BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **36***(2) (2013), 511–523*

Growth Property and Integral Representation of Harmonic Functions in a Cone

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Abstract. Our aim in this paper is to deal with the growth property at infinity for modified Poisson integrals in an n-dimensional cone. We also generalize the integral representation of harmonic functions in a half space of $\mathbf{R}^n (n \geq 2)$ to the conical case.

2010 Mathematics Subject Classification: 31B10, 31C05

Keywords and phrases: Growth property, integral representation, harmonic function, cone.

1. Introduction and results

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of all real numbers and the set of all positive real numbers, respectively. We denote by $\mathbf{R}^n (n \geq 2)$ the n-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $P = (X, x_n)$, $X = (x_1, x_2, \ldots, x_{n-1})$. The Euclidean distance of two points P and Q in $\mathbf{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin O of $\mathbf{R}^n$ is simply denoted by $|P|$. The boundary and the closure of a set $\mathbf{S}$ in $\mathbf{R}^n$ are denoted by $\partial \mathbf{S}$ and $\overline{\mathbf{S}}$, respectively.

We introduce a system of spherical coordinates (r, Θ), $\Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to cartesian coordinates $(X, x_n) = (x_1, x_2, \ldots, x_{n-1}, x_n)$ by $x_n = r\cos\theta_1$. For $P \in \mathbf{R}^n$ and $R > 0$, Let $B(P, R)$ denote the open ball with center at P and radius R in $\mathbf{R}^n$. $S_R = B(O, R)$. The unit sphere and the upper half unit sphere in $\mathbf{R}^n (n \geq 2)$ are denoted by $\mathbf{S}_1$ and $\mathbf{S}_1^+$, respectively. For simplicity, a point $(1, \Theta)$ on $\mathbf{S}_1$ and the set $\{\Theta; (1, \Theta) \in \Gamma\}$ for a set Γ, $\Gamma \subset \mathbf{S}_1$, are often identified with Θ and Γ, respectively. For two sets $\Lambda \subset \mathbf{R}_+$ and $\Gamma \subset \mathbf{S}_1$, the set $\{(r, \Theta) \in \mathbf{R}^n; r \in \Lambda, (1, \Theta) \in \Gamma\}$ in $\mathbf{R}^n$ is simply denoted by $\Lambda \times \Gamma$. In particular, the half space $\mathbf{R}_+ \times \mathbf{S}_1^+ = \{(X, x_n) \in \mathbf{R}^n; x_n > 0\}$ will be denoted by $\mathbf{T}_n$. By $C_n(\Gamma)$, we denote the set $\mathbf{R}_+ \times \Gamma$ in $\mathbf{R}^n$ with the domain Γ on $\mathbf{S}_1$. We call it a cone. Then T_n is a special cone obtained by putting $\Gamma = \mathbf{S}_1^+$. We denote the sets $I \times \Gamma$ and $I \times \partial\Gamma$ with an interval on $\mathbf{R}$ by $C_n(\Gamma; I)$ and $S_n(\Gamma; I)$. By $S_n(\Gamma; R)$ we denote $C_n(\Gamma) \cap S_R$. By $S_n(\Gamma)$ we denote $S_n(\Gamma; (0, +\infty))$ which is $\partial C_n(\Gamma) - \{O\}$.

Communicated by S. Ponnusamy.
Received: January 23, 2011; *Revised:* August 23, 2011.

Furthermore, we denote by $d\sigma_Q$ (resp. dS_R) the $(n-1)$-dimensional volume elements induced by the Euclidean metric on $\partial C_n(\Gamma)$ (resp. S_R) and by dw the elements of the Euclidean volume in $\mathbf{R}^n$. Let $\Gamma \subset \mathbf{S}_1$, Δ be the Laplace operator in $\mathbf{R}^n$ and Δ^* be a Laplace-Beltrami (spherical part of the Laplace) on the unit sphere. It is known (see, e.g. [7, p. 41]) that

$$(1.1)\qquad \begin{aligned} \Delta^*\varphi(\Theta)+\lambda\varphi(\Theta)&=0 \quad \text{in} \quad \Gamma,\\ \varphi(\Theta)&=0 \quad \text{on} \quad \partial\Gamma, \end{aligned}$$

has the non-decreasing sequence of positive eigenvalues of (1.1) in the domain Γ, repeating accordingly to their multiplicities, and the corresponding eigenfunctions are denoted, respectively, by λ_i and $\varphi_i(\Theta)$, $i=1,2,3,\ldots$. Especially, we denote the least positive eigenvalue of (1.1) by λ_1 and the normalized positive eigenfunction to λ_1 by $\varphi_1(\Theta)$, $\int_\Gamma |\varphi_1(\Theta)|^2 dS_1 = 1$.

To make simplify our consideration in the following, we put a rather strong assumption on Γ: if $n \geq 3$, then Γ is a $C^{2,\alpha}$-domain $(0<\alpha<1)$ on $\mathbf{S}_1$ surrounded by a finite number of mutually disjoint closed hypersurfaces (e.g. see [4, p. 88–89] for the definition of $C^{2,\alpha}$-domain). Then $\varphi_i \in C^2(\overline{\Gamma})$ $(i=1,2,3,\ldots)$ and $\partial\varphi_1/\partial n > 0$ on $\partial\Gamma$ (here and below, $\partial/\partial n$ denotes differentiation along the interior normal). Further, there exist three positive constants c_1, c_2 and c_3 such that

$$(1.2)\qquad |\varphi_i(\Theta)| \leq c_1 i^{\frac{1}{2}} \quad (\Theta\in\Gamma, i=1,2,3,\ldots)$$

and

$$(1.3)\qquad c_2 \operatorname{dist}(\Theta,\partial\Gamma) \leq \varphi_1(\Theta) \leq c_3 \operatorname{dist}(\Theta,\partial\Gamma) \quad (\Theta\in\Gamma)$$

(by modifying Miranda's method [5, p. 7–8], we can prove this inequality).

The set of sequential eigenfunctions corresponding to the same value of λ_i in the sequence $\varphi_i(\Theta)$ $(i=1,2,3,\ldots)$ makes an orthonormal basis for the eigenspace of the eigenvalue λ_i. Hence for each $\Gamma \subset S_1$ there is a sequence $\{k_j\}$ of positive integers such that $k_1=1$, $\lambda_{k_j} < \lambda_{k_{j+1}}$, $\lambda_{k_j} = \lambda_{k_j+1} = \lambda_{k_j+2} = \ldots = \lambda_{k_{j+1}-1}$ and $\{\varphi_{k_j}, \varphi_{k_j+1}, \ldots, \varphi_{k_{j+1}-1}\}$ is an orthonormal basis for the eigenspace of the eigenvalue λ_{k_j} $(j=1,2,3,\ldots)$. By I_{k_m} we denote the set of all positive integers less than k_m $(m=1,2,3,\ldots)$. In spite of the fact $I_{k_1}=\varnothing$, the summation over I_{k_1} of a function $S(k)$ of a variable k will be used by promising $\sum_{i\in I_{k_1}} S(i)=0$.

We note that each function

$$r^{\aleph_i^\pm}\varphi_i(\Theta) \quad (i=1,2,3,\ldots)$$

is harmonic in $C_n(\Gamma)$, belongs to the class $C^2(C_n(\Gamma)\backslash\{O\})$ and vanishes on $S_n(\Gamma)$, where

$$2\aleph_i^\pm = -n+2\pm\sqrt{(n-2)^2+4\lambda_i} \quad (i=1,2,3,\ldots).$$

If $\Gamma=\mathbf{S}_1^+$, then $\aleph_1^+=1$, $\aleph_1^-=1-n$ and $\varphi_1(\Theta)=(2nw_n^{-1})^{1/2}\cos\theta_1$, where w_n is the surface area $2\pi^{n/2}(\Gamma(n/2))^{-1}$ of $\mathbf{S}_1$. In the sequel, for the sake of brevity, we shall write φ instead of φ_1, $\aleph^\pm$ instead of $\aleph_1^\pm$ and χ instead of $\aleph_1^+ - \aleph_1^-$.

Let $G_{C_n(\Gamma)}(P,Q)$ $(P=(r,\Theta), Q=(t,\Phi)\in C_n(\Gamma))$ be the Green function of $C_n(\Gamma)$. Then the ordinary Poisson kernel relative to $C_n(\Gamma)$ is defined by

$$P_{C_n(\Gamma)}(P,Q)=\frac{1}{c_n}\frac{\partial}{\partial n_Q}G_{C_n(\Gamma)}(P,Q), \quad c_n=\begin{cases}2\pi & \text{if } n=2\\ (n-2)w_n & \text{if } n\geq 3,\end{cases}$$

where $Q \in S_n(\Gamma)$ and $\partial/\partial n_Q$ denotes the differentiation at Q along the inward normal into $C_n(\Gamma)$.

Let $F(\Theta)$ be a function on Γ. The integral

$$\int_\Gamma F(\Theta)\varphi_i(\Theta)dS_1,$$

is denoted by $N_i(F)$, when it exists. For a non-negative integer m and two points $P=(r,\Theta)\in C_n(\Gamma)$, $Q=(t,\Phi)\in S_n(\Gamma)$, we put

$$\widetilde{K}_{C_n(\Gamma),m}(P,Q)=\begin{cases}0 & \text{if } 0<t<1,\\ K_{C_n(\Gamma),m}(P,Q) & \text{if } 1\le t<\infty,\end{cases}$$

where

$$K_{C_n(\Gamma),m}(P,Q)=\sum_{i\in I_{k_{m+1}}} 2^{\aleph_i^+ +n-1}N_i(P_{C_n(\Gamma)}((1,\Theta),(2,\Phi)))r^{\aleph_i^+}t^{-\aleph_i^+ -n+1}\varphi_i(\Theta). \tag{1.4}$$

To obtain the modified Poisson integral representation in a cone, as in [9], we use the following modified kernel function defined by

$$P_{C_n(\Gamma),m}(P,Q)=P_{C_n(\Gamma)}(P,Q)-\widetilde{K}_{C_n(\Gamma),m}(P,Q), \tag{1.5}$$

where $P\in C_n(\Gamma)$ and $Q\in S_n(\Gamma)$.

Remark 1.1. Suppose $\Gamma=S_1^+$, $P=(r,\Theta)=(X,x_n)\in T_n$ and $Q=(t,\Phi)=(Y,0)\in\partial T_n$ satisfying $r<t$. Then we have $\aleph_{k_i}^+=i$ $(i=1,2,3,\ldots)$ and

$$P_{T_n,m}(P,Q)=\begin{cases}P_{T_n}(P,Q)=2w_n^{-1}x_n|P-Q|^{-n} & \text{if } 0<t<1,\\ P_{T_n}(P,Q)-2w_n^{-1}\sum_{i=0}^{m-1}x_nr^it^{-n-i}C_i^{\frac{n}{2}}(\cos\eta) & \text{if } 1\le t<\infty,\end{cases} \tag{1.6}$$

where $C_i^{n/2}(\cdot)$ is the Gegenbauer polynomial of degree i and η is the angle between $M=(X,0)$ and $N=(Y,0)$ defined by

$$\cos\eta=\frac{(M,N)}{|M||N|}$$

(see [9, Remarks 1, 2 and 3]).

Write

$$U_{C_n(\Gamma),m}(P)=\int_{S_n(\Gamma)}P_{C_n(\Gamma),m}(P,Q)u(Q)d\sigma_Q,$$

where $u(Q)$ is a continuous function on $\partial C_n(\Gamma)$.

For real numbers $\beta\ge 1$, we denote $\mathscr{A}_{\Gamma,\beta}$ the class of all measurable functions $f(t,\Phi)$ $(Q=(t,\Phi)=(Y,y_n)\in C_n(\Gamma))$ satisfying the following inequality

$$\int_{C_n(\Gamma)}\frac{|f(t,\Phi)|\varphi}{1+t^{n+\aleph_{k_{[\beta]}}^+ +\{\beta\}}}dw<\infty \tag{1.7}$$

and the class $\mathscr{B}_{\Gamma,\beta}$, consists of all measurable functions $g(t,\Phi)$ $(Q=(t,\Phi)=(Y,y_n)\in S_n(\Gamma))$ satisfying

$$\int_{S_n(\Gamma)}\frac{|g(t,\Phi)|}{1+t^{n+\aleph_{k_{[\beta]}}^+ +\{\beta\}-2}}\frac{\partial\varphi}{\partial n}d\sigma_Q<\infty, \tag{1.8}$$

where $[\beta]$ is the integral part of β and $\beta=[\beta]+\{\beta\}$.

We will also consider the class of all continuous functions $u(t,\Phi)$ $((t,\Phi)\in\overline{C_n(\Gamma)})$ harmonic in $C_n(\Gamma)$ with $u^+(t,\Phi)=\max(u(t,\Phi),0)\in\mathscr{A}_{\Gamma,\beta}$ $((t,\Phi)\in C_n(\Gamma))$ and $u^+(t,\Phi)\in\mathscr{B}_{\Gamma,\beta}$ $((t,\Phi)\in S_n(\Gamma))$ is denoted by $\mathscr{C}_{\Gamma,\beta}$.

Remark 1.2. If we denote $\Gamma=S_1^+$ and $\alpha=\beta-1$ in (1.7)–(1.8), by Remark 1.1 we have

$$\int_{T_n}\frac{y_n|f(Y,y_n)|}{1+t^{n+\alpha+2}}dQ<\infty \quad\text{and}\quad \int_{\partial T_n}\frac{|g(Y,0)|}{1+t^{n+\alpha}}dY<\infty,$$

which yield that $\mathscr{C}_{S_1^+,\alpha+1}$ is equivalent to $(CH)_\alpha$ in the notation of [3].

Recently, Siegel-Talvila (cf. [8, Corollary 2.1]) proved the following result.

Theorem 1.1. *If u is a continuous function on ∂T_n satisfying*

$$\int_{\partial T_n}\frac{|u(t,\Phi)|}{1+t^{n+m}}dQ<\infty,$$

then the function $U_{T_n,m}(P)$ satisfies

$$U_{T_n,m}\in C^2(T_n)\cap C^0(\overline{T_n}),$$
$$\Delta U_{T_n,m}=0 \ \text{ in } \ T_n, \quad U_{T_n,m}=u \ \text{ on } \ \partial T_n,$$
$$\lim_{r\to\infty,P=(r,\Theta)\in T_n} r^{-m-1}\cos^{n-1}\theta_1 U_{T_n,m}(P)=0.$$

Our first aim is to be concerned with the growth property of $U_{C_n(\Gamma),m}$.

Theorem 1.2. *If $\gamma+\aleph^+-1>0$, $\gamma-n+1\le\aleph^+_{k_{m+1}}<\gamma-n+2$ and u is a continuous function on $\partial C_n(\Gamma)$ satisfying*

$$\int_{S_n(\Gamma)}\frac{|u(t,\Phi)|}{1+t^{\gamma}}d\sigma_Q<\infty, \tag{1.9}$$

then the function $U_{C_n(\Gamma),m}(P)$ satisfies

$$U_{C_n(\Gamma),m}\in C^2(C_n(\Gamma))\cap C^0(\overline{C_n(\Gamma)}),$$
$$\Delta U_{C_n(\Gamma),m}=0 \ \text{ in } \ C_n(\Gamma), \quad U_{C_n(\Gamma),m}=u \ \text{ on } \ \partial C_n(\Gamma),$$
$$\lim_{r\to\infty,P=(r,\Theta)\in C_n(\Gamma)} r^{n-\gamma-1}\varphi^{n-1}(\Theta)U_{C_n(\Gamma),m}(P)=0.$$

The following Corollary 1.1 generalizes the growth property of $U_{T_n,m}$ to the conical case.

Corollary 1.1. *If u is a continuous function on $\partial C_n(\Gamma)$ satisfying*

$$\int_{S_n(\Gamma)}\frac{|u(t,\Phi)|}{1+t^{n+\aleph^+_{k_{m+1}}-1}}d\sigma_Q<\infty, \tag{1.10}$$

then

$$\lim_{r\to\infty,P=(r,\Theta)\in C_n(\Gamma)} r^{-\aleph^+_{k_{m+1}}}\varphi^{n-1}(\Theta)U_{C_n(\Gamma),m}(P)=0.$$

By the boundedness of $\varphi(\Theta)$, we immediately obtain

Corollary 1.2. *If u is a continuous function on $\partial C_n(\Gamma)$ satisfying (1.10), then*

$$\lim_{r\to\infty,P=(r,\Theta)\in C_n(\Gamma)} r^{-\aleph^+_{k_{m+1}}}\int_{\Gamma}|U_{C_n(\Gamma),m}(P)|\varphi(\Theta)dS_1=0. \tag{1.11}$$

An integral representation of harmonic functions in a half space, due to Deng (see [3]) is the following

Theorem 1.3. *If $u \in \mathscr{C}_{S_1^+,\alpha+1}$ $(\alpha \geq 0)$, m is an integer such that $m < \alpha \leq m+1$ and $P_{T_n,m}$ is defined by (1.6), then the following properties hold:*

(I) *If $\alpha = 0$, then the integral*

$$\int_{\partial T_n} P_{T_n}(P,Q)u(Q)d\sigma_Q$$

is absolutely convergent, it represents a harmonic function $U_{T_n}(P)$ in T_n and can be continuously extended to $\overline{T_n}$ such that $u(P) = U_{T_n}(P)$ for $P = (r,\Theta) = (X,0) \in \partial T_n$ and there exists a constant b such that $u(P) = bx_n + U_{T_n}(P)$ for $P = (r,\Theta) = (X,x_n) \in T_n$.

(II) *If $\alpha > 0$, then the integral*

$$\int_{\partial T_n} P_{T_n,m}(P,Q)u(Q)d\sigma_Q$$

is absolutely convergent, it represents a harmonic function $U_{T_n,m}(P)$ in T_n and can be continuously extended to $\overline{T_n}$ such that $u(P) = U_{T_n,m}(P)$ for $P = (r,\Theta) = (X,0) \in \partial T_n$,

$$\lim_{R\to\infty} R^{-\alpha-1} \sup\{|x_n^{n-1} U_{T_n,m}(RP)| : P = (1,\Theta) = (X,x_n) \in T_n\} = 0$$

and there exists a harmonic polynomial $Q_{T_n,m}(P)$ of degree not greater than m which vanishes on the boundary ∂T_n such that $u(P) = U_{T_n,m}(P) + Q_{T_n,m}(P)$ for $P = (r,\Theta) = (X,x_n) \in T_n$.

As an application of Theorem 1.2, we give the following result, which is a generalization of Theorem 1.3.

Theorem 1.4. *If $u \in \mathscr{C}_{\Gamma,\beta}$, m is an integer such that $\aleph^+_{k_m} < \aleph^+_{k_{[\beta]}} + \{\beta\} \leq \aleph^+_{k_{m+1}}$ and $P_{C_n(\Gamma),m}$ is defined by (1.5), then the following properties hold:*

(I) *If $\beta = 1$, then the integral*

$$\int_{S_n(\Gamma)} P_{C_n(\Gamma),1}(P,Q)u(Q)d\sigma_Q,$$

is absolutely convergent, it represents a harmonic function $U_{C_n(\Gamma),0}(P)$ in $C_n(\Gamma)$ and can be continuously extended to $\overline{C_n(\Gamma)}$ such that $U_{C_n(\Gamma),0}(P) = u(P)$ for $P = (r,\Theta) \in S_n(\Gamma)$ and there exists a constant c such that $u(P) = cr\varphi(\Theta) + U_{C_n(\Gamma),0}(P)$ for $P = (r,\Theta) \in C_n(\Gamma)$.

(II) *If $\beta > 1$, then*

(i) *$u(t,\Phi) \in \mathscr{B}_{\Gamma,\beta}$ $((t,\Phi) \in S_n(\Gamma))$, i.e.*

$$\int_{S_n(\Gamma)} \frac{|u(t,\Phi)|}{1+t^{n+\aleph^+_{k_{[\beta]}}+\{\beta\}-2}} \frac{\partial\varphi}{\partial n} d\sigma_Q < \infty. \tag{1.12}$$

(ii) *The integral*

$$\int_{S_n(\Gamma)} P_{C_n(\Gamma),m}(P,Q)u(Q)d\sigma_Q,$$

is absolutely convergent, it represents a harmonic function $U_{C_n(\Gamma),m}(P)$ *in* $C_n(\Gamma)$ *and can be continuously extended to* $\overline{C_n(\Gamma)}$ *such that* $U_{C_n(\Gamma),m}(P) = u(P)$ *for* $P = (r,\Theta) \in S_n(\Gamma)$.

(iii) *There exists a harmonic polynomial* $h(P) = \sum_{i=1}^{k_{m+1}-1} A_i r^{\aleph_i^+} \varphi_i(\Theta)$ *vanishing continuously on* $\partial C_n(\Gamma)$ *such that* $u(P) = U_{C_n(\Gamma),m}(P) + h(P)$ *for* $P = (r,\Theta) \in C_n(\Gamma)$*, where* A_i $(i = 1,2,3,\dots,k_{m+1}-1)$ *is a constant.*

2. Lemmas

Throughout this paper, Let M denote various constants independent of the variables in questions, which may be different from line to line.

Lemma 2.1.

(i)

$$P_{C_n(\Gamma)}(P,Q) \le M r^{\aleph^-} t^{\aleph^+ - 1} \varphi(\Theta)$$

(ii)

$$\left(\text{resp. } P_{C_n(\Gamma)}(P,Q) \le M r^{\aleph^+} t^{\aleph^- - 1} \varphi(\Theta)\right)$$

for any $P = (r,\Theta) \in C_n(\Gamma)$ *and any* $Q = (t,\Phi) \in S_n(\Gamma)$ *satisfying* $0 < t/r \le 4/5$ (*resp.* $0 < r/t \le 4/5$);

(iii)

$$P_{C_n(\Gamma)}(P,Q) \le M \frac{\varphi(\Theta)}{t^{n-1}} + M \frac{r\varphi(\Theta)}{|P-Q|^n}$$

for any $P = (r,\Theta) \in C_n(\Gamma)$ *and any* $Q = (t,\Phi) \in S_n(\Gamma;(4/5r,5/4r))$.

Proof. These immediately follow from [1, Lemma 4 and Remark] and (1.3). ∎

Lemma 2.2. [9, Lemma 3]. *For a non-negative integer m, we have*

$$|P_{C_n(\Gamma)}(P,Q) - K_{C_n(\Gamma),m}(P,Q)| \le M(2r)^{\aleph^+_{k_{m+1}}} t^{-\aleph^+_{k_{m+1}} - n + 1}$$

for any $P = (r,\Theta) \in C_n(\Gamma)$ *and any* $Q = (t,\Phi) \in S_n(\Gamma)$ *satisfying* $0 < r/t < 1/2$*, where* M *is a constant independent of* P*,* Q *and* m*.*

Lemma 2.3. [9, Lemma 5]. *If u is a locally integrable and upper semi-continuous function on* $\partial C_n(\Gamma)$*. For any fixed* $P \in C_n(\Gamma)$*,* $V(P,Q)$ $(Q \in \partial C_n(\Gamma))$ *is a locally integrable function on* $\partial C_n(\Gamma)$*. Put*

$$W(P,Q) = P_{C_n(\Gamma)}(P,Q) - V(P,Q) \quad (P \in C_n(\Gamma), Q \in \partial C_n(\Gamma)).$$

Suppose that the following conditions (I) and (II) are satisfied:

(I) *For any* $Q' \in \partial C_n(\Gamma)$ *and any* $\varepsilon > 0$*, there exist a neighborhood* $B(Q')$ *of* Q' *in* $\mathbf{R}^n$ *and a number* R $(0 < R < \infty)$ *such that*

$$\int_{S_n(\Gamma;[R,\infty))} |W(P,Q)||u(Q)| d\sigma_Q < \varepsilon \tag{2.1}$$

for any $P = (r,\Theta) \in C_n(\Gamma) \cap B(Q')$.

(II) *For any* $Q' \in \partial C_n(\Gamma)$ *and any number* R $(0 < R < \infty)$,

$$\limsup_{P \to Q', P \in C_n(\Gamma)} \int_{S_n(\Gamma;(0,R))} |V(P,Q)||u(Q)| d\sigma_Q = 0. \tag{2.2}$$

Then

$$\limsup_{P\to Q', P\in C_n(\Gamma)} \int_{S_n(\Gamma)} W(P,Q)u(Q)d\sigma_Q \le u(Q')$$

for any $Q' \in \partial C_n(\Gamma)$.

The following Lemma generalizes the Carleman's formula (referring to the holomorphic functions in the half space, see [2]) to the subharmonic functions in smooth cones $\mathbf{R}^n$ (see [6, Theorem 1]).

Lemma 2.4. *If* $R > r > 0$ *and* $u(t,\Phi)$ *is a subharmonic function on a domain containing* $C_n(\Gamma;(r,R))$, *then*

$$\begin{aligned}&\int_{C_n(\Gamma;(r,R))} \left(\frac{1}{t^{-\aleph^-}} - \frac{t^{\aleph^+}}{R^\chi}\right)\varphi\Delta u dw \\ (2.3)\quad &= \chi\int_{S_n(\Gamma;R)} \frac{u\varphi}{R^{1-\aleph^-}} dS_R + \int_{S_n(\Gamma;(r,R))} u\left(\frac{1}{t^{-\aleph^-}} - \frac{t^{\aleph^+}}{R^\chi}\right)\frac{\partial\varphi}{\partial n} d\sigma_Q + d_1(r) + \frac{d_2(r)}{R^\chi},\end{aligned}$$

where

$$d_1(r) = \int_{S_n(\Gamma;r)} \frac{\aleph^-}{r^{1-\aleph^-}} u\varphi - \frac{\varphi}{r^{-\aleph^-}}\frac{\partial u}{\partial n} dS_r$$

and

$$d_2(r) = \int_{S_n(\Gamma;r)} r^{\aleph^+}\varphi\frac{\partial u}{\partial n} - \frac{\aleph^+ u\varphi}{r^{1-\aleph^+}} dS_r.$$

Lemma 2.5. [10, Theorem 3.3]. *Let* $m\ (\ge 1)$ *be a positive integer and* $h(r,\Theta)$ *be a harmonic function in* $C_n(\Gamma)$ *vanishing continuously on* $\partial C_n(\Gamma)$. *If*

$$\liminf_{r\to\infty} r^{-\aleph^+_{k_{m+1}}} \int_\Gamma h^+(r,\Theta)\varphi(\Theta)dS_1 = 0,$$

then

$$h(r,\Theta) = \sum_{i=1}^{k_{m+1}-1} A_i r^{\aleph^+_i}\varphi_i(\Theta),$$

where $A_i\ (i = 1,2,3,\ldots,k_{m+1}-1)$ *is a constant.*

3. Proof of Theorem 1.2

For any fixed $P = (r,\Theta) \in C_n(\Gamma)$, take a number R satisfying $R > \max(1,2r)$. By $\gamma - \aleph^+_{k_{m+1}} - n + 1 \le 0$, Lemma 2.2 and (1.9) we have

$$\begin{aligned}\int_{S_n(\Gamma;(R,\infty))} |P_{C_n(\Gamma),m}(P,Q)||u(Q)|d\sigma_Q &\le M(2r)^{\aleph^+_{k_{m+1}}} \int_{S_n(\Gamma;(R,\infty))} t^{-\aleph^+_{k_{m+1}}-n+1}|u(t,\Phi)|d\sigma_Q \\ &\le Mr^{\gamma-n+1}\int_{S_n(\Gamma;(R,\infty))} |u(t,\Phi)|t^{-\gamma}d\sigma_Q \\ (3.1)\quad &\le Mr^{\gamma-n+1} < \infty.\end{aligned}$$

Hence $U_{C_n(\Gamma),m}(P)$ is absolutely convergent and finite for any $P \in C_n(\Gamma)$. Thus $U_{C_n(\Gamma),m}(P)$ is harmonic on $C_n(\Gamma)$.

Next we prove that $\lim_{P\in C_n(\Gamma),P\to Q'} U_{C_n(\Gamma),m}(P) = u(Q')$ for any $Q' = (t',\Phi') \in \partial C_n(\Gamma)$. Setting $V(P,Q) = \widetilde{K}_{C_n(\Gamma),m}(P,Q)$, which is locally integrable on $\partial C_n(\Gamma)$ for any fixed $P \in C_n(\Gamma)$. Then we apply Lemma 2.3 to $u(Q)$ and $-u(Q)$.

For any $\varepsilon > 0$ and a positive number δ, by (3.1) we can choose a number R, $R > \max\{1, 2(t'+\delta)\}$ such that (2.1) holds, where $P \in C_n(\Gamma)\cap B(Q',\delta)$. Since $\lim_{\Theta\to\Phi'} \varphi_i(\Theta) = 0$ $(i = 1,2,3\ldots)$ as $P = (r,\Theta) \to Q' = (t',\Phi') \in S_n(\Gamma)$, $\lim_{P\in C_n(\Gamma),P\to Q'} \widetilde{K}_{C_n(\Gamma),m}(P,Q) = 0$, where $Q \in S_n(\Gamma)$ and $Q' \in \partial C_n(\Gamma)$. Then (2.2) holds.

For ε mentioned above, there exists $R_\varepsilon > 1$ such that

$$\int_{S_n(\Gamma;(R_\varepsilon,\infty))} \frac{|u(t,\Phi)|}{1+t^\gamma} d\sigma_Q < \varepsilon.$$

Take any fixed point $P = (r,\Theta) \in C_n(\Gamma)$ such that $r > 5/4R_\varepsilon$, write

$$U_{C_n(\Gamma),m}(P) \le U_1(P)+U_2(P)+U_3(P)+U_4(P)+U_5(P)+U_6(P)+U_7(P),$$

where

$$U_1(P) = \int_{S_n(\Gamma;(0,1])} |P_{C_n(\Gamma)}(P,Q)||u(Q)|d\sigma_Q,$$
$$U_2(P) = \int_{S_n(\Gamma;(1,R_\varepsilon])} |P_{C_n(\Gamma)}(P,Q)||u(Q)|d\sigma_Q,$$
$$U_3(P) = \int_{S_n\left(\Gamma;\left(R_\varepsilon,\frac{4}{5}r\right]\right)} |P_{C_n(\Gamma)}(P,Q)||u(Q)|d\sigma_Q,$$
$$U_4(P) = \int_{S_n\left(\Gamma;\left(\frac{4}{5}r,\frac{5}{4}r\right)\right)} |P_{C_n(\Gamma)}(P,Q)||u(Q)|d\sigma_Q,$$
$$U_5(P) = \int_{S_n\left(\Gamma;\left[\frac{5}{4}r,2r\right]\right)} |P_{C_n(\Gamma)}(P,Q)||u(Q)|d\sigma_Q,$$
$$U_6(P) = \int_{S_n(\Gamma;[1,2r])} |\widetilde{K}_{C_n(\Gamma),m}(P,Q)||u(Q)|d\sigma_Q$$

and

$$U_7(P) = \int_{S_n(\Gamma;(2r,\infty))} |P_{C_n(\Gamma),m}(P,Q)||u(Q)|d\sigma_Q.$$

We first obtain the following growth estimates by $\gamma+\aleph^+-1>0$ and Lemma 2.1 (i)

$$U_2(P) \le Mr^{\aleph^-}\varphi(\Theta)\int_{S_n(\Gamma;(1,R_\varepsilon])} t^{\aleph^+-1}|u(t,\Phi)|d\sigma_Q$$

$$\le Mr^{\aleph^-}R_\varepsilon^{\gamma+\aleph^+-1}\varphi(\Theta)\int_{S_n(\Gamma;(1,R_\varepsilon])} |u(t,\Phi)|t^{-\gamma}d\sigma_Q \le Mr^{\aleph^-}R_\varepsilon^{\gamma+\aleph^+-1}\varphi(\Theta). \tag{3.2}$$

$$U_1(P) \le Mr^{\aleph^-}\varphi(\Theta). \tag{3.3}$$

$$U_3(P) \le M\varepsilon r^{\gamma-n+1}\varphi(\Theta). \tag{3.4}$$

If $\aleph^+_{k_{m+1}} \ge \gamma-n+1$, then $\gamma-n-\aleph^++1 \le 0$. By Lemma 2.1 (ii)

$$U_5(P) \le Mr^{\aleph^+}\varphi(\Theta)\int_{S_n\left(\Gamma;\left[\frac{5}{4}r,\infty\right)\right)} t^{\aleph^--1}|u(t,\Phi)|d\sigma_Q$$

$$\le Mr^{\gamma-n+1}\varphi(\Theta)\int_{S_n\left(\Gamma;\left[\frac{5}{4}r,\infty\right)\right)} |u(t,\Phi)|t^{-\gamma}d\sigma_Q \le M\varepsilon r^{\gamma-n+1}\varphi(\Theta). \tag{3.5}$$

By Lemma 2.1 (iii), we consider the inequality

$$U_4(P) \le U_{41}(P) + U_{42}(P),$$

where

$$U_{41}(P) = M\varphi(\Theta)\int_{S_n(\Gamma;(\frac{4}{5}r,\frac{5}{4}r))} \frac{|u(t,\Phi)|}{t^{n-1}} d\sigma_Q,$$

$$U_{42}(P) = Mr\varphi(\Theta)\int_{S_n(\Gamma;(\frac{4}{5}r,\frac{5}{4}r))} \frac{|u(t,\Phi)|}{|P-Q|^n} d\sigma_Q.$$

We first have

$$\begin{aligned} U_{41}(P) &\le M\varphi(\Theta)\int_{S_n(\Gamma;(\frac{4}{5}r,\frac{5}{4}r))} t^{\aleph^+ + \aleph^- -1}|u(t,\Phi)|d\sigma_Q \\ &\le Mr^{\aleph^+}\varphi(\Theta)\int_{S_n(\Gamma;(\frac{4}{5}r,\infty))} t^{\aleph^- -1}|u(t,\Phi)|d\sigma_Q \le M\varepsilon r^{\gamma-n+1}\varphi(\Theta), \end{aligned} \tag{3.6}$$

which is similar to the estimate of $U_5(P)$.

Next, we shall estimate $U_{42}(P)$. Take a sufficiently small positive number k such that

$$S_n\left(\Gamma;\left(\frac{4}{5}r,\frac{5}{4}r\right)\right) \subset \bigcup_{P=(r,\Theta)\in\Pi(k)} B\left(P,\frac{1}{2}r\right),$$

where

$$\Pi(k) = \left\{P=(r,\Theta)\in C_n(\Gamma);\ \inf_{(1,z)\in\partial\Gamma}|(1,\Theta)-(1,z)|<k,\ 0<r<\infty\right\},$$

and divide $C_n(\Gamma)$ into two sets $\Pi(k)$ and $C_n(\Gamma)-\Pi(k)$. If $P=(r,\Theta)\in C_n(\Gamma)-\Pi(k)$, then there exists a positive k' such that $|P-Q|\ge k'r$ for any $Q\in S_n(\Gamma)$, and hence

$$U_{42}(P) \le M\varphi(\Theta)\int_{S_n(\Gamma;(\frac{4}{5}r,\frac{5}{4}r))} \frac{|u(t,\Phi)|}{t^{n-1}} d\sigma_Q \le M\varepsilon r^{\gamma-n+1}\varphi(\Theta), \tag{3.7}$$

which is similar to the estimate of $U_{41}(P)$.

We shall consider the case $P=(r,\Theta)\in\Pi(k)$. Now put

$$H_i(P) = \left\{Q\in S_n\left(\Gamma;\left(\frac{4}{5}r,\frac{5}{4}r\right)\right);\ 2^{i-1}\delta(P)\le|P-Q|<2^i\delta(P)\right\},$$

where $\delta(P)=\inf_{Q\in\partial C_n(\Gamma)}|P-Q|$. Since $S_n(\Gamma)\cap\{Q\in\mathbf{R}^n : |P-Q|<\delta(P)\}=\varnothing$, we have

$$U_{42}(P) = M\sum_{i=1}^{i(P)}\int_{H_i(P)} r\varphi(\Theta)\frac{|u(t,\Phi)|}{|P-Q|^n}d\sigma_Q,$$

where $i(P)$ is a positive integer satisfying $2^{i(P)-1}\delta(P)\le r/2<2^{i(P)}\delta(P)$. Since $r\varphi(\Theta)\le M\delta(P)$ ($P=(r,\Theta)\in C_n(\Gamma)$), similar to the estimate of $U_{41}(P)$ we obtain

$$\begin{aligned} \int_{H_i(P)} r\varphi(\Theta)\frac{|u(t,\Phi)|}{|P-Q|^n}d\sigma_Q &\le \int_{H_i(P)} r\varphi(\Theta)\frac{|u(t,\Phi)|}{(2^{i-1}\delta(P))^n}d\sigma_Q \\ &\le M2^{(1-i)n}\varphi^{1-n}(\Theta)\int_{H_i(P)} t^{1-n}|u(t,\Phi)|d\sigma_Q \le M\varepsilon r^{\gamma-n+1}\varphi^{1-n}(\Theta), \end{aligned}$$

for $i=0,1,2,\dots,i(P)$. So

$$U_{42}(P) \le M\varepsilon r^{\gamma-n+1}\varphi^{1-n}(\Theta). \tag{3.8}$$

Since in the case $m=0$, $U_6(P)\equiv 0$, we only consider the case $U_6(P)$ for $m\geq 1$. From (1.2) and (1.4), we see

$$U_6(P)\leq ML\sum_{i\in I_{k_{m+1}}} i2^{\aleph_i^{+}+n-1}q_i(r),$$

where

$$L=\max_{\Theta\in\Gamma,\Phi\in\partial\Gamma} P_{C_n(\Gamma)}((1,\Theta),(2,\Phi)),$$

and

$$q_i(r)=r^{\aleph_i^+}\int_{S_n(\Gamma;[1,2r))} t^{-\aleph_i^+-n+1}|u(t,\Phi)|d\sigma_Q.$$

To estimate $q_i(r)$, we write

$$q_i(r)\leq q_i'(r)+q_i''(r),$$

where

$$q_i'(r)=r^{\aleph_i^+}\int_{S_n(\Gamma;[1,R_\varepsilon])} t^{-\aleph_i^+-n+1}|u(t,\Phi)|d\sigma_Q,$$
$$q_i''(r)=r^{\aleph_i^+}\int_{S_n(\Gamma;(R_\varepsilon,2r))} t^{-\aleph_i^+-n+1}|u(t,\Phi)|d\sigma_Q.$$

By $\gamma-\aleph_{k_{m+1}}^{+}-n+2>0$, we have the following estimates

$$\begin{aligned} q_i'(r)&=r^{\aleph_i^+}\int_{S_n(\Gamma;[1,R_\varepsilon])} t^{-\aleph_{k_{m+1}}^+}t^{\aleph_{k_{m+1}}^+-\aleph_i^+-n+1}|u(t,\Phi)|d\sigma_Q\\ &\leq r^{\aleph_{k_{m+1}}^+-1}\int_{S_n(\Gamma;[1,R_\varepsilon])} t^{-\aleph_{k_{m+1}}^+-n+2}|u(t,\Phi)|d\sigma_Q\leq Mr^{\aleph_{k_{m+1}}^+-1}R_\varepsilon^{\gamma-\aleph_{k_{m+1}}^+-n+2}. \end{aligned}$$

$$q_i''(r)\leq M\varepsilon r^{\gamma-n+1}.$$

Thus we can conclude that

$$q_i(r)\leq M\varepsilon r^{\gamma-n+1},$$

which yields

$$U_6(P)\leq M\varepsilon r^{\gamma-n+1}. \tag{3.9}$$

We obtain by $r-\aleph_{k_{m+1}}^{+}-n+1\leq 0$ and Lemma 2.2

$$\begin{aligned} U_7(P)&\leq M_1(2r)^{\aleph_{k_{m+1}}^+}\int_{S_n(\Gamma;(2r,\infty))} t^{-\aleph_{k_{m+1}}^+-n+1}|u(t,\Phi)|d\sigma_Q\\ &\leq Mr^{\gamma-n+1}\int_{S_n(\Gamma;(2r,\infty))}|u(t,\Phi)|t^{-\gamma}d\sigma_Q\leq M\varepsilon r^{\gamma-n+1}. \end{aligned} \tag{3.10}$$

Combining (3.2)–(3.10), we complete the proof of Theorem 1.2.

4. Proof of Theorem 1.4

To prove (II). We apply the formula (2.3) with $R>r=1$ to $u=u^+-u^-$ in $C_n(\Gamma;(1,R))$, where $u^+=\max\{u,0\}$ and $u^-=(-u)^+$.

$$\begin{aligned}&m_+(R)+\int_{S_n(\Gamma;(1,R))}u^+\left(\frac{1}{t^{-\aleph^-}}-\frac{t^{\aleph^+}}{R^\chi}\right)\frac{\partial\varphi}{\partial n}d\sigma_Q+d_1+\frac{d_2}{R^\chi}\\ &=m_-(R)+\int_{S_n(\Gamma;(1,R))}u^-\left(\frac{1}{t^{-\aleph^-}}-\frac{t^{\aleph^+}}{R^\chi}\right)\frac{\partial\varphi}{\partial n}d\sigma_Q,\end{aligned}\tag{4.1}$$

where

$$m_\pm(R)=\chi\int_{S_n(\Gamma;R)}\frac{u^\pm\varphi}{R^{1-\aleph^-}}dS_R,$$

$$d_1=\int_{S_n(\Gamma;1)}\aleph^-u\varphi-\varphi\frac{\partial u}{\partial n}dS_1,\quad d_2=\int_{S_n(\Gamma;1)}\varphi\frac{\partial u}{\partial n}-\aleph^+u\varphi dS_1.$$

Since $u\in\mathscr{C}_{\Gamma,\beta}$, we obtain by (1.7)

$$\begin{aligned}\frac{1}{\chi}\int_1^\infty\frac{m_+(R)}{R^{\aleph^+_{k[\beta]}+\{\beta\}-\aleph^++1}}dR&=\int_{C_n(\Gamma;(1,\infty))}\frac{u^+\varphi}{t^{n+\aleph^+_{k[\beta]}+\{\beta\}}}d\sigma_Q\\ &\le 2\int_{C_n(\Gamma)}\frac{u^+\varphi}{1+t^{n+\aleph^+_{k[\beta]}+\{\beta\}}}d\sigma_Q<\infty.\end{aligned}\tag{4.2}$$

From (1.8), we conclude that

$$\begin{aligned}&\int_1^\infty\frac{1}{R^{\aleph^+_{k[\beta]}+\{\beta\}-\aleph^++1}}\int_{S_n(\Gamma;(1,R))}u^+\left(\frac{1}{t^{-\aleph^-}}-\frac{t^{\aleph^+}}{R^\chi}\right)\frac{\partial\varphi}{\partial n}d\sigma_QdR\\ &=\int_{S_n(\Gamma;(1,\infty))}u^+t^{\aleph^+}\int_t^\infty\frac{1}{R^{\aleph^+_{k[\beta]}+\{\beta\}-\aleph^++1}}\left(\frac{1}{t^\chi}-\frac{1}{R^\chi}\right)dR\frac{\partial\varphi}{\partial n}d\sigma_Q\\ &\le\frac{\chi}{\chi+1}\int_{S_n(\Gamma;(1,\infty))}\frac{u^+}{t^{n+\aleph^+_{k[\beta]}+\{\beta\}-2}}\frac{\partial\varphi}{\partial n}d\sigma_Q\\ &\le 2\frac{\chi}{\chi+1}\int_{S_n(\Gamma)}\frac{u^+}{1+t^{n+\aleph^+_{k[\beta]}+\{\beta\}-2}}\frac{\partial\varphi}{\partial n}d\sigma_Q<\infty.\end{aligned}\tag{4.3}$$

Combining (4.1), (4.2) and (4.3), we obtain

$$\begin{aligned}&\int_1^\infty\frac{1}{R^{\aleph^+_{k[\beta]}+\frac{\{\beta\}}{2}-\aleph^++1}}\int_{S_n(\Gamma;(1,R))}u^-\left(\frac{1}{t^{-\aleph^-}}-\frac{t^{\aleph^+}}{R^\chi}\right)\frac{\partial\varphi}{\partial n}d\sigma_QdR\\ &\le\int_1^\infty\frac{m_+(R)}{R^{\aleph^+_{k[\beta]}+\frac{\{\beta\}}{2}-\aleph^++1}}dR+\int_1^\infty\frac{1}{R^{\aleph^+_{k[\beta]}+\frac{\{\beta\}}{2}-\aleph^++1}}\int_{S_n(\Gamma;(1,R))}u^+\left(\frac{1}{t^{-\aleph^-}}-\frac{t^{\aleph^+}}{R^\chi}\right)\frac{\partial\varphi}{\partial n}d\sigma_QdR\\ &\quad+\int_1^\infty\frac{1}{R^{\aleph^+_{k[\beta]}+\frac{\{\beta\}}{2}-\aleph^++1}}(d_1+\frac{d_2}{R^\chi})dR\\ &<\infty.\end{aligned}$$

Set

$$\mathscr{H}(\beta)=\lim_{t\to\infty}\frac{\int_t^\infty R^{-\aleph^+_{k_{[\beta]}}-\frac{\{\beta\}}{2}+\aleph^+-1}\left(\frac{1}{t\chi}-\frac{1}{R\chi}\right)dR}{t^{-n-\aleph^+_{k_{[\beta]}}-\{\beta\}-\aleph^++2}}.$$

By the L'hospital's rule, we have

$$\mathscr{H}(\beta)=\begin{cases}\frac{\chi}{(\aleph^+_{k_{[\beta]}}-\aleph^+)(n+\aleph^+_{k_{[\beta]}}+\aleph^+-2)} & \text{if } \{\beta\}=0,\\ +\infty & \text{if } \{\beta\}\neq 0,\end{cases}$$

which yields that there exists a positive constant A such that for any $t\geq 1$,

$$\int_t^\infty \frac{t^{\aleph^+}}{R^{\aleph^+_{k_{[\beta]}}+\frac{\{\beta\}}{2}-\aleph^++1}}\left(\frac{1}{t\chi}-\frac{1}{R\chi}\right)dR\geq \frac{A}{t^{n+\aleph^+_{k_{[\beta]}}+\{\beta\}-2}}.$$

Then

$$\begin{aligned}&A\int_{S_n(\Gamma;(1,\infty))}\frac{u^-}{t^{n+\aleph^+_{k_{[\beta]}}+\{\beta\}-2}}\frac{\partial\varphi}{\partial n}d\sigma_Q\\ &\leq \int_{S_n(\Gamma;(1,\infty))}u^-t^{\aleph^+}\int_t^\infty\frac{1}{R^{\aleph^+_{k_{[\beta]}}+\frac{\{\beta\}}{2}-\aleph^++1}}\left(\frac{1}{t\chi}-\frac{1}{R\chi}\right)dR\frac{\partial\varphi}{\partial n}d\sigma_Q\\ &<\infty,\end{aligned}$$

which shows that (1.12) holds. Notice that $\aleph^+_{k_m}<\aleph^+_{k_{[\beta]}}+\{\beta\}\leq\aleph^+_{k_{m+1}}$ and condition (1.12) is stronger than (1.10). So the proofs of (ii) are similar to them in Theorem 1.2. Here we omit them.

Finally we consider the function $u(P)-U_{C_n(\Gamma),m}(P)$, which is harmonic in $C_n(\Gamma)$ and vanishes continuously on $\partial C_n(\Gamma)$. Since

$$0\leq\left(u(P)-U_{C_n(\Gamma),m}(P)\right)^+\leq u^+(P)+\left(U_{C_n(\Gamma),m}\right)^-(P) \tag{4.4}$$

for any $P\in C_n(\Gamma)$. Further, (1.7) gives that

$$\liminf_{r\to\infty} r^{-\aleph^+_{k_{m+1}}}\int_\Gamma u^+(P)\varphi(\Theta)dS_1=0. \tag{4.5}$$

From Lemma 2.5, (1.11), (4.4) and (4.5), the conclusion (iii) holds. If $u\in\mathscr{C}_{\Gamma,1}$, then $u\in\mathscr{C}_{\Gamma,\beta}$ for each $\beta>1$, so there exists a constant c_1 such that

$$u(P)=c_1r\varphi(\Theta)+U_{C_n(\Gamma),1}(P)$$

for all $P\in C_n(\Gamma)$. So if we take $c=c_1-\int_{S_n(\Gamma;[1,\infty))}P_{C_n(\Gamma)}(0,Q)u(Q)d\sigma_Q$, we see that $u(P)=cr\varphi(\Theta)+U_{C_n(\Gamma),0}(P)$ holds for all $P\in C_n(\Gamma)$. Then we complete the proof of Theorem 1.4.

Acknowledgement. This work is supported by the National Natural Science Foundation of China (No. 11226093, 11271045) and Specialized Research Fund for the Doctoral Program of Higher Education (No. 20100003110004).

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