Schauder degree $d_{LS}[I-\Phi_\delta(\cdot,\lambda),B(R_*),0]$ is well defined for any $\lambda\in[0,1]$, and

$$d_{LS}[I-\Phi_\delta(u,\lambda),B(R_*),0]=d_{LS}[I-\Phi_\delta(u,0),B(R_*),0].$$

Since $\Phi_\delta(u,0)=\Psi_\phi(u,1)$, from the proof of Theorem 3.1, we can see that the right hand side is nonzero. Thus (3.6) with (1.2)–(1.4) has at least one solution when δ is small enough. This completes the proof. ∎

3.2. Case (ii)

In this subsection, we consider the existence of solutions for system (1.1) with (1.2), (1.5) and (1.4). When f satisfies sub-$(p^- -1)$ growth condition, we have

Theorem 3.3. *If f satisfies sub-$(p^- -1)$ growth condition, we also assume that*

$$\begin{aligned}&\sum_{i=1}^{k}|A_i(u,v)|\le C_1(1+|u|+|v|)^{\frac{q^+-1}{p^+-1}},\\ &\sum_{i=1}^{k}|D_i(u,v)|\le C_2(1+|u|+|v|)^{\alpha_i^+},\quad \forall\,(u,v)\in\mathbb{R}^N\times\mathbb{R}^N,\end{aligned}$$

where

$$\alpha_i\le\frac{q^+-1}{p(t_i)-1},\quad \text{and}\quad p(t_i)-1\le q^+-\alpha_i,\quad i=1,\cdots,k,$$

then problem (1.1) *with* (1.2), (1.4) *and* (1.5) *has at least a solution.*

Proof. Similar to the proof of [30, Theorem 3.4], we can prove that

$$\sum_{i=1}^{k}|B_i(u,v)|\le C_2(1+|u|+|v|)^{q^+-1},\quad \forall\,(u,v)\in\mathbb{R}^N\times\mathbb{R}^N.$$ ∎

When f satisfies general growth condition, we have

Theorem 3.4. *If ϕ satisfies sub-($p^- - 1$) growth condition, and we also assume*

$$\sum_{i=1}^{k} |A_i(u,v)| \le C_1(1+|u|+|v|)^{\frac{q^+-1}{p^+-1}}$$

$$\sum_{i=1}^{k} |D_i(u,v)| \le C_2(1+|u|+|v|)^{\alpha_i^+}, \quad \forall\, (u,v) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where $\alpha_i \le (q^+-1)/(p(t_i)-1)$, and $p(t_i)-1 \le q^+ - \alpha_i, i = 1,\cdots,k$, then problem (3.6) *with* (1.2), (1.4) *and* (1.5) *has at least one solution when parameter δ is small enough.*

Proof. Similar to the proof of Theorem 3.2 and Theorem 3.3, we omit it here. ∎

3.3. Case (iii)

In this subsection, we will consider the existence of nonnegative solutions. For any $x = (x^1,\cdots,x^N) \in \mathbb{R}^N$, the notation $x \ge 0$ means $x^j \ge 0$ for any $j = 1,\cdots,N$.

Theorem 3.5. *We assume*

(1^0) $f(t,x,y,s,z) \le 0$, $\forall\, (t,x,y,s,z) \in J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$;

(2^0) *For any $i = 1,\cdots,k$, $B_i(u,v) \le 0, \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N$;*

(3^0) *For any $i = 1,\cdots,k$, $j = 1,\cdots,N$, $A_i^j(u,v)v^j \ge 0$, $\forall\, (u,v) \in \mathbb{R}^N \times \mathbb{R}^N$;*

(4^0) *$\sigma < 1$ or $\sigma = 1$ and $g(t) > 0$ a.e. on J.*

Then every solution of (1.1)–(1.4) *is nonnegative.*

Proof. Let u be a solution of (1.1)–(1.4). From Lemma 2.4, we have

$$u(t) = \sum_{t_i<t} A_i + F\left\{\varphi^{-1}\left[t,(w(t))^{-1}\left(\rho_1(u) + \sum_{t_i<t} B_i + F(N_f(u))\right)\right]\right\}(t), \quad \forall\, t \in J.$$

We claim that $\rho_1(u) \ge 0$. If it is false, then there exists some $j \in \{1,\cdots,N\}$ such that $\rho_1^j(u) < 0$. It follows from (1^0) and (2^0) that

$$\left[\rho_1(u) + \sum_{t_i<t} B_i + F(N_f(u))(t)\right]^j < 0, \quad \forall\, t \in J. \tag{3.8}$$

Thus (3.8) and condition (3^0) hold

$$A_i^j \le 0, \quad i = 1,\cdots,k. \tag{3.9}$$

Similar to the proof before Lemma 2.2, from the boundary value conditions, we have

$$\begin{aligned} 0 = &\int_0^1 (1-\sigma)\varphi^{-1}\left[t,(w(t))^{-1}\left(\rho_1(u) + \sum_{t_i<t} B_i + F(N_f(u))(t)\right)\right] dt \\ &+ \int_0^1 g(t)\left\{\int_t^1 \varphi^{-1}\left[t,(w(t))^{-1}\left(\rho_1(u) + \sum_{t_i<t} B_i + F(N_f(u))(t)\right)\right] dt\right\} dt \\ &+ \sum_{i=1}^{k} A_i\left(1 - \int_{t_i}^1 g(t)dt\right). \end{aligned} \tag{3.10}$$

From (3.8) and (3.9), we get a contradiction to (3.10). Thus $\rho_1(u) \ge 0$.

We claim that

$$\rho_1(u)+\sum_{i=1}^{k}B_i+F(N_f)(1)\le 0. \tag{3.11}$$

If it is false, then there exists some $j\in\{1,\cdots,N\}$ such that

$$\left[\rho_1(u)+\sum_{i=1}^{k}B_i+F(N_f)(1)\right]^j>0.$$

It follows from (1^0) and (2^0) that

$$\left[\rho_1(u)+\sum_{t_i<t}B_i+F(N_f(u))(t)\right]^j>0,\quad \forall\, t\in J. \tag{3.12}$$

Thus (3.12) and condition (3^0) hold

$$A_i^j\ge 0,\quad i=1,\cdots,k. \tag{3.13}$$

From (3.12) and (3.13), we get a contradiction to (3.10). Thus (3.11) is valid.

Denote

$$\Theta(t)=\rho_1(u)+\sum_{t_i<t}B_i+F(N_f(u))(t),\quad \forall\, t\in J'.$$

Obviously, $\Theta(0)=\rho_1(u)\ge 0$, $\Theta(1)\le 0$, and $\Theta(t)$ is decreasing, i.e., $\Theta(t')\le\Theta(t'')$ for any $t',t''\in J$ with $t'\ge t''$. For any $j=1,\cdots,N$, there exists $\zeta_j\in J$ such that

$$\Theta^j(t)\ge 0,\quad \forall\, t\in(0,\zeta_j),\qquad \text{and}\qquad \Theta^j(t)\le 0,\quad \forall\, t\in(\zeta_j,T).$$

Together with condition (3^0) this implies that $u^j(t)$ is increasing on $[0,\zeta_j]$, and $u^j(t)$ is decreasing on $(\zeta_j,T]$. Thus

$$\min\left\{u^j(0),u^j(1)\right\}=\inf_{t\in J}u^j(t),\quad j=1,\cdots,N.$$

For any fixed $j\in\{1,\cdots,N\}$, if

$$u^j(0)=\inf_{t\in J}u^j(t),$$

from (1.4), we have $u^j(0)=0$. Thus $u^j\ge 0$.

If

$$u^j(1)=\inf_{t\in J}u^j(t), \tag{3.14}$$

from (1.4) and (3.14), we have $(1-\sigma)u^j(1)=\int_0^1 g(t)\left[u^j(t)-u^j(1)\right]dt\ge 0$.

If $\sigma<1$, then $u^j(1)\ge 0$. If $\sigma=1$, we have $\int_0^1 g(t)[u^j(t)-u^j(1)]dt=0$, which together with condition (4^0) implies that

$$u^j(t)\equiv u^j(1)=u^j(0)=0.$$

Thus $u(t)\ge 0,\forall\, t\in[0,T]$. The proof is completed. ∎

Corollary 3.1. *Under the conditions of Theorem 3.1, we also assume*

(1^0) $f(t,x,y,s,z)\le 0$, $\forall\,(t,x,y,s,z)\in J\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N$ *with* $x,s,z\ge 0$*;*

(2^0) *For any* $i=1,\cdots,k$, $B_i(u,v)\le 0$, $\forall\,(u,v)\in\mathbb{R}^N\times\mathbb{R}^N$ *with* $u\ge 0$*;*

(3^0) *For any* $i=1,\cdots,k$, $j=1,\cdots,N$, $A_i^j(u,v)v^j\ge 0$, $\forall\,(u,v)\in\mathbb{R}^N\times\mathbb{R}^N$ *with* $u\ge 0$*;*

(4^0) *$\sigma < 1$ or $\sigma = 1$ and $g(t) > 0$ a.e. on J;*
(5^0) *For any $t \in [0,1]$ and $s \in [0,1]$, $k_*(t,s) \geq 0$, $h_*(t,s) \geq 0$.*

Then (1.1)–(1.4) *has a nonnegative solution.*

Proof. Define
$$M(u) = (M_*(u^1), \cdots, M_*(u^N)),$$
where
$$M_*(u) = \begin{cases} u, & u \geq 0 \\ 0, & u < 0. \end{cases}$$

Denote
$$\widetilde{f}(t,u,v,S(u),T(u)) = f\left(t, M(u), v, S(M(u)), T(M(u))\right), \quad \forall\ (t,u,v) \in J \times \mathbb{R}^N \times \mathbb{R}^N,$$
then $\widetilde{f}(t,u,v,S(u),T(u))$ satisfies Caratheodory condition, and $\widetilde{f}(t,u,v,S(u),T(u)) \leq 0$ for any $(t,u,v) \in J \times \mathbb{R}^N \times \mathbb{R}^N$.

For any $i = 1, \cdots, k$, we denote
$$\widetilde{A}_i(u,v) = A_i(M(u),v), \quad \widetilde{B}_i(u,v) = B_i(M(u),v), \quad \forall\ (u,v) \in \mathbb{R}^N \times \mathbb{R}^N,$$
then $\widetilde{A}_i$ and $\widetilde{B}_i$ are continuous, and satisfy
$$\begin{aligned} \widetilde{B}_i(u,v) \leq 0, \quad &\forall\ (u,v) \in \mathbb{R}^N \times \mathbb{R}^N, \text{ for any } i = 1, \cdots, k, \\ \widetilde{A}_i^j(u,v)v^j \geq 0, \quad &\forall\ (u,v) \in \mathbb{R}^N \times \mathbb{R}^N, \text{ for any } i = 1, \cdots, k,\ j = 1, \cdots, N. \end{aligned}$$
It is not hard to check that

(2^0)$'$ $\lim_{|u|+|v|\to+\infty}\left(\widetilde{f}(t,u,v,S(u),T(u))/(|u|+|v|)^{q(t)-1}\right) = 0$, for $t \in J$ uniformly, where $q(t) \in C(J,\mathbb{R})$, and $1 < q^- \leq q^+ < p^-$;

(3^0)$'$ $\sum_{i=1}^k \left|\widetilde{A}_i(u,v)\right| \leq C_1(1+|u|+|v|)^{(q^+-1)/(p^+-1)}, \quad \forall\ (u,v) \in \mathbb{R}^N \times \mathbb{R}^N$;

(4^0)$'$ $\sum_{i=1}^k \left|\widetilde{B}_i(u,v)\right| \leq C_2(1+|u|+|v|)^{q^+-1}, \quad \forall\ (u,v) \in \mathbb{R}^N \times \mathbb{R}^N$.

Consider
$$(3.15) \qquad \begin{cases} -\triangle_{p(t)}u + \widetilde{f}(t,u,(w(t))^{\frac{1}{p(t)-1}}u',\ S(u),T(u)) = 0, & t \in J', \\ \lim\limits_{t\to t_i^+} u(t) - \lim\limits_{t\to t_i^-} u(t_i) = \widetilde{A}_i(\lim\limits_{t\to t_i^-} u(t),\ \lim\limits_{t\to t_i^-}(w(t))^{\frac{1}{p(t)-1}}u'(t)), & i = 1, \cdots, k, \\ \lim\limits_{t\to t_i^+} w(t)\varphi(t,u'(t)) - \lim\limits_{t\to t_i^-} w(t)\varphi(t,u'(t)) & \\ \qquad = \widetilde{B}_i(\lim\limits_{t\to t_i^-} u(t), \lim\limits_{t\to t_i^-}(w(t))^{\frac{1}{p(t)-1}}u'(t)), & i = 1, \cdots, k, \\ u(0) = 0,\ u(1) = \int_0^1 g(t)u(t)dt. \end{cases}$$

It follows from Theorem 3.1 and Theorem 3.5 that (3.15) have a nonnegative solution u. Since $u \geq 0$, we have $M(u) = u$, and then
$$\widetilde{f}\left(t,u,(w(t))^{\frac{1}{p(t)-1}}u',S(u),T(u)\right) = f\left(t,u,(w(t))^{\frac{1}{p(t)-1}}u',S(u),T(u)\right),$$
$$\widetilde{A}_i\left(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-}(w(t))^{\frac{1}{p(t)-1}}u'(t)\right) = A_i\left(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-}(w(t))^{\frac{1}{p(t)-1}}u'(t)\right),$$

$$\widetilde{B}_i\left(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-} (w(t))^{\frac{1}{p(t)-1}} u'(t)\right) = B_i\left(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-} (w(t))^{\frac{1}{p(t)-1}} u'(t)\right).$$

Thus u is a nonnegative solution of (1.1)–(1.4). This completes the proof. ∎

Note 3.1. Similarly, one can get the existence of nonnegative solutions of (3.6) with (1.2)–(1.4).

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