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(a,b,c)-Koszul Algebras

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Abstract. Given fixed integers a, b and c with $a > c > b > 1$, the notion of (a,b,c)-*Koszul algebra* is introduced, which is another extension of Koszul algebras and includes some Artin-Schelter regular algebras of global dimension five as special examples. Some criteria for a standard graded algebra to be (a,b,c)-Koszul are given. Further, the Yoneda algebras and the H-Galois graded extensions of (a,b,c)-Koszul algebras are discussed, where H is a finite dimensional semisimple and cosemisimple Hopf algebra. Moreover, the so-called *(generalized)* (a,b,c)-*Koszul modules* are introduced and some basic properties are also provided.

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1. Introduction

Noncommutative graded algebras play an important role in algebra, topology, and mathematical physics, etc. Maybe one of the most interesting class of such algebras is the class of Koszul algebras, which were originally defined by Priddy in 1970 (see [23]) and studied by many people since then, such as [2,3,11,20–22], etc. In the last decade, several extensions of this theory to some more general cases have been developed (see [5–7,9,12,16–19], etc.). In some sense, the development of the classification of Artin-Schelter regular algebras of lower global dimension accelerates and promotes the progress of Koszul theory. More precisely, motivated by the classification problem of Artin-Schelter regular algebras of global dimension three (see [1]), Berger introduced the notion of *nonquadratic Koszul algebra* (see [5]) in 2001, which was often called *d-Koszul algebra* later (such as [4, 8, 10, 12–14, 24, 25], etc.), where $d \geq 2$ is a fixed integer. Inspired by the classification problem of Artin-Schelter regular algebras of global dimension four (see [15]), Si and Lu introduced the notion of *bi-Koszul algebra* in 2008 (see [16]). From [15], one can see clearly that it is too complicated to classify Artin-Schelter regular algebras of global dimension four completely and of course, let alone the case of dimension five.

In this paper, the notion of (a,b,c)-*Koszul algebra* is introduced, which is another extension of Koszul algebras and determined by a triple of integers (a,b,c) with $a > c > b > 1$.

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It should be noted that such algebras include Koszul algebras, d-Koszul algebras and some Artin-Schelter regular algebras of global dimension five as special examples. Motivated by [4], the notion of generalized (a,b,c)-Koszul module is also introduced, which can be regarded as a natural generalization of (a,b,c)-Koszul modules. The whole paper is arranged as follows: In Section 2, we give some notation, definitions and examples. In Section 3, we give some criteria for a standard graded algebra to be (a,b,c)-Koszul. In Section 4, for a finite dimensional semisimple and cosemisimple Hopf algebra H, the Koszulity of the graded right H-module algebra $A=\bigoplus_{n\geq 0}A_n$ and the coinvariant graded subalgebra $B=A^{coH}$ of A is studied. It turns out that we can judge the Koszulity of A in terms of B and vice versa. In the last section, we turn to study (a,b,c)-Koszul modules and define the so-called generalized (a,b,c)-Koszul modules, some basic properties of (a,b,c)-Koszul modules and generalized (a,b,c)-Koszul modules are given and further, follow the ideals of Backelin and Fröberg (see [2]), we obtain many Koszul modules from a given (a,b,c)-Koszul module.

2. Notation, definitions and examples

Throughout, $\mathbb{N}$ denotes the set of natural numbers, $\Bbbk$ is a fixed base field and the phrase "standard graded algebra" means a positively graded $\Bbbk$-algebra $A=\bigoplus_{i\geq 0}A_i$ with (a) $A_0=\Bbbk\times\cdots\times\Bbbk$, a finite product of $\Bbbk$; (b) $A_i\cdot A_j=A_{i+j}$ for all $0\leq i,j<\infty$; and (c) $\dim_{\Bbbk}A_i<\infty$ for all $i\geq 0$. Obviously, condition (b) implies that A is generated by A_1 over A_0.

Proposition 2.1. [11] Let A be a standard graded algebra. Then there exists a finite quiver $\Gamma=(\Gamma_0,\Gamma_1)$ and a graded ideal I in $\Bbbk\Gamma$ with $I\subset\sum_{n\geq 2}(\Bbbk\Gamma)_n$ such that $A\cong\Bbbk\Gamma/I$ as graded algebras, where Γ_0 denotes the set of vertices of Γ and Γ_1 the set of arrows of Γ.

Under the above assumptions, it is easy to see that the graded Jacobson radical of A, which we denote by J, is $\bigoplus_{i\geq 1}A_i$ and for any finitely generated graded left A-module M (It is sufficient that M is bounded below in fact, i.e., $M_i=0$ for all $i<i_0$ for some integer i_0), M has a graded projective resolution

$$\cdots\longrightarrow Q_n\xrightarrow{d_n}\cdots\longrightarrow Q_1\xrightarrow{d_1}Q_0\xrightarrow{d_0}M\longrightarrow 0$$

in the category of finitely generated graded left A-modules such that "$\ker d_i\subseteq JQ_i$" for all $i\geq 0$, i.e., the resolution is "minimal".

Let $Gr(A)$ denote the category of graded left A-modules and $gr(A)$ denote the category of finitely generated graded left A-modules. Endowed with the Yoneda product, $\bigoplus_{i\geq 0}\mathrm{Ext}_A^i(A_0,A_0)$ is a bigraded algebra. Let M be a finitely generated graded left A-module. Then $\bigoplus_{i\geq 0}\mathrm{Ext}_A^i(M,A_0)$ is a bigraded $\bigoplus_{i\geq 0}\mathrm{Ext}_A^i(A_0,A_0)$-module. For simplicity, we write

$$E(A)=\bigoplus_{i\geq 0}\mathrm{Ext}_A^i(A_0,A_0),\ \ \mathscr{E}(M)=\bigoplus_{i\geq 0}\mathrm{Ext}_A^i(M,A_0)$$

and call $E(A)$ the *Yoneda algebra* of A, and $\mathscr{E}(M)$ the *Ext module* of M.

Given integers a, b, c with $a>c>b>1$, we introduce a set function $\delta_{b,c}^a:\mathbb{N}\to\mathbb{N}$ by

$$\delta_{b,c}^a(n)=\begin{cases} ka, & n=4k;\\ ka+1, & n=4k+1;\\ ka+b, & n=4k+2;\\ ka+c, & n=4k+3,\end{cases}$$

where $k\in\mathbb{N}$.

Definition 2.2. Let A be a standard graded $\Bbbk$-algebra and $M = \bigoplus_{i\geq 0} M_i \in gr(A)$. We call M an (a,b,c)-*Koszul module* provided that M admits a minimal graded projective resolution

$$\cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0,$$

such that each Q_n is generated in degree $\delta^a_{b,c}(n)$ for all $n \geq 0$. In particular, the standard graded algebra A will be called an (a,b,c)-*Koszul algebra* if the trivial left A-module A_0 is an (a,b,c)-Koszul module.

Let $\mathscr{K}^a_{b,c}(A)$ denote the category of (a,b,c)-Koszul modules.

We now give some examples.

Example 2.3. The following list some trivial examples of (a,b,c)-Koszul algebras:

(1) Koszul algebras (see [2,3,20,21,23], etc.) are $(4,2,3)$-Koszul algebras.
(2) d-Koszul algebras (see [5,10], etc.) are $(2d,d,d+1)$-Koszul algebras.
(3) The opposite algebra of an (a,b,c)-Koszul algebra is also an (a,b,c)-Koszul algebra, we omit the details here since it is similar to that of Proposition 2.2.1 of [3].
(4) Let A be an (a,b,c)-Koszul algebra and M a finitely 1-generated graded left A-module. Let

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A_0 \longrightarrow 0$$

and

$$\cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

be the corresponding minimal graded projective resolutions in $gr(A)$. If in addition, $\Omega^3(M)[-a]$ is an (a,b,c)-Koszul module and Q_i is generated in degree $\delta^a_{b,c}(i+1)$ for $i = 1,\ 2$, then the triangular graded algebra

$$\mathrm{Alg}^A_M = \begin{pmatrix} A & M \\ 0 & \Bbbk \end{pmatrix},$$

with the new grading:

$$(\mathrm{Alg}^A_M)_0 = \begin{pmatrix} A_0 & 0 \\ 0 & \Bbbk \end{pmatrix}, \quad \text{and} \quad (\mathrm{Alg}^A_M)_i = \begin{pmatrix} A_i & M_i \\ 0 & 0 \end{pmatrix} \ (\forall\, i \geq 1),$$

is an (a,b,c)-Koszul algebra, which can be seen easily from the minimal graded Alg^A_M-projective resolution in $gr(\mathrm{Alg}^A_M)$:

$$\cdots \to \begin{pmatrix} P_n \\ 0 \end{pmatrix} \oplus \begin{pmatrix} Q_{n-1} \\ 0 \end{pmatrix} \to \cdots \to \begin{pmatrix} P_0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M \\ \Bbbk \end{pmatrix} \to (\mathrm{Alg}^A_M)_0 \to 0.$$

Example 2.4. The following are two concrete examples of (a,b,c)-Koszul algebras.

(1) Let Γ be the quiver:

$$\bullet^1 \underset{\beta_1}{\overset{\alpha_1}{\leftrightarrows}} \bullet^2 \underset{\beta_2}{\overset{\alpha_2}{\leftrightarrows}} \bullet^3 \underset{\beta_3}{\overset{\alpha_3}{\leftrightarrows}} \bullet^4.$$

Let

$$A = \frac{\Bbbk\Gamma}{\langle \alpha_i\beta_i = \beta_{i+1}\alpha_{i+1},\ \alpha_{i+1}\alpha_i,\ \beta_i\beta_{i+1} : i = 1,\ 2,\ 3\rangle}.$$

Then under a routine computation, one can get a minimal projective resolution of the trivial A-module $\Bbbk^{\oplus 4}$:

$\cdots \to (A\oplus P_2\oplus P_3)[7] \to (A\oplus P_2\oplus P_3)[6] \to A[5] \to (A\oplus P_2\oplus P_3)[3] \to (A\oplus P_2\oplus P_3)[2] \to (A\oplus P_2\oplus P_3)[1] \to A \to \Bbbk^{\oplus 4} \to 0$, where P_i denotes the simple A-module related to the vertex i. Thus A is a $(5,2,3)$-Koszul algebra.

(2) Let $A = \Bbbk\langle x,y\rangle/(x^3y-3x^2yx+3xyx^2-yx^3,\ 2xyxy-x^2y^2-2yxyx+y^2x^2,\ xy^3+3y^2xy-y^3x-3yxy^2)$, where $\deg(x)=\deg(y)=1$. Under a routine computation, the Hilbert series of A is

$$H_A(z) = \frac{1}{(1-z)^2(1-z^2)(1-z^3)^2} = \frac{1}{1-2z+3z^4-3z^6+2z^9-z^{10}},$$

which implies a minimal graded projective resolution of the trivial module $\Bbbk$

$$0 \to A[10] \to A[9]^{\oplus 2} \to A[6]^{\oplus 3} \to A[4]^{\oplus 3} \to A[1]^{\oplus 2} \to A \to \Bbbk \to 0.$$

Therefore, A is an Artin-Schelter regular algebra of global dimension five and a $(9,4,6)$-Koszul algebra.

Definition 2.5. Let A be an (a,b,c)-Koszul algebra. If $a > c+1 > b+2 > 4$ and $a > 2b$, then A is called a *nontrivial* (a,b,c)-Koszul algebra.

Example 2.6. The following are some examples of nontrivial (a,b,c)-Koszul algebras:

(1) The opposite algebra of a nontrivial (a,b,c)-Koszul algebra is again a nontrivial (a,b,c)-Koszul algebra.
(2) In (4) of Example 2.3, if A is a nontrivial (a,b,c)-Koszul algebra and $\Omega^3(M)[-a]$ is a nontrivial (a,b,c)-Koszul module. Then so is the triangular algebra Alg_M^A.
(3) The algebra in (2) of Example 2.4 is a nontrivial $(9,4,6)$-Koszul algebra.

3. (a,b,c)-Koszul algebras

In this section, we will give some criteria for a standard graded algebra to be (a,b,c)-Koszul.

Proposition 3.1. The following are equivalent for a standard graded algebra A:

(1) A is (a,b,c)-Koszul;
(2) $\mathrm{Ext}_A^n(A_0,A_0) = \mathrm{Ext}_A^n(A_0,A_0)_{-\delta_{b,c}^a(n)}$ for all $n \geq 2$;
(3) $\mathrm{Tor}_n^A(A_0,A_0) = \mathrm{Tor}_n^A(A_0,A_0)_{\delta_{b,c}^a(n)}$ for all $n \geq 2$.

Proof. We omit the details since it is similar to that of Proposition 2.1.3 of [3]. ∎

Lemma 3.1. *[26] Let A be a standard graded algebra and $A^e := A\otimes_\Bbbk A^{op}$ its enveloping algebra. Let* $\mathbf{r}$ *be the graded Jacobson radical of A^e and $f: P \to Q$ be a homomorphism of finitely generated A^e-projective modules. Then $Im f \subseteq \mathbf{r}Q$ if and only if for each simple A-module S, we have $Im(f\otimes_A 1_S) \subseteq J(Q\otimes_A S)$.*

Proposition 3.2. Let A be a standard graded algebra. Then A is an (a,b,c)-Koszul algebra if and only if A is an (a,b,c)-Koszul A^e-module.

Proof. Let

$$\mathscr{P}_*: \cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

be a minimal graded projective A^e-resolution of A in $gr(A^e)$. Then by Lemma 3.1, $\mathscr{P}_*$ is minimal if and only if $\mathscr{P}_*\otimes_A A_0$:

$$\cdots \longrightarrow P_n\otimes_A A_0 \longrightarrow \cdots \longrightarrow P_1\otimes_A A_0 \longrightarrow P_0\otimes_A A_0 \longrightarrow A\otimes_A A_0 \cong A_0 \longrightarrow 0$$

is a minimal graded projective resolution of A_0 in $gr(A)$. Further, for all $i \geq 0$, P_i is generated in degree s as a graded A^e-module if and only if $P_i \otimes_A A_0$ is generated in degree s as a graded left A-module, where $s \in \mathbb{Z}$, which completes the proof. ∎

Proposition 3.3. Let $A = \Bbbk\Gamma/I$ be a standard graded algebra and

$$\cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A_0 \longrightarrow 0$$

a minimal graded projective resolution of the trivial A-module A_0 in $gr(A)$. Then the following statements are equivalent:

(1) A is an (a,b,c)-Koszul algebra;
(2) $\ker d_n \subseteq J^{\delta^a_{b,c}(n+1)-\delta^a_{b,c}(n)}P_n$ and $J\ker d_n = \ker d_n \cap J^{\delta^a_{b,c}(n+1)-\delta^a_{b,c}(n)+1}P_n$ for all $n \geq 0$;
(3) for any fixed $n \geq 1$ and $1 \leq i \leq n$, $P_i = \bigoplus_{l\geq 1} Ae_{i_l}[-\delta^a_{b,c}(i)]$, the component of $d_i(e_{i_l})$ in some Ae_{i-1_m} is in $A_{\delta^a_{b,c}(i)-\delta^a_{b,c}(i-1)}$, $\ker d_n \subseteq J^{\delta^a_{b,c}(n+1)-\delta^a_{b,c}(n)}P_n$ and $J\ker d_n = \ker d_n \cap J^{\delta^a_{b,c}(n+1)-\delta^a_{b,c}(n)+1}P_n$.

Proof. (1) $\Rightarrow$ (2) Suppose that A is an (a,b,c)-Koszul algebra. Then for all $n \geq 0$, P_n is generated in degree $\delta^a_{b,c}(n)$. Note that $d_{n+1}(P_{n+1}) = \ker d_n$, which implies that $\ker d_n$ is generated in degree $\delta^a_{b,c}(n+1)$. But recall that P_n is generated in degree $\delta^a_{b,c}(n)$, hence the elements of degree $\delta^a_{b,c}(n+1)$ of P_n are in $J^{\delta^a_{b,c}(n+1)-\delta^a_{b,c}(n)}P_n$. Thus for all $n \geq 0$, $\ker d_n \subseteq J^{\delta^a_{b,c}(n+1)-\delta^a_{b,c}(n)}P_n$. Now it is clear that $J\ker f_n \subseteq \ker f_n \cap J^{\delta^a_{b,c}(n+1)-\delta^a_{b,c}(n)+1}P_n$. Let $x \in \ker f_n \cap J^{\delta^a_{b,c}(n+1)-\delta^a_{b,c}(n)+1}P_n$ be a homogeneous element of degree i. It is easy to see that $i \geq \delta^a_{b,c}(n+1)+1$. If $x \notin J\ker d_n$, then x is a generator of $\ker d_n$, which implies that $\ker d_n$ is generated in degree larger than $\delta^a_{b,c}(n+1)+1$ since the degree of x is larger than $\delta^a_{b,c}(n+1)+1$, which contradicts to that $\ker d_n$ is generated in degree $\delta^a_{b,c}(n+1)$. Therefore, $x \in J\ker d_n$ and $J\ker f_n \supseteq \ker f_n \cap J^{\delta^a_{b,c}(n+1)-\delta^a_{b,c}(n)+1}P_n$.

(2) $\Rightarrow$ (1) First we claim that for all $n \geq 0$, $(P_n)_j = 0$ for all $j < \delta^a_{b,c}(n)$. Do it by induction on n. First we prove that $(P_0)_j = 0$ for $j < \delta^a_{b,c}(0) = 0$. If not, since P_0 is a finitely generated graded module, there exists a smallest $j_0 < \delta^a_{b,c}(0)$ such that $(P_0)_{j_0} \neq 0$. Let $x \neq 0$ be a homogeneous element of P_0 of degree j_0. Then $d_0(x) = 0$ since $d_0(x) \in (A_0)_{j_0}$ and $A_0 = (A_0)_0$, which implies that $x \in \ker d_0 \subset JP_0$, which contradicts to the choice of j_0. Now suppose that $(P_{n-1})_j = 0$ for all $j < \delta^a_{b,c}(n-1)$. Similarly, assume that there exists a smallest $j_0' < \delta^a_{b,c}(n)$ such that $(P_n)_{j_0'} \neq 0$. Let $x \neq 0$ be a homogeneous element of P_n of degree j_0'. Note that $d_n(x) \in Imd_n = \ker d_{n-1} \subseteq J^{\delta^a_{b,c}(n)-\delta^a_{b,c}(n-1)}P_{n-1}$, we have $d_n(x) = 0$ since $J^{\delta^a_{b,c}(n)-\delta^a_{b,c}(n-1)}P_{n-1}$ is supported in $\{i | i \geq \delta^a_{b,c}(n)\}$. Therefore, $x \in \ker d_n \subseteq J^{\delta^a_{b,c}(n+1)-\delta^a_{b,c}(n)}P_n$, which also contradicts to the choice of j_0'.

Now we claim that for any $x \in (P_n)_i$ with $i > \delta^a_{b,c}(n)$, then $x \in J^sP_n$ for some $s > 0$. If we prove this claim, then it is clear that for all $n \geq 0$, P_n is generated in degree $\delta^a_{b,c}(n)$. In fact, we can prove this by induction on n. Note that A_0 is generated in degree 0, thus $d_0(x) \in JA_0 = J$, which implies that $x \in d_0^{-1}(J) = JP_0 + \ker d_0 \subseteq JP_0$. Therefore, P_0 is generated in degree 0. Suppose that for any $x \in (P_{n-1})_i$ with $i > \delta^a_{b,c}(n-1)$, then we have $x \in J^sP_{n-1}$ for some $s > 0$ and P_{n-1} is generated in degree $\delta^a_{b,c}(n-1)$. By the condition $J\ker f_{n-1} = \ker f_{n-1} \cap J^{\delta^a_{b,c}(n)-\delta^a_{b,c}(n-1)+1}P_{n-1}$, we have $\ker d_{n-1}$ is generated in degree $\delta^a_{b,c}(n)$, which implies that P_n is generated in degree $\delta^a_{b,c}(n)$ for all $n \geq 0$. Of course, for any $x \in (P_n)_i$ with $i > \delta^a_{b,c}(n)$, we have $x \in J^s P_n$ for some $s > 0$.

(1) and (2) $\Rightarrow$ (3) Suppose that A is an (a,b,c)-Koszul algebra. Then for each $i \geq 0$, P_i is generated in degree $\delta^a_{b,c}(i)$. Thus all e_{i_l} are of degree $\delta^a_{b,c}(i)$, which implies that $d_i(e_{i_l}) \in (P_{i-1})_{\delta^a_{b,c}(i)}$. But P_{i-1} is generated in degree $\delta^a_{b,c}(i-1)$, so $(P_{i-1})_{\delta^a_{b,c}(i)} \subseteq A_{\delta^a_{b,c}(i)-\delta^a_{b,c}(i-1)}$ $(P_{i-1})_{\delta^a_{b,c}(i-1)}$. Now (3) is clear by (2).

(3) $\Rightarrow$ (1) By an induction on n, it suffices to prove that P_0 is generated in degree $\delta^a_{b,c}(0)$ and $\ker d_0$ is generated in degree $\delta^a_{b,c}(1)$, which is similar to the proof of (2) $\Rightarrow$ (1) and we omit the details. ∎

Proposition 3.4. Let A be a standard graded algebra and $M \in \mathscr{K}^a_{b,c}(A)$. Then the the Ext module $\mathscr{E}(M)$ is generated in degree 0 as a graded $E(A)$-module if and only if A is an (a,b,c)-Koszul algebra.

Proof. Let $\mathscr{P}_*$ and $\mathscr{Q}_*$ be the minimal graded projective resolutions of A_0 and M, respectively. We have Q_n is generated in degree $\delta^a_{b,c}(n)$ for each $n \geq 0$ since $M \in \mathscr{K}^a_{b,c}(A)$.

($\Rightarrow$) By the hypothesis, we have $\mathrm{Ext}^n_A(M,A_0) = \mathrm{Ext}^n_A(A_0,A_0) \cdot \mathrm{Ext}^0_A(M,A_0)$ for all $n \geq 1$. By Proposition 3.1, we have $\mathrm{Ext}^n_A(M,A_0) = \mathrm{Ext}^n_A(M,A_0)_{-\delta^a_{b,c}(n)}$ for each $n \geq 0$ since $M \in \mathscr{K}^a_{b,c}(A)$. Thus $\mathrm{Ext}^n_A(A_0,A_0) = \mathrm{Ext}^n_A(A_0,A_0)_{-\delta^a_{b,c}(n)}$ for all $n \geq 0$, which implies that A is an (a,b,c)-Koszul algebra by Proposition 3.1.

($\Leftarrow$) By Proposition 3.5 of [10]. ∎

Now combining Propositions 3.1, 3.2, 3.3 and 3.4, we have obtained the following criteria for (a,b,c)-Koszul algebras:

Theorem 3.5. *Let $A = \bigoplus_{i\geq 0} A_i$ be a standard graded algebra and*

$$\cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A_0 \longrightarrow 0$$

be a minimal graded projective resolution of the trivial A-module A_0 in gr(A). Then the following statements are equivalent:

(1) *A is (a,b,c)-Koszul;*
(2) *The opposite algebra A^{opp} of A is (a,b,c)-Koszul;*
(3) *A is (a,b,c)-Koszul as a graded A^e-module, where $A^e := A \otimes_{\Bbbk} A^{op}$ denotes the enveloping algebra of A;*
(4) $\mathrm{Ext}^n_A(A_0,A_0) = \mathrm{Ext}^n_A(A_0,A_0)_{-\delta^a_{b,c}(n)}$ *for all $n \geq 2$;*
(5) $\mathrm{Tor}^A_n(A_0,A_0) = \mathrm{Tor}^A_n(A_0,A_0)_{\delta^a_{b,c}(n)}$ *for all $n \geq 2$;*
(6) *We have* $\ker d_n \subseteq J^{\delta^a_{b,c}(n+1)-\delta^a_{b,c}(n)} P_n$ *and* $J \ker d_n = \ker d_n \cap J^{\delta^a_{b,c}(n+1)-\delta^a_{b,c}(n)+1} P_n$ *for all $n \geq 0$;*
(7) *For any fixed integer $n \geq 1$ and $1 \leq i \leq n$, $P_i = \oplus_{l\geq 1} Ae_{i_l}[-\delta^a_{b,c}(i)]$, the component of $d_i(e_{i_l})$ in some Ae_{i-1_m} is in $A_{\delta^a_{b,c}(i)-\delta^a_{b,c}(i-1)}$,* $\ker d_n \subseteq J^{\delta^a_{b,c}(n+1)-\delta^a_{b,c}(n)} P_n$ *and* $J \ker d_n = \ker d_n \cap J^{\delta^a_{b,c}(n+1)-\delta^a_{b,c}(n)+1} P_n$*;*
(8) *Let $M \in \mathscr{K}^a_{b,c}(A)$. Then the the Ext module $\mathscr{E}(M)$ is generated in degree 0 as a graded $E(A)$-module.*

Theorem 3.6. *Let A be a standard graded algebra and $E(A)$ its Yoneda algebra. If A is a nontrivial (a,b,c)-Koszul algebra, then as a graded $\Bbbk$-algebra, $E(A)$ is minimally generated in ext-degrees $0,1,2,3$ and 4.*

Proof. Suppose that A is a nontrivial (a,b,c)-Koszul algebra, then A_0 has a minimal graded projective resolution in $gr(A)$

$$\cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A_0 \longrightarrow 0$$

such that for all $n \geq 0$, P_n is generated in degree $\delta^a_{b,c}(n)$. Note that $\delta^a_{b,c}(n) = \delta^a_{b,c}(4) + \delta^a_{b,c}(n-4)$ for all $n \geq 4$. By Proposition 3.6 of [10], we know that for all $n > 4$, we have $\mathrm{Ext}^n_A(A_0,A_0) = \mathrm{Ext}^4_A(A_0,A_0)\cdot\mathrm{Ext}^{n-4}_A(A_0,A_0)$, which implies that $E(A)$ can be generated in degrees 0, 1, 2, 3 and 4.

To finish the proof we only need to prove the "minimally generating property", which can be seen clearly from the following analysis:

(i) $(\mathrm{Ext}^1_A(A_0,A_0))^2 \subseteq \mathrm{Ext}^2_A(A_0,A_0)_{-2} = 0$ since $\mathrm{Ext}^2_A(A_0,A_0) = \mathrm{Ext}^2_A(A_0,A_0)_{-b}$ and $b > 2$;

(ii) $\mathrm{Ext}^1_A(A_0,A_0)\cdot\mathrm{Ext}^2_A(A_0,A_0) \subseteq \mathrm{Ext}^3_A(A_0,A_0)_{-b-1} = 0$ since $\mathrm{Ext}^3_A(A_0,A_0) = \mathrm{Ext}^3_A(A_0,A_0)_{-c}$ and $c > b+1$;

(iii) $\mathrm{Ext}^1_A(A_0,A_0)\cdot\mathrm{Ext}^3_A(A_0,A_0) \subseteq \mathrm{Ext}^4_A(A_0,A_0)_{-c-1} = 0$ since $\mathrm{Ext}^4_A(A_0,A_0) = \mathrm{Ext}^4_A(A_0,A_0)_{-a}$ and $a > c+1$;

(iv) $(\mathrm{Ext}^2_A(A_0,A_0))^2 \subseteq \mathrm{Ext}^4_A(A_0,A_0)_{-2b} = 0$ since $\mathrm{Ext}^4_A(A_0,A_0) = \mathrm{Ext}^4_A(A_0,A_0)_{-a}$ and $a > 2b$. ∎

Theorem 3.7. *Let A be a standard graded algebra and $E(A)$ its Yoneda algebra. If the following conditions are satisfied:*

(1) *$E(A)$ is minimally generated in the ext-degrees $0,1,2,3$ and 4;*

(2) *$\mathrm{Ext}^2_A(A_0,A_0) = \mathrm{Ext}^2_A(A_0,A_0)_{-b}$, $\mathrm{Ext}^3_A(A_0,A_0) = \mathrm{Ext}^3_A(A_0,A_0)_{-c}$ and $\mathrm{Ext}^4_A(A_0,A_0) = \mathrm{Ext}^4_A(A_0,A_0)_{-a}$;*

(3) *$\mathrm{Ext}^2_A(A_0,A_0)\cdot\mathrm{Ext}^3_A(A_0,A_0) = 0$ and $(\mathrm{Ext}^3_A(A_0,A_0))^2 = 0$,*

then A is a nontrivial (a,b,c)-Koszul algebra.

Proof. By (1), we have that $\mathrm{Ext}^4_A(A_0,A_0)$ and $\mathrm{Ext}^2_A(A_0,A_0)$ can't be generated in lower degrees, which imply $a > c+1 > b+2 > 4$ and $a > 2b$.

Let

$$\cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A_0 \longrightarrow 0$$

be a minimal graded projective resolution of the trivial A-module A_0 in $gr(A)$. In order to prove A is a nontrivial (a,b,c)-Koszul algebra, it suffices to prove, for all $n \geq 0$, that P_n is generated in degree $\delta^a_{b,c}(n)$. Note that A is a standard graded algebra, which implies that P_0 is generated in degree $\delta^a_{b,c}(0) = 0$ and P_1 is generated in degree $\delta^a_{b,c}(1) = 1$. By condition (2), we have P_i is generated in degree $\delta^a_{b,c}(i)$ for $i = 2,\ 3,\ 4$.

By (3) and the conditions $a > c+1 > b+2 > 4$ and $a > 2b$, we have the fact that

$$\sum_{s+t=4k+j,\ s,t\neq j,4k} \mathrm{Ext}^s_A(A_0,A_0)\cdot\mathrm{Ext}^t_A(A_0,A_0) = 0.$$

Now let $i > 4$, writing $i = 4k + j$, where $0 \leq j < 4$. We have

$$
\begin{aligned}
\mathrm{Ext}_A^{4k+j}(A_0,A_0) &=^{(a)} \sum_{s+t=4k+j} \mathrm{Ext}_A^s(A_0,A_0)\cdot \mathrm{Ext}_A^t(A_0,A_0)\\
&= \sum_{s+t=4k+j,\, s,t\neq j,4k} \mathrm{Ext}_A^s(A_0,A_0)\cdot \mathrm{Ext}_A^t(A_0,A_0)\\
&\quad + \mathrm{Ext}_A^j(A_0,A_0)\cdot \mathrm{Ext}_A^{4k}(A_0,A_0) + \mathrm{Ext}_A^{4k}(A_0,A_0)\cdot \mathrm{Ext}_A^j(A_0,A_0)\\
&=^{(b)} \mathrm{Ext}_A^j(A_0,A_0)\cdot (\mathrm{Ext}_A^4(A_0,A_0))^k + (\mathrm{Ext}_A^4(A_0,A_0))^k \cdot \mathrm{Ext}_A^j(A_0,A_0)\\
&=^{(c)} \mathrm{Ext}_A^j(A_0,A_0)_{-\delta_{b,c}^a(j)}\cdot (\mathrm{Ext}_A^4(A_0,A_0)_{-\delta_{b,c}^a(4)})^k\\
&\quad + (\mathrm{Ext}_A^4(A_0,A_0)_{-\delta_{b,c}^a(4)})^k \cdot \mathrm{Ext}_A^j(A_0,A_0)_{-\delta_{b,c}^a(j)}\\
&=^{(d)} \mathrm{Ext}_A^j(A_0,A_0)_{-\delta_{b,c}^a(j)}\cdot \mathrm{Ext}_A^{4k}(A_0,A_0)_{-k\delta_{b,c}^a(4)}\\
&\quad + \mathrm{Ext}_A^{4k}(A_0,A_0)_{-k\delta_{b,c}^a(4)} \cdot \mathrm{Ext}_A^j(A_0,A_0)_{-\delta_{b,c}^a(j)}\\
&=^{(e)} \mathrm{Ext}_A^j(A_0,A_0)_{-\delta_{b,c}^a(j)}\cdot \mathrm{Ext}_A^{4k}(A_0,A_0)_{-\delta_{b,c}^a(4k)}\\
&\quad + \mathrm{Ext}_A^{4k}(A_0,A_0)_{-\delta_{b,c}^a(4k)} \cdot \mathrm{Ext}_A^j(A_0,A_0)_{-\delta_{b,c}^a(j)}\\
&=^{(f)} \mathrm{Ext}_A^{4k+j}(A_0,A_0)_{-\delta_{b,c}^a(4k)-\delta_{b,c}^a(j)}\\
&=^{(g)} \mathrm{Ext}_A^{4k+j}(A_0,A_0)_{-\delta_{b,c}^a(4k+j)},
\end{aligned}
$$

where (a) is by that $E(A)$ is a positively graded algebra under the Yoneda product, (b) is by the above fact and Proposition 3.6 of [10], (c) is by the condition (2), (d) is by Proposition 3.6 of [10], (e) is by the definition of $\delta_{b,c}^a$, (f) is by Proposition 3.6 of [10] and (g) is by the definition of $\delta_{b,c}^a$. Now by Proposition 3.1, we have that A is a nontrivial (a,b,c)-Koszul algebra. ∎

4. The H-Galois graded extensions of (a,b,c)-Koszul algebras

In this section, for a special given standard graded algebra A, we will find a proper graded subalgebra B of A, such that the (a,b,c)-Koszulity of A and B can be determined by each other.

We refer to [14] for the related notions, such as H-Galois graded extension, etc.

Lemma 4.1. *[14] Let H be a finite dimensional semisimple and cosemisimple Hopf algebra and A/B be an H-Galois graded extension. If $A=\bigoplus_{i\geq 0}A_i$ is a standard graded algebra, then A_0/B_0 is an H-Galois extension.*

Lemma 4.2. *[14] Let H be a finite dimensional semisimple and cosemisimple Hopf algebra, $A=\bigoplus_{n\geq 0}A_n$ be a graded right H-module algebra and $B=A^{coH}$, the coinvariant subalgebra of A. Suppose that A/B is an H-Galois graded extension. Then we have an isomorphism of bigraded algebras*

$$(\bigoplus_{i\geq 0}\mathrm{Ext}_B^i(A_0,A_0)) \cong (\bigoplus_{i\geq 0}\mathrm{Ext}_A^i(A_0,A_0))\#H,$$

where the bigrading of $(\bigoplus_{i\geq 0}\mathrm{Ext}_A^i(A_0,A_0))\#H$ is induced from that of $\bigoplus_{i\geq 0}\mathrm{Ext}_A^i(A_0,A_0)$.

Theorem 4.1. *Let H be a finite dimensional semisimple and cosemisimple Hopf algebra, $A = \bigoplus_{n\geq 0} A_n$ be a graded right H-module algebra such that A is a standard graded algebra, and let $B = A^{coH}$, the coinvariant subalgebra of A. Suppose that A/B is an H-Galois graded extension. Then B is (a,b,c)-Koszul if and only if A is (a,b,c)-Koszul.*

Proof. ($\Rightarrow$) By Lemma 4.1, A_0/B_0 is an H-Galois extension since A/B is an H-Galois graded extension. Note that $A_0\#H$ and B_0, A_0 and $(A_0\#H)\#H^*$ are both Morita equivalent, and H is a finite dimensional semisimple and cosemisimple Hopf algebra, we have that A_0 is semisimple since B_0 is semisimple and $A_0 = B_0 \oplus S$ as right B_0-modules for some semisimple B_0-module S. By the hypothesis, B is an (a,b,c)-Koszul algebra, by Proposition 3.1, which is equivalent to that $\mathrm{Ext}^i_B(B_0,B_0) = \mathrm{Ext}^i_B(B_0,B_0)_{-\delta^a_{b,c}(i)}$ for all $i \geq 0$. Note that S is a direct summand of $B_0 \oplus B_0 \oplus \cdots \oplus B_0$, a finite copies of B_0, which implies that $\mathrm{Ext}^i_B(B_0,S) = \mathrm{Ext}^i_B(B_0,S)_{-\delta^a_{b,c}(i)}$, $\mathrm{Ext}^i_B(S,B_0) = \mathrm{Ext}^i_B(S,B_0)_{-\delta^a_{b,c}(i)}$ and $\mathrm{Ext}^i_B(S,S) = \mathrm{Ext}^i_B(S,S)_{-\delta^a_{b,c}(i)}$ for all $i \geq 0$. Also observe that we have the following isomorphism

$$\mathrm{Ext}^i_B(A_0,A_0) = \mathrm{Ext}^i_B(B_0,B_0) \oplus \mathrm{Ext}^i_B(B_0,S) \oplus \mathrm{Ext}^i_B(S,B_0) \oplus \mathrm{Ext}^i_B(S,S)$$

for all $i \geq 0$, which implies that $\mathrm{Ext}^i_B(A_0,A_0) = \mathrm{Ext}^i_B(A_0,A_0)_{-\delta^a_{b,c}(i)}$ for all $i \geq 0$. By Lemma 4.2, we have $\mathrm{Ext}^i_A(A_0,A_0)\#H = (\mathrm{Ext}^i_A(A_0,A_0)\#H)_{-\delta^a_{b,c}(i)}$ for all $i \geq 0$. By the definition of the bigrading of $\mathrm{Ext}^i_A(A_0,A_0)\#H$, we obtain that $\mathrm{Ext}^i_A(A_0,A_0) = \mathrm{Ext}^i_A(A_0,A_0)_{-\delta^a_{b,c}(i)}$ for all $i \geq 0$. By Proposition 3.1, A is an (a,b,c)-Koszul algebra.

($\Leftarrow$) Suppose that A is an (a,b,c)-Koszul algebra, by Proposition 3.1, which is equivalent to $\mathrm{Ext}^i_A(A_0,A_0) = \mathrm{Ext}^i_A(A_0,A_0)_{-\delta^a_{b,c}(i)}$ for all $i \geq 0$. By Lemma 4.2, we have $\mathrm{Ext}^i_B(A_0,A_0) = \mathrm{Ext}^i_B(A_0,A_0)_{-\delta^a_{b,c}(i)}$ for all $i \geq 0$. Similarly, we have $A_0 = B_0 \oplus S$ as right B_0-modules for some semisimple B_0-module S, thus we have

$$\mathrm{Ext}^i_B(A_0,A_0) = \mathrm{Ext}^i_B(B_0,B_0) \oplus \mathrm{Ext}^i_B(B_0,S) \oplus \mathrm{Ext}^i_B(S,B_0) \oplus \mathrm{Ext}^i_B(S,S)$$

for all $i \geq 0$, which implies $\mathrm{Ext}^i_B(B_0,B_0) = \mathrm{Ext}^i_B(B_0,B_0)_{-\delta^a_{b,c}(i)}$ for all $i \geq 0$. By Proposition 3.1, B is an (a,b,c)-Koszul algebra. ∎

5. (Generalized) (a,b,c)-Koszul modules

We mainly focus on (generalized) (a,b,c)-Koszul modules in this section.

Lemma 5.1. *Let A be a standard graded algebra and*

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence in $gr(A)$. Then the following statements are equivalent:

(1) *If M and N are generated in a single degree s, then so is K;*
(2) *$JK = K \cap JM$;*

(3) *We have the following commutative diagram*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^1(K) & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(N) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P_0 & \longrightarrow & L_0 & \longrightarrow & Q_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

with P_0*,* L_0 *and* Q_0 *are graded projective covers.*

Proof. (1) $\Rightarrow$ (2) A routine check.

(2) $\Rightarrow$ (3) Clearly, we have the exact sequence

$$0 \longrightarrow K/JK \longrightarrow M/JM \longrightarrow N/JN \longrightarrow 0$$

since $JK = K \cap JM$. Note that for any $X \in gr(A)$, $A \otimes_{A_0} X/JX \longrightarrow X \longrightarrow 0$ is a projective cover. Now setting $P_0 := A \otimes_{A_0} K/JK$, $L_0 := A \otimes_{A_0} M/JM$ and $Q_0 := A \otimes_{A_0} N/JN$. We have the following exact sequence $0 \longrightarrow P_0 \longrightarrow Q_0 \longrightarrow L_0 \longrightarrow 0$ since A_0 is semisimple. Therefore, (3) is true by the “Snake Lemma”.

(3) $\Rightarrow$ (1) Suppose that we have the given commutative diagram. Then $L_0 \cong P_0 \oplus Q_0$ as graded left A-modules. By assumption, L_0 is generated in degree s since M is generated in degree s, which implies that P_0 is generated in degree s. Thus, K is generated in degree s since $P_0 \to K \to 0$ is a graded projective cover, (1) follows. ∎

Proposition 5.1. Let A be a standard graded algebra and

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence in $gr(A)$. Then $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ for all $i \geq 0$ if and only if the minimal Horseshoe Lemma holds. That is, given a diagram

$$\begin{array}{ccccccccc} & & \mathscr{P}_* & & & & \mathscr{Q}_* & & \\ & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \\ & & 0 & & & & 0 & & \end{array}$$

with $\mathscr{P}_*$ and $\mathscr{Q}_*$ being minimal graded projective resolutions of K and N, respectively. Then we can complete it into the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{P}_* & \longrightarrow & \mathscr{L}_* & \longrightarrow & \mathscr{Q}_* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

such that $\mathscr{L}_* \longrightarrow M \longrightarrow 0$ is also a minimal graded projective resolution.

Proof. $(\Rightarrow)$ By Lemma 5.1, for all $i \geq 0$, we have the the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^{i+1}(K) & \longrightarrow & \Omega^{i+1}(M) & \longrightarrow & \Omega^{i+1}(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_i & \longrightarrow & L_i & \longrightarrow & Q_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^{i}(K) & \longrightarrow & \Omega^{i}(M) & \longrightarrow & \Omega^{i}(N) & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $P_i \to \Omega^i(K) \to 0$, $L_i \to \Omega^i(M) \to 0$ and $Q_i \to \Omega^i(N) \to 0$ are graded projective covers. Now pasting these commutative diagrams together, we finish the proof.

$(\Leftarrow)$ In particular, we obtain a lot of commutative diagrams as above for all $i \geq 0$. Now by Lemma 5.1, we are done. ∎

Proposition 5.2. Let $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence in $gr(A)$ and A a standard graded algebra. Then

(1) If K, $N \in \mathscr{K}^a_{b,c}(A)$, then so is M.
(2) If K, $M \in \mathscr{K}^a_{b,c}(A)$, then so is N.
(3) If M, $N \in \mathscr{K}^a_{b,c}(A)$, then $K \in \mathscr{K}^a_{b,c}(A)$ if and only if the minimal Horseshoe Lemma holds.

Proof. (1) By Lemma 5.1 and Proposition 5.1, we have the following commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{P}_* & \longrightarrow & \mathscr{L}_* & \longrightarrow & \mathscr{Q}_* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where the vertical columns are minimal graded projective resolutions. Then L_n is generated in degree $\delta^a_{b,c}(n)$ for all $n \geq 0$ since $(\mathscr{L}_*)_n = (\mathscr{P}_* \oplus \mathscr{Q}_*)_n = P_n \oplus Q_n$. By the hypothesis, P_n and Q_n are both generated in degree $\delta^a_{b,c}(n)$ for all $n \geq 0$, which implies that $M \in \mathscr{K}^a_{b,c}(A)$.

(2) Note that K, M and N are all generated in degree zero as graded left A-modules, by Lemma 5.1, we have the commutative diagram as in (3) of Lemma 5.1. By hypothesis, $\Omega^1(K)$ and $\Omega^1(M)$ are generated in degree $\delta^a_{b,c}(1)$, which implies that $\Omega^1(N)$ is also generated in degree $\delta^a_{b,c}(1)$. Thus, by Lemma 5.1 and Proposition 5.1 several times, repeating the above arguments, we can get the above commutative diagram. Note that Q_n and L_n are generated in the same single degree and $M \in \mathscr{K}^a_{b,c}(A)$, which implies that $N \in \mathscr{K}^a_{b,c}(A)$.

(3) Suppose $K \in \mathscr{K}^a_{b,c}(A)$, then by Lemma 5.1 and similar to the proof of (1), we can obtain the desired commutative diagram. Conversely, let $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence with M and N being (a,b,c)-Koszul modules. Then by the hypothesis, we have the above commutative diagram and $L_n \cong P_n \oplus Q_n$ for all $n \geq 0$. Note that $M \in \mathscr{K}^a_{b,c}(A)$, then $P_n \oplus Q_n$ is generated in degree $\delta^a_{b,c}(n)$ for all $n \geq 0$, which implies that for all $n \geq 0$, P_n is generated in degree $\delta^a_{b,c}(n)$. Therefore, $K \in \mathscr{K}^a_{b,c}(A)$. ∎

Proposition 5.3. Let $M \in \mathscr{K}^a_{b,c}(A)$. Then

(1) $\Omega^{4k}(M)[-ka] \in \mathscr{K}^a_{b,c}(A)$ for all $k \in \mathbb{N}$;
(2) If A is an (a,b,c)-Koszul algebra, then $\Omega^{4k-1}(JM)[-ka] \in \mathscr{K}^a_{b,c}(A)$ for all $k \in \mathbb{N}$.

Proof. (1) can be obtained by truncating a minimal graded projective resolution of M at an appropriate place and we omit the details.

For (2), consider the exact sequence $0 \longrightarrow JM \longrightarrow M \longrightarrow M/JM \longrightarrow 0$, we have the exact sequence

$$0 \longrightarrow \Omega^1(M) \longrightarrow \Omega^1(M/JM) \longrightarrow JM \longrightarrow 0$$

such that each term is generated in degree $\delta^a_{b,c}(1)$. By Lemma 5.1, we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^2(M) & \longrightarrow & \Omega^2(M/JM) & \longrightarrow & \Omega^1(JM) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_1 & \longrightarrow & Q_1 \oplus L_0 & \longrightarrow & L_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(M/JM) & \longrightarrow & JM & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the vertical columns are graded projective covers. Repeating the above procedures, for all $k \geq 0$, we get the following exact sequences

$$0 \longrightarrow \Omega^{4k}(M) \longrightarrow \Omega^{4k}(M/JM) \longrightarrow \Omega^{4k-1}(JM) \longrightarrow 0,$$

which implies the following exact sequences

$$0 \longrightarrow \Omega^{4k}(M)[-ka] \longrightarrow \Omega^{4k}(M/JM)[-ka] \longrightarrow \Omega^{4k-1}(JM)[-ka] \longrightarrow 0.$$

Note that M/JM is an (a,b,c)-Koszul module since A is an (a,b,c)-Koszul algebra. Now by (1), we have $\Omega^{4k}(M)[-ka]$ and $\Omega^{4k}(M/JM)[-ka]$ are (a,b,c)-Koszul modules. Therefore, $\Omega^{4k-1}(JM)[-ka] \in \mathscr{K}^a_{b,c}(A)$ by Proposition 5.2. ∎

Proposition 5.4. Let A be a standard graded algebra. Then the following are equivalent for a finitely generated graded left A-module M:

(1) M is (a,b,c)-Koszul;
(2) $\mathrm{Ext}^n_A(M,A_0) = \mathrm{Ext}^n_A(M,A_0)_{-\delta^a_{b,c}(n)}$ for all $n \geq 0$;
(3) $\mathrm{Tor}^A_n(M,A_0) = \mathrm{Tor}^A_n(M,A_0)_{\delta^a_{b,c}(n)}$ for all $n \geq 0$.

Proposition 5.5. Let A be an (a,b,c)-Koszul algebra and $M \in gr_0(A)$. Then $M \in \mathscr{K}^a_{b,c}(A)$ if and only if the Ext module $\mathscr{E}(M)$ is generated in degree 0 as a graded $E(A)$-module.

Proof. Let $\mathscr{P}_*$ and $\mathscr{Q}_*$ be the minimal graded projective resolutions of A_0 and M, respectively. By the hypothesis, for all $n \geq 0$, P_n is generated in degree $\delta^a_{b,c}(n)$. By Proposition 3.5 of [10], the Ext module $\mathscr{E}(M)$ is generated by $\mathrm{Ext}^0_A(M,A_0)$ as a graded $E(A)$-module if and only if for all $n \geq 0$, Q_n is generated in degree $\delta^a_{b,c}(n)$, which is equivalent to $M \in \mathscr{K}^a_{b,c}(A)$. ∎

From now on, A denotes an (a,b,c)-Koszul algebra and $M \in \mathscr{K}^a_{b,c}(A)$. Set

$$\mathbb{E}^{[l]}(A) := \bigoplus_{i\geq 0} \mathrm{Ext}^{4ki+l}_A(A_0,A_0) \text{ and } \mathscr{E}^{[l]}(M) := \bigoplus_{i\geq 0} \mathrm{Ext}^{4ki+l}_A(M,A_0),\ (k \in \mathbb{N},\ l = 0,1,2,3).$$

Lemma 5.2. *Let $\mathbb{E}^{[l]}(A)$ and $\mathscr{E}^{[l]}(M)$ be defined as above. Then*

(1) $\mathbb{E}^{[0]}(A) := \bigoplus_{i\geq 0} \mathrm{Ext}_A^{4ki}(A_0,A_0)$ *is a standard graded subalgebra of* $E(A)$ *for each* $k \geq 0$*;*

(2) $\mathbb{E}^{[l]}(A)$ *and* $\mathscr{E}^{[l]}(M)$ *can be viewed as 0-generated graded* $\mathbb{E}^{[0]}(A)$*-modules for* $l = 0,1,2,3$.

Proof. (1) $\mathbb{E}^{[0]}(A)$ is a graded subalgebra of $E(A)$ can be obtained directly from $(\mathbb{E}^{[0]}(A))_i \cdot (\mathbb{E}^{[0]}(A))_j = \mathrm{Ext}_A^{4ki}(A_0,A_0)\cdot \mathrm{Ext}_A^{4kj}(A_0,A_0) \subseteq \mathrm{Ext}_A^{4k(i+j)}(A_0,A_0) = (\mathbb{E}^{[0]}(A))_{i+j}$. Thus to complete the proof, it suffices to prove that $\mathbb{E}^{[0]}(A)$ is generated in degrees 0 and 1. Note that $\delta^a_{b,c}(4ki) = \delta^a_{b,c}(4ki-4k) + \delta^a_{b,c}(4k)$. By Proposition 3.6 of [10], we have $(\mathbb{E}^{[0]}(A))_i = (\mathbb{E}^{[0]}(A))_1 \cdot (\mathbb{E}^{[0]}(A))_{i-1}$, which implies that $\mathbb{E}^{[0]}(A)$ can be generated by $(\mathbb{E}^{[k]}(A))_0$ and $(\mathbb{E}^{[k]}(A))_1$.

(2) We only consider the case of $\mathscr{E}^{[l]}(M)$ since $\mathbb{E}^{[l]}(A) = \mathscr{E}^{[l]}(A_0)$. Note that for all $i,j \geq 0$, we have $(\mathbb{E}^{[0]}(A))_i \cdot (\mathscr{E}^{[l]}(M))_j = \mathrm{Ext}_A^{4ki}(A_0,A_0)\cdot \mathrm{Ext}_A^{4kj+l}(M,A_0) \subseteq \mathrm{Ext}_A^{4k(i+j)+l}(M,A_0) = (\mathscr{E}^{[l]}(M))_{i+j}$, which implies that $\mathscr{E}^{[l]}(M)$ can be viewed as a graded $\mathbb{E}^{[0]}(A)$-module, and $\mathscr{E}^{[l]}(M) = \mathbb{E}^{[0]}(A)\cdot \mathscr{E}^{[l]}(M)_0$ can be obtained by $(\mathscr{E}^{[l]}(M))_i = \mathrm{Ext}_A^{4ki+l}(M,A_0) \overset{(a)}{=} \mathrm{Ext}_A^{4ki}(A_0,A_0)\cdot \mathrm{Ext}_A^{l}(M,A_0) = (\mathbb{E}^{[0]}(A))_i \cdot (\mathscr{E}^{[l]}(M))_0$, where $i \geq 1$ and (a) is implied by Proposition 5.5. ∎

Proposition 5.6. $\mathscr{E}^{[0]}(M)$ is a Koszul $\mathbb{E}^{[0]}(A)$-module for each $k \geq 0$. In particular, $\mathbb{E}^{[0]}(A)$ is a Koszul algebra for each $k \geq 0$.

Proof. It suffices to prove $\mathscr{E}^{[0]}(M)$ is a Koszul module since $\mathbb{E}^{[0]}(A) = \mathscr{E}^{[0]}(A_0)$.

Consider the exact sequence $0 \longrightarrow JM \longrightarrow M \longrightarrow M/JM \longrightarrow 0$. Similar to the proof of Proposition 5.3 (2), for all k, $n \geq 0$, we have the exact sequences

$$0 \longrightarrow \Omega^{4kn}(M) \longrightarrow \Omega^{4kn}(M/JM) \longrightarrow \Omega^{4kn-1}(JM) \longrightarrow 0,$$

which imply the following exact sequences

$$0 \longrightarrow \mathrm{Hom}_A(\Omega^{4kn-1}(JM),A_0) \longrightarrow \mathrm{Hom}_A(\Omega^{4kn}(M/JM),A_0) \longrightarrow \mathrm{Hom}_A(\Omega^{4kn}(M),A_0) \longrightarrow 0$$

since A_0 is semisimple as a graded left A-module. Therefore, for all k, $n \geq 0$, we have the exact sequences

$$0 \longrightarrow \mathrm{Ext}_A^{4kn-1}(JM,A_0) \longrightarrow \mathrm{Ext}_A^{4kn}(M/JM,A_0) \longrightarrow \mathrm{Ext}_A^{4kn}(M,A_0) \longrightarrow 0$$

such that all terms in the above exact sequences are supported in shifting-degree $\delta^a_{b,c}(4kn) = kna$. Clearly, we have the exact sequences

$$0 \longrightarrow \mathrm{Ext}_A^{4k(n-1)}(\Omega^{4k-1}(JM),A_0) \longrightarrow \mathrm{Ext}_A^{4kn}(M/JM,A_0) \longrightarrow \mathrm{Ext}_A^{4kn}(M,A_0) \longrightarrow 0$$

for all k, $n \geq 0$ since $\mathrm{Ext}_A^{4kn-1}(JM,A_0) = \mathrm{Ext}_A^{4k(n-1)}(\Omega^{4k-1}(JM),A_0)$. Thus, we obtain the following exact sequences

$$0 \to \mathscr{E}^{[0]}(\Omega^{4k-1}(JM))[1] \to \bigoplus_{n\geq 1} \mathrm{Ext}_A^{4kn}(M/JM,A_0) \to \bigoplus_{n\geq 1} \mathrm{Ext}_A^{4kn}(M,A_0) \to 0.$$

Now we claim that $\mathscr{E}^{[0]}(M/JM)$ is a graded projective cover of $\mathscr{E}^{[0]}(M)$ and it is generated in degree $\delta^a_{b,c}(0) = 0$. In fact, $\mathscr{E}^{[0]}(M/JM)$ is an $\mathbb{E}^{[0]}(A)$-projective module since M/JM is semisimple. It is trivial that M/JM is an (a,b,c)-Koszul module since A is an (a,b,c)-Koszul algebra. By Proposition 5.5, $\mathscr{E}^{[0]}(M/JM)$ is generated in degree 0 as a graded $\mathbb{E}^{[0]}(A)$-module, and by the above exact sequence, it is the graded projective cover of $\mathscr{E}^{[0]}(M)$.

Therefore the first syzygy is $\bigoplus_{n>0}\mathrm{Ext}_A^{4k(n-1)}(\Omega^{4k-1}(JM),A_0)$, and $\Omega^{4k-1}(JM)$ is generated in degree $\delta^a_{b,c}(4k)=ka$ and by Proposition 5.3, $\Omega^{4k-1}(JM)[-ka]$ is again an (a,b,c)-Koszul module. Inductively, we complete the proof. ∎

Remark 5.1. Motivated by Lemma 5.2 and Proposition 5.6, one can ask a natural question:

- Are these $\mathbb{E}^{[0]}(A)$-modules $\mathbb{E}^{[l]}(A)$ and $\mathscr{E}^{[l]}(M)$ Koszul?

We can give a sufficient condition for the above question to be positive in terms of generalized (a,b,c)-Koszul modules.

Definition 5.7. Let A be a standard graded algebra and $M=\bigoplus_{i\geq 0}M_i$ a finitely generated graded left A-module. Let

$$\cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

be a minimal graded projective resolution of M. Then M is called an *generalized* (a,b,c)-*Koszul module* if for each $n\geq 0$, Q_n is generated in degrees in $\Delta^a_{b,c}(n)$, where $a,\ b,\ c\in\mathbb{N}$ satisfy $a>c>b>1$ and $\Delta^a_{b,c}$ is a set function from $\mathbb{N}$ to $\mathbb{N}$ defined by

$$\Delta^a_{b,c}(n)=\begin{cases}\{ka\}, & n=4k;\\ \{ka+1,ka+2,\cdots,ka+b-1\}, & n=4k+1;\\ \{ka+b,ka+b+1,\cdots,ka+c-1\}, & n=4k+2;\\ \{ka+c,ka+c+1,\cdots,ka+a-1\}, & n=4k+3,\end{cases}$$

and $k\in\mathbb{N}$.

Example 5.8. Let Γ be the following quiver:

$$\alpha \circlearrowright 1 \xrightarrow{\beta} 2 \xrightarrow{\gamma} 3 \xrightarrow{\varepsilon} 4.$$

Let I be the graded homogeneous ideal generated by α^3 and $\alpha\beta\gamma$ and $A:=\frac{\Bbbk\Gamma}{I}$. Now consider the minimal graded projective resolution of the simple module $S_1=\Bbbk$ related to the vertex 1, under a routine computation, we obtain the following minimal graded projective resolution $\mathscr{P}_{1*}:\cdots\to(Ae_1\oplus Ae_3)[6]\to Ae_1[4]\oplus Ae_3[5]\to(Ae_1\oplus Ae_3)[3]\to(Ae_1\oplus Ae_2)[1]\to Ae_1\to S_1\to 0$. Note that $\ker((Ae_1\oplus Ae_3)[6]\to Ae_1[4]\oplus Ae_3[5])=\ker((Ae_1\oplus Ae_3)[3]\to(Ae_1\oplus Ae_2)[1])[3]=(A\alpha\oplus A\beta\gamma)[6]$, thus we get a clear periodic minimal graded projective resolution of S_1. Thus S_1 is a generalized $(6,3,4)$-Koszul module.

Example 5.9. Let A be an (a,b,c)-Koszul algebra and M an (a,b,c)-Koszul module. Then all the syzygies of M, $\Omega^i(M)[-\delta^a_{b,c}(i)]$ $(\forall i\geq 0)$, are generalized (a,b,c)-Koszul modules.

Lemma 5.3. *Let A be a standard graded algebra and M a finitely generated graded left A-module. Then we have $\mathscr{E}^{[l]}(M)\cong\mathscr{E}^{[0]}(\Omega^l(M))$ as graded $\mathbb{E}^{[0]}$-modules.*

Proof. Let

$$\cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

be a minimal graded projective resolution of M in $gr(A)$. Then $\Omega^l(M)$ has the following minimal graded projective resolution in $gr(A)$

$$\cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_{l+1} \longrightarrow Q_l \longrightarrow \Omega^l(M) \longrightarrow 0.$$

Note that A_0 is semisimple, we have

$$\mathscr{E}^{[l]}(M) = \bigoplus_{i\geq 0} \mathrm{Ext}_A^{4ki+l}(M, A_0) = \bigoplus_{i\geq 0} \mathrm{Hom}_A(Q_{4ki+l}, A_0)$$

and

$$\mathscr{E}^{[0]}(\Omega^l(M)) = \bigoplus_{i\geq 0} \mathrm{Ext}_A^{4ki}(\Omega^l(M), A_0) = \bigoplus_{i\geq 0} \mathrm{Hom}_A(Q_{4ki+l}, A_0),$$

which complete the proof. ∎

Theorem 5.10. *Let A be an (a,b,c)-Koszul algebra and X be any generalized (a,b,c)-Koszul module. Suppose that $\mathscr{E}^{[0]}(X)$ is a Koszul $\mathbb{E}^{[0]}(A)$-module. Then all $\mathbb{E}^{[0]}(A)$-modules $\mathbb{E}^{[l]}(A)$ and $\mathscr{E}^{[l]}(M)$, $(l = 1,\ 2,\ 3)$ are Koszul, where M is an (a,b,c)-Koszul module.*

Proof. By Example 5.9, we have $\Omega^i(M)[-\delta^a_{b,c}(i)]$ is a generalized (a,b,c)-Koszul module since M is an (a,b,c)-Koszul module . By the hypothesis, $\mathscr{E}^{[0]}(\Omega^i(M)[-\delta^a_{b,c}(i)])$ is a Koszul $\mathbb{E}^{[0]}(A)$-module for all $i \geq 0$. By Lemma 5.3, $\mathscr{E}^{[l]}(M) \cong \mathscr{E}^{[0]}(\Omega^l(M)) = \mathscr{E}^{[0]}(\Omega^l(M)[-\delta^a_{b,c}(l)])$ for $l = 1,\ 2,\ 3$. Therefore, all $\mathscr{E}^{[l]}(M)$ $(l = 1,\ 2,\ 3)$, are Koszul $\mathbb{E}^{[0]}(A)$-modules. ∎

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