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Some New Approach to the Computation for Fixed Point Index and Applications

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Abstract. In this paper, some new methods of computation of a fixed point index for A-proper semilinear operators are given. As applications, the existence of positive solutions for nonlinear first order periodic boundary value problem is discussed.

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1. Introduction and preliminaries

Cremins [3] developed a fixed point index for A-proper maps of the form $L-N$ using a method similar to that of Fitzpatrick and Petryshyn [7]. The fixed point index, like the topological degree, is a useful means for determining the existence of solutions to nonlinear problems. It is of particular interest when considering the existence of positive solutions. In this case the maps are defined on relatively open subsets of a cone which may have empty interior and, consequently, topological degree theory is not directly applicable.

The main purpose of this paper is to establish some theorems about computation of the fixed point index for A-proper maps of the form $L-N$ employing partial order relation. As applications, we obtain the existence of positive solutions for nonlinear first order periodic boundary value problem. Next, we will state the definitions that are used in the remainder of the paper.

Let X and Y be Banach spaces, D a linear subspace of X, $\{X_n\} \subset D$, and $\{Y_n\} \subset Y$ sequences of oriented finite dimensional subspaces such that $Q_n y \to y$ in Y for every y and $\mathrm{dist}(x, X_n) \to 0$ for every $x \in D$ where $Q_n : Y \to Y_n$ and $P_n : X \to X_n$ are sequences of continuous linear projections. The projection scheme $\Gamma = \{X_n, Y_n, P_n, Q_n\}$ is then said to be admissible for maps from $D \subset X$ to Y.

Definition 1.1. [17] *A map $T : D \subset X \to Y$ is called approximation-proper (abbreviated A-proper) at a point $y \in Y$ with respect to Γ if $T_n \equiv Q_n T|_{D\cap X_n}$ is continuous for each $n \in \mathbb{N}$ and whenever $\{x_{n_j} | x_{n_j} \in D \cap X_{n_j}\}$ is bounded with $T_{n_j} x_{n_j} \to y$, then there exists a subsequence*

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$\{x_{n_{j_k}}\}$ *such that* $x_{n_{j_k}} \to x \in D$*, and* $Tx = y$*. T is said to be A-proper on a set D if it is A-proper at all points of D.*

Now let K be a cone in an infinite dimensional Banach space X with projection scheme Γ such that $Q_n(K) \subseteq K$ for every $n \in \mathbb{N}$. Let $\rho : X \to K$ be an arbitrary retraction and $\Omega \subset X$ an open bounded set such that $\Omega_K = \Omega \cap K \neq \emptyset$. Let $T : \overline{\Omega}_K \to K$ be such that $I - T$ is A-proper at 0. Write $K_n = K \cap X_n = Q_n K$ and $\Omega_n = \Omega_K \cap X_n$. Then $Q_n \rho : X_n \to K_n$ is a finite dimensional retraction.

Definition 1.2. [17] *If $Tx \neq x$ on $\partial\Omega_K$, then we define*

$$\mathrm{ind}_K(T, \Omega) = \{k \in \mathbb{Z} \cup \{\pm\infty\} : i_{K_{n_j}}(Q_{n_j} T, \Omega_{n_j}) \to k \text{ for some } n_j \to \infty\},$$

that is, the index is the set of limit points of $i_{K_{n_j}}(Q_{n_j} T, \Omega_{n_j})$, where $i_{K_{n_j}}(Q_{n_j} T, \Omega_{n_j})$ is the finite dimensional index.

Let $L : \mathrm{dom}L \subset X \to Y$ be a Fredholm operator of index zero and $P : X \to X$, $Q : Y \to Y$ be continuous projectors such that ImP=Ker$L$, Ker$Q$=ImL and X=Ker$L \bigoplus$ Ker$P$, Y=Im$L \bigoplus$ImQ. The restriction of L to dom$L \cap$Ker$P$, denote L_1, is a bijection onto ImL with continuous inverse L_1^{-1} : Im$L \to$dom$L\cap$Ker$P$. Since dimImQ=dimKer$L$, there exists a continuous bijection J : Im$Q \to$ Ker$L$. Then X becomes an ordered Banach space under the partial ordering $\leq$ which is induced by K. K is said to be normal if there exists a positive constant γ such that $\theta \leq x \leq y$ implies $\|x\| \leq \gamma\|y\|$. K is called total if $X = \overline{K - K}$. For the concepts and the properties about the cone we refer to [1, 2, 5, 9]. If we let $H = L + J^{-1}P$, then $H : \mathrm{dom}L \subset X \to Y$ is a linear bijection with bounded inverse. Thus $K_1 = H(K \cap \mathrm{dom}L)$ is a cone in the Banach space Y.

Let $\Omega \subset X$ be open and bounded with $\Omega_K \cap \mathrm{dom}L \neq \emptyset$, $L : \mathrm{dom}L \subset X \to Y$ a bounded Fredholm operator of index zero, $N : \overline{\Omega}_K \cap \mathrm{dom}L \to Y$ a bounded continuous nonlinear operator such that $L - N$ is A-proper at 0.

We now recall the definition of the index of A-proper maps of the form $L - N$ acting on cones.

Definition 1.3. [3] *Let ρ_1 be a retraction from Y to K_1 and assume $Q_n K_1 \subset K_1$, $P + JQN + L_1^{-1}(I - Q)N$ maps K to K and $Lx \neq Nx$ on $\partial\Omega_K$. We define the fixed point index of $L - N$ over Ω_K as*

$$\mathrm{ind}_K([L, N], \Omega) = \mathrm{ind}_{K_1}(T, U),$$

where $U = H(\Omega_K)$, $T : Y \to Y$ be defined as $Ty = (N + J^{-1}P)H^{-1}y$ for each $y \in Y$, and the index on the right is that of Definition 1.2.

For convenience, we recall some properties of ind_K.

Proposition 1.1. [3] *Let L : dom$L \to Y$ be Fredholm of index zero, $\Omega \subset X$ be open and bounded. Assume that $P + JQN + L_1^{-1}(I - Q)N$ maps K to K, and $Lx \neq Nx$ on $\partial\Omega_K$. Then we have*

- (P_1) *(Existence property) if* $\mathrm{ind}_K([L, N], \Omega) \neq \{0\}$*, then there exists $x \in \Omega_K$ such that $Lx = Nx$.*
- (P_2) *(Normalization property) if $x_0 \in \Omega_K$, then* $\mathrm{ind}_K([L, -J^{-1}P + \hat{y}_0], \Omega) = \{1\}$*, where $\hat{y}_0 = Hx_0$ and $\hat{y}_0(y) = y_0$ for every $y \in H(\Omega_K)$.*

(P_3) *(Additivity property) if $Lx \neq Nx$ for $x \in \overline{\Omega}_K \backslash (\Omega_1 \cup \Omega_2)$, where Ω_1 and Ω_2 are disjoint relatively open subsets of Ω_K, then*

$$\mathrm{ind}_K([L,N],\Omega) \subseteq \mathrm{ind}_K([L,N],\Omega_1) + \mathrm{ind}_K([L,N],\Omega_2)$$

with equality if either of indices on the right is a singleton.

(P_4) *(Homotopy invariance property) if $L - N(\lambda,x)$ is an A-proper homotopy on Ω_K for $\lambda \in [0,1]$ and $(N(\lambda,x) + J^{-1}P)H^{-1} : K_1 \to K_1$ and $0 \notin (L - N(\lambda,x))(\partial\Omega_K)$ for $\lambda \in [0,1]$, then $\mathrm{ind}_K([L,N(\lambda,x)],\Omega) = \mathrm{ind}_{K_1}(T_\lambda, U)$ is independent of $\lambda \in [0,1]$, where $T_\lambda = (N(\lambda,x) + J^{-1}P)H^{-1}$.*

The following lemmas will be used in this paper.

Lemma 1.1. [4] *If $L : \mathrm{dom}L \to Y$ is Fredholm of index zero, Ω is an open bounded set, and $\Omega_K \cap \mathrm{dom}L \neq \emptyset$, $\theta \in \Omega \subset X$, and let $L - \lambda N$ be A-proper for $\lambda \in [0,1]$. Assume that N is bounded and $P + JQN + L_1^{-1}(I-Q)N$ maps K to K. If $Lx \neq \mu Nx - (1-\mu)J^{-1}Px$ on $\partial\Omega_K$ for $\mu \in [0,1]$, then*

$$\mathrm{ind}_K([L,N],\Omega) = \{1\}.$$

Lemma 1.2. *If $L : \mathrm{dom}L \to Y$ is Fredholm of index zero, Ω is an open bounded set, and $\Omega_K \cap \mathrm{dom}L \neq \emptyset$, and let $L - \lambda N$ be A-proper for $\lambda \in [0,1]$. Assume that N is bounded and $P + JQN + L_1^{-1}(I-Q)N$ maps K to K. If there exists $e \in K_1 \backslash \{\theta\}$, such that*

$$Lx - Nx \neq \mu e, \tag{1.1}$$

for every $x \in \partial\Omega_K$ and all $\mu \geq 0$, then

$$\mathrm{ind}_K([L,N],\Omega) = \{0\}.$$

Proof. Choose a real number l such that

$$l > \sup_{x\in\Omega} \frac{\|Lx - Nx\|}{\|e\|}, \tag{1.2}$$

and define $N(\mu,x) : [0,1] \times \overline{\Omega}_K \to Y$ by

$$N(\mu,x) = Nx + l\mu e.$$

Trivially, $(N(\mu,x) + J^{-1}P)H^{-1} : K_1 \to K_1$ and from (1.1) we obtain

$$Nx + l\mu e \neq Lx, \quad \text{for any } (\mu,x) \in [0,1] \times \partial\Omega_K.$$

Again, by homotopy invariance property in Proposition 1.1, we have

$$\mathrm{ind}_K([L,N(0,x)],\Omega) = \mathrm{ind}_K([L,N],\Omega) = \mathrm{ind}_K([L,N(1,x)],\Omega).$$

However

$$\mathrm{ind}_K([L,N(1,x)],\Omega) = \{0\}.$$

In fact, if $\mathrm{ind}_K([L,N(1,x)],\Omega) \neq \{0\}$, the existence property in Proposition 1.1 implies that there exists $x_0 \in \Omega_K$ such that

$$Lx_0 = Nx_0 + le.$$

Then

$$l = \frac{\|Lx_0 - Nx_0\|}{\|e\|},$$

which contradicts (1.2). So $\mathrm{ind}_K([L,N],\Omega) = \{0\}$. ∎

Remark 1.1. The original condition of [3, Theorem 5] was given with $\theta \neq e \in L(K \cap \mathrm{dom}L)$ instead of $e \in K_1 \backslash \{\theta\}$. The modification is necessary since otherwise it can not guarantee that $(N + \mu e + J^{-1}P)H^{-1} : K_1 \to K_1$.

Let $B : X \to X$ be a bounded linear operator. B is said to be positive if $B(K) \subset K$. In this case, B is an increasing operator, namely for $x, y \in X$, $x \leq y$ implies $Bx \leq By$.

2. Computation of fixed point index

Throughout this section, we denote by $\leq_1$ for partial ordering in K_1 and $\leq$ for partial ordering in K. We shall give some methods of computing the fixed point index for A-proper semilinear operators by using the cone theory.

Theorem 2.1. *If $L : \mathrm{dom}L \to Y$ is Fredholm of index zero, and let $L - \lambda N$ be A-proper for $\lambda \in [0,1]$. Assume that N is bounded and $P + JQN + L_1^{-1}(I-Q)N$ maps K to K, where K is a normal cone in X. Suppose that*

(i) *there exist a positive bounded linear operator $B : K \to K$ and $u_0 \in K_1 \backslash \{\theta\}$, such that*
$$(N + J^{-1}P)x \leq_1 (L + J^{-1}P)Bx + u_0, \quad \text{for any } x \in K;$$
(ii) *$r(B) < 1$, where $r(B)$ is the spectral radius of B.*

Then there exists $R_0 > 0$ such that for $R > R_0$, the fixed point index
$$\mathrm{ind}_K([L,N], B_R) = \{1\},$$
where $B_R = \{x \in X : \|x\| < R\}$.

Proof. Setting
$$W = \{x \in K : Lx = \mu Nx - (1-\mu)J^{-1}Px, \quad \mu \in [0,1]\},$$
we claim that W is bounded. For $x \in W$, then there exists $\mu \in [0,1]$ such that $(L + J^{-1}P)x = \mu(N + J^{-1}P)x$. From condition (i) we have
$$(L + J^{-1}P)x = \mu(N + J^{-1}P)x \leq_1 (N + J^{-1}P)x \leq_1 (L + J^{-1}P)Bx + u_0.$$
Applying $(L + J^{-1}P)^{-1}$ to the above inequality, we obtain
$$x \leq Bx + (L + J^{-1}P)^{-1}u_0.$$
This shows that
$$(I - B)x \leq (L + J^{-1}P)^{-1}u_0. \tag{2.1}$$
The condition (ii) gives $r(B) < 1$. From (ii) we obtain $(I - B)^{-1}$ is a bounded linear operator and maps K into K. On account of (2.1), we arrive at
$$\theta \leq x \leq (I - B)^{-1}(L + J^{-1}P)^{-1}u_0,$$
which together with the normality of K leads to
$$\|x\| \leq \gamma \|(I - B)^{-1}(L + J^{-1}P)^{-1}u_0\|, \quad \forall\, x \in W,$$
where γ is the normal constant of K. This shows that W is bounded. ∎

Let $R_0 = \sup_{x \in W} \|x\|$. For $R > R_0$, we have
$$Lx \neq \mu Nx - (1-\mu)J^{-1}Px, \quad \forall\, x \in \partial B_R \cap K, \quad \mu \in [0,1]. \tag{2.2}$$
Using Lemma 1.1, we infer by (2.2) that the conclusion is true.

Theorem 2.2. *If $L : \mathrm{dom}L \to Y$ is Fredholm of index zero, $\Omega \subset X$ is an open bounded set, and $\Omega_K \cap \mathrm{dom}L \neq \emptyset$, and let $L - \lambda N$ be A-proper for $\lambda \in [0,1]$. Assume that N is bounded, $P + JQN + L_1^{-1}(I-Q)N$ maps K to K, and $Lx \neq Nx$ on $\partial\Omega_K$. If there exist a positive bounded linear operator $B : K \to K$ and $u_1 \in K\backslash\{\theta\}$, such that*

$$Bu_1 \geq u_1, \tag{2.3}$$

and

$$(N + J^{-1}P)x \geq_1 (L + J^{-1}P)Bx, \quad \text{for any } x \in \partial\Omega_K, \tag{2.4}$$

then the fixed point index

$$\mathrm{ind}_K([L,N],\Omega) = \{0\}.$$

Proof. We show that

$$Lx - Nx \neq \mu u^*, \quad \forall\, x \in \partial\Omega_K, \quad \mu \geq 0, \tag{2.5}$$

where $u^* = (L + J^{-1}P)u_1 \in K_1 \setminus \{\theta\}$.

If otherwise, there exist $x_1 \in \partial\Omega_K$, and $\mu_1 \geq 0$, such that $Lx_1 - Nx_1 = \mu_1 u^*$. Thus $\mu_1 > 0$ and

$$(L + J^{-1}P)x_1 = (N + J^{-1}P)x_1 + \mu_1 u^* \geq_1 \mu_1 u^* = \mu_1(L + J^{-1}P)u_1. \tag{2.6}$$

Applying $(L + J^{-1}P)^{-1}$ to (2.6), we have $x_1 \geq \mu_1 u_1$. Put

$$\mu^* = \sup\{\mu \,|\, x_1 \geq \mu u_1\}.$$

It is easy to see that $\mu^* \geq \mu_1 > 0$ and $x_1 \geq \mu^* u_1$. Applying $(L + J^{-1}P)B : K \to K_1$ to this relation, we obtain $(L + J^{-1}P)Bx_1 \geq \mu^*(L + J^{-1}P)Bu_1$. This, together with (2.3)–(2.4), (2.6), implies that

$$\begin{aligned}
(L + J^{-1}P)x_1 &= (N + J^{-1}P)x_1 + \mu_1 u^* \geq_1 (L + J^{-1}P)Bx_1 + \mu_1 u^* \\
&\geq_1 \mu^*(L + J^{-1}P)Bu_1 + \mu_1 u^* \geq_1 \mu^*(L + J^{-1}P)u_1 + \mu_1(L + J^{-1}P)u_1 \\
&= (\mu^* + \mu_1)(L + J^{-1}P)u_1.
\end{aligned}$$

Operating on both sides of the last inequality by $(L + J^{-1}P)^{-1}$, we obtain $x_1 \geq (\mu^* + \mu_1)u_1$, which contradicts the definition of μ^*. Hence (2.5) is true and we have from Lemma 1.2 that

$$\mathrm{ind}_K([L,N],\Omega) = \{0\}. \qquad \blacksquare$$

Theorem 2.3. *If $L : \mathrm{dom}L \to Y$ is Fredholm of index zero, $\Omega \subset X$ is an open bounded set, K is a total cone in X, and $\Omega_K \cap \mathrm{dom}L \neq \emptyset$, and let $L - \lambda N$ be A-proper for $\lambda \in [0,1]$. Assume that N is bounded, $P + JQN + L_1^{-1}(I-Q)N$ maps K to K, and $Lx \neq Nx$ on $\partial\Omega_K$. Suppose that*

(i) *there exists a positive completely continuous linear operator $B : K \to K$, such that*

$$(N + J^{-1}P)x \geq_1 (L + J^{-1}P)Bx, \quad \forall\ x \in \partial\Omega_K;$$

(ii) *$r(B) \geq 1$, where $r(B)$ is the spectral radius of B.*

Then the fixed point index

$$\mathrm{ind}_K([L,N],\Omega) = \{0\}.$$

Proof. It follows from Krein–Rutman theorem (see [11]) that, there exists $u_1 \in K\backslash\{\theta\}$, such that $Bu_1 = r(B)u_1$, which, together condition (ii), implies $Bu_1 \geq u_1$.

The rest of the proof is the same as that of Theorem 2.2. ∎

Note the fact that $(L+J^{-1}P)^{-1}(N+J^{-1}P) = P+JQN+L_1^{-1}(I-Q)N$ (see [3, Lemma 2], and [13, 14]). An immediate consequence of Theorems 2.1, 2.2, 2.3 are the following corollaries.

Corollary 2.1. *Let all conditions of Theorem 2.1 hold except for (i), which is replaced by the following hypotheses:*

(i) *there exist a positive bounded linear operator $B : K \to K$ and $u_0 \in K\backslash\{\theta\}$, such that*

$$(P+JQN+L_1^{-1}(I-Q)N)x \leq Bx+u_0, \quad \textit{for any } x \in K.$$

Then there exists $R_0 > 0$ such that for $R > R_0$, the fixed point index

$$\mathrm{ind}_K([L,N],B_R) = \{1\},$$

where $B_R = \{x \in X : \|x\| < R\}$.

Corollary 2.2. *Let all conditions of Theorem 2.2 hold except for (2.4), which is replaced by the following hypotheses:*

$$(P+JQN+L_1^{-1}(I-Q)N)x \geq Bx, \quad \textit{for any } x \in \partial\Omega_K,$$

then the fixed point index

$$\mathrm{ind}_K([L,N],\Omega) = \{0\}.$$

Corollary 2.3. *Let all conditions of Theorem 2.3 hold except for (i), which is replaced by the following hypotheses:*

(i) *there exists a positive completely continuous linear operator $B : K \to K$, such that*

$$(P+JQN+L_1^{-1}(I-Q)N)x \geq Bx, \quad \forall\, x \in \partial\Omega_K,$$

then the fixed point index

$$\mathrm{ind}_K([L,N],\Omega) = \{0\}.$$

The next theorem gives some sufficient conditions for the existence of a positive solution to a semilinear equation in cones. It is worth mentioning that existence of the positive or nonnegative solutions of a coincidence equation $Lx = Nx$ was also discussed by recent papers of Cremins [4], O'Regan and Zima [16], Zima [23], Infante and Zima [10] and the earlier papers [6, 8, 15, 19–21].

Theorem 2.4. *If $L : \mathrm{dom}L \to Y$ is Fredholm of index zero, and let $L-\lambda N$ be A-proper for $\lambda \in [0,1]$. Assume that N is bounded, $P+JQN+L_1^{-1}(I-Q)N$ maps K to K, and K is a normal total cone in X. Suppose there exist a positive bounded linear operator $B : K \to K$ with $r(B) < 1$, $u_0 \in K\backslash\{\theta\}$, and a positive completely continuous linear operator $B_1 : K \to K$ with $r(B_1) \geq 1$, such that*

(i) $(P+JQN+L_1^{-1}(I-Q)N)x \leq Bx+u_0, \quad \forall\, x \in K;$

(ii) $(P+JQN+L_1^{-1}(I-Q)N)x \geq B_1x, \quad \forall\, x \in \partial B_r \cap K$, *where $B_r = \{x \in X : \|x\| < r\}$.*

Then there exists $x^* \in \mathrm{dom}L \cap K \setminus \{\theta\}$ such that $Lx^* = Nx^*$.

Proof. It follows from Corollary 2.1 and condition (i) that there exists $R > 0$ with $R > r$ such that

$$\mathrm{ind}_K([L,N],B_R) = \{1\}. \tag{2.7}$$

We assume $Lx \neq Nx$ on $\partial B_r \cap K \cap \mathrm{dom}L$; otherwise the conclusion follows. Using Corollary 2.3 we get from condition (ii) that

$$\mathrm{ind}_K([L,N],B_r) = \{0\}. \tag{2.8}$$

By (2.7), (2.8), and the additivity property in Proposition 1.1 we obtain

$$\mathrm{ind}_K([L,N],B_R\backslash B_r) = \mathrm{ind}_K([L,N],B_R) - \mathrm{ind}_K([L,N],B_r) = \{1\} - \{0\} = \{1\} \neq \{0\},$$

which completes the proof from the existence property in Proposition 1.1. ∎

3. Applications to periodic problem

In this section, we will apply Theorem 2.4 to the following first order periodic boundary value problem (PBVP)

$$\begin{cases} x'(t) = f(t,x(t)), & t \in (0,1), \\ x(0) = x(1), \end{cases} \tag{3.1}$$

where $f : [0,1] \times [0,+\infty) \to \mathbb{R}$ is a continuous function.

Consider the Banach spaces $X = Y = C[0,1]$ endowed with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$. Define the cone K in X by $K = \{x \in X : x(t) \geq 0,\ t \in [0,1]\}$, then K is a normal total cone of X (see [5]). Define L be the linear operator from $\mathrm{dom}L \subset X$ to Y with

$$\mathrm{dom}L = \left\{x \in X : x' \in C[0,1], x(0) = x(1)\right\}, \text{ and } Lx(t) = x'(t),\ x \in \mathrm{dom}L \text{ and } t \in [0,1].$$

We define $N : X \to Y$ by setting

$$Nx(t) = f(t,x(t)), \quad t \in [0,1],$$

then PBVP (3.1) can be written as $Lx = Nx$. It is easy to check that

$$\mathrm{Ker}L = \{x \in \mathrm{dom}L : x(t) \equiv c \text{ on } [0,1], c \in \mathbb{R}\}, \quad \mathrm{Im}L = \left\{y \in Y : \int_0^1 y(s)ds = 0\right\},$$
$$\dim \mathrm{Ker}L = \mathrm{codim}\ \mathrm{Im}L = 1,$$

so that L is a Fredholm operator of index zero.

Next, define the projections $P : X \to X$ by

$$Px = \int_0^1 x(s)ds,$$

and $Q : Y \to Y$ by

$$Qy = \int_0^1 y(s)ds.$$

Furthermore, we define the isomorphism $J : \mathrm{Im}Q \to \mathrm{Im}P$ as $Jy = y$. Note that for $y \in \mathrm{Im}L$, the inverse operator $L_1^{-1} : \mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P$ of $L|_{\mathrm{dom}L\cap\mathrm{Ker}P} : \mathrm{dom}L \cap \mathrm{Ker}P \to \mathrm{Im}L$ is given by $(L_1^{-1}y)(t) = \int_0^1 G(t,s)y(s)ds$, where

$$G(t,s) = \begin{cases} s+1, & 0 \leq s < t \leq 1, \\ s, & 0 \leq t \leq s \leq 1. \end{cases}$$

In fact, for $y \in \mathrm{Im}L$, we have

$$(LL_1^{-1})y(t) = [(L_1^{-1}y)(t)]' = y(t)$$

and for $x \in \mathrm{dom}L \cap \mathrm{Ker}P$, we know

$$\begin{aligned}(L_1^{-1}L)x(t) &= \int_0^1 G(t,s)x'(s)ds = \int_0^t (s+1)x'(s)ds + \int_t^1 sx'(s)ds \\ &= (t+1)x(t) - x(0) - \int_0^t x(s)ds + x(1) - tx(t) - \int_t^1 x(s)ds \\ &= x(t) - x(0) + x(1) - \int_0^1 x(s)ds,\end{aligned}$$

in view of $x \in \mathrm{dom}L \cap \mathrm{Ker}P, x(0) = x(1), \int_0^1 x(s)ds = 0$, thus

$$(L_1^{-1}L)x(t) = x(t).$$

This shows that $(L_1^{-1}y)(t) = \int_0^1 G(t,s)y(s)ds$.

For notational convenience, we set $H(t,s) = 1 + G(t,s) - \int_0^1 G(t,s)ds$ or

$$H(t,s) = \begin{cases} \frac{3}{2} - (t-s), & 0 \le s < t \le 1, \\ \frac{1}{2} + (s-t), & 0 \le t \le s \le 1. \end{cases}$$

Notice that $1/2 \le H(t,s) \le 3/2$, $\forall\, t,s \in [0,1]$, and $H(0,s) = H(1,s)$, $\forall\, s \in [0,1]$.

We can now state and prove our result on the existence of a positive solution for the PBVP (3.1).

Theorem 3.1. *Suppose*

(H_1) $f(t,x) \ge -2/3x$, *for all* $t \in [0,1]$, $x \ge 0$,
(H_2) $\liminf_{x\to 0^+} \min_{t\in[0,1]} f(t,x)/x > 0$,
(H_3) $\limsup_{x\to +\infty} \max_{t\in[0,1]} f(t,x)/x < 0$.

Then the PBVP (3.1) has at least one positive solution.

Proof. First, we note that L, as so defined, is Fredholm of index zero, L_1^{-1} is compact by Arzela-Ascoli theorem and thus $L - \lambda N$ is A-proper for $\lambda \in [0,1]$ by (a) of [6, Lemma 2].

For each $x \in K$, then by condition (H_1) that

$$\begin{aligned}&Px + JQNx + L_1^{-1}(I-Q)Nx \\ &= \int_0^1 x(s)ds + \int_0^1 f(s,x(s))ds + \int_0^1 G(t,s)\left(f(s,x(s)) - \int_0^1 f(s,x(s))ds\right)ds \\ &= \int_0^1 x(s)ds + \int_0^1 H(t,s)f(s,x(s))ds \ge \int_0^1 \left(1 - \frac{2}{3}H(t,s)\right)x(s)ds \ge 0.\end{aligned}$$

Thus $(P + JQN + L_1^{-1}(I-Q)N)(K) \subset K$.

It follows from condition (H_2) that there exist $\varepsilon > 0$ and $r > 0$ such that

$$f(t,x) \ge \varepsilon x, \quad \forall\, t \in [0,1], \quad 0 < x \le r. \tag{3.2}$$

Let $B_1 x = (1+\varepsilon/2)\int_0^1 x(s)ds$. Then $B_1 : X \to X$ is a positive completely continuous linear operator. One can easily show that $r(B_1) = 1 + \varepsilon/2 > 1$. From (3.2), we get

$$Px + JQNx + L_1^{-1}(I-Q)Nx = \int_0^1 x(s)ds + \int_0^1 H(t,s)f(s,x(s))ds$$

$$\geq \int_0^1 x(s)ds + \frac{\varepsilon}{2}\int_0^1 x(s)ds = \left(1+\frac{\varepsilon}{2}\right)\int_0^1 x(s)ds$$
$$= B_1 x, \quad \forall\, x \in K,\ \|x\| \leq r.$$

This implies that condition (ii) of Theorem 2.4 is satisfied.

It follows from condition (H_3) that there exist $0<\sigma<2/3$ and $R>r>0$ such that

$$f(t,x) \leq -\sigma x, \quad \forall\, t \in [0,1], \quad x \geq R. \tag{3.3}$$

(3.3) implies

$$f(t,x) \leq -\sigma x + M, \quad \forall\, t \in [0,1], \quad x \geq 0, \tag{3.4}$$

where

$$M = \sup_{t\in[0,1],\ 0\leq x\leq R} |f(t,x)|.$$

Take $u_0 = 3M/2$, $Bx = (1-3\sigma/2)\int_0^1 x(s)ds$. One can see that $B: X \to X$ is a positive bounded linear operator. It is clear to see that $r(B) = 1-3\sigma/2 < 1$. Thus, by (3.4), we have

$$\begin{aligned}
&Px + JQNx + L_1^{-1}(I-Q)Nx \\
&= \int_0^1 x(s)ds + \int_0^1 H(t,s)f(s,x(s))ds \leq \int_0^1 x(s)ds + \frac{3}{2}\int_0^1 (-\sigma x(s)+M)ds \\
&\leq \int_0^1 \left(1-\frac{3\sigma}{2}\right)x(s)ds + \frac{3M}{2} = \left(1-\frac{3\sigma}{2}\right)\int_0^1 x(s)ds + \frac{3M}{2} \\
&= Bx + u_0, \quad \forall\, x \in K.
\end{aligned}$$

This means that condition (i) of Theorem 2.4 is verified. ∎

Thus all conditions of Theorem 2.4 are satisfied and there exists $x \in K\backslash\{\theta\}$ such that $Lx = Nx$ and the assertion follows.

Remark 3.1. It should be noted that the periodic boundary value problem for the first order differential equations has been studied earlier by many authors under various conditions on $f(t,x)$ we refer the reader to [5,8,10,12,16,20,22,23]. The method we used here is different in essence from other papers and Theorem 3.1 of this paper is also new.

The following example shows that all the hypotheses of Theorem 3.1 can be satisfied.

Example 3.1. Consider the first order periodic boundary value problem (PBVP)

$$\begin{cases} x'(t) = f(x(t)), & t \in (0,1), \\ x(0) = x(1), \end{cases} \tag{3.5}$$

where $f: [0,+\infty) \to (-\infty,+\infty)$ is defined by

$$f(x) = \begin{cases} x^2 + x, & 0 \leq x \leq 1, \\ x+1, & 1 < x \leq 2, \\ -\frac{1}{2}x + 4, & x > 2. \end{cases}$$

After simple computations, we get

(1) $f(x) \geq -2/3x$, $\forall\, x \geq 0$,
(2) $\liminf_{x\to 0^+} f(x)/x = 1 > 0$,
(3) $\limsup_{x\to+\infty} f(x)/x = -1/2 < 0$.

Then by Theorem 3.1, the PBVP (3.5) has at least one positive solution.

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