BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **36**(2) (2013), 525–540

Strong Convergence Theorems for Variational Inequality Problems and Fixed Point Problems in Banach Spaces

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Abstract. In this paper, we introduce a new iterative algorithm for finding a common element of the set of solutions of a general variational inequality problem for finite inverse-strongly accretive mappings and the set of common fixed points of a countable family of strict pseudocontractive mappings in a Banach space. We obtain a strong convergence theorem under some suitable conditions. Our results improve and extend the recent ones announced by many others in the literature.

2010 Mathematics Subject Classification: 47H09, 47H10

Keywords and phrases: Strong convergence, variational inequality, fixed point, strict pseudocontractive mapping.

1. Introduction

Throughout this paper, we denote by E and E^* a real Banach space and the dual space of E, respectively. Let C be a subset of E and T be a self-mapping of C. We use $F(T)$ to denote the fixed points of T.

The duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2,\ \|x^*\| = \|x\| \right\}, \quad \forall x \in E.$$

If E is a Hilbert space, then $J = I$, where I is the identity mapping. It is well-known that if E is smooth, then J is single-valued, which is denoted by j.

Recall that $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

$T : C \to C$ is said to be Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

$T : C \to C$ is said to be a λ-strict pseudo-contractive, if there exists a constant $\lambda > 0$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x - y\|^2 - \lambda \|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C. \tag{1.3}$$

Communicated by Tomonari Suzuki.
Received: February 23, 2011; *Revised:* August 17, 2011.

Remark 1.1. From (1.3) we can prove that if T is λ-strict pseudo-contractive, then T is Lipschitz continuous with the Lipschitz constant $L=(1+\lambda)/\lambda$.

A mapping $A:C\to E$ is said to be accretive if there exists $j(x-y)\in J(x-y)$ such that

$$\langle Ax-Ay, j(x-y)\rangle \geq 0, \quad \forall x,y\in C. \tag{1.4}$$

A mapping $A:C\to E$ is said to be α-inverse strongly accretive if there exist $j(x-y)\in J(x-y)$ and $\alpha>0$ such that

$$\langle Ax-Ay, j(x-y)\rangle \geq \alpha \left\|Ax-Ay\right\|^2, \quad \forall x,y\in C. \tag{1.5}$$

A mapping $f:C\to C$ is said to be a contraction if there exists a constant $\alpha\in(0,1)$ such that

$$\|f(x)-f(y)\| \leq \alpha\|x-y\|, \quad \forall x,y\in C. \tag{1.6}$$

We use the notation Π_C to denote the collection of all contractions on C.

Variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral, free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. Several numerical methods have been developed for solving variational inequalities and related optimization problems, see [4, 6–9, 12–17, 20, 23, 25, 26] and the references therein.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A,B:C\to H$ be two mappings. In 2008, Ceng *et al.* [4] considered the following problem of finding $(x^*,y^*)\in C\times C$ such that

$$\begin{cases}\langle \lambda Ay^*+x^*-y^*, x-x^*\rangle \geq 0, & \forall x\in C,\\ \langle \mu Bx^*+y^*-x^*, x-y^*\rangle \geq 0, & \forall x\in C,\end{cases} \tag{1.7}$$

which is called a general system of variational inequalities, where $\lambda>0$ and $\mu>0$ are two constants. In particular, if $A=B$, then problem (1.7) reduces to finding $(x^*,y^*)\in C\times C$ such that

$$\begin{cases}\langle \lambda Ay^*+x^*-y^*, x-x^*\rangle \geq 0, & \forall x\in C,\\ \langle \mu Ax^*+y^*-x^*, x-y^*\rangle \geq 0, & \forall x\in C.\end{cases} \tag{1.8}$$

Ceng *et al.* [4] introduced a relaxed extragradient method for finding a common element of the set of solutions of problem (1.7) for two inverse-strongly monotone mappings and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Let $x_1=v\in C$ and let $\{x_n\}$ and $\{y_n\}$ be given by

$$\begin{cases}y_n=P_C(x_n-\mu Bx_n),\\ x_{n+1}=a_nv+b_nx_n+(1-a_n-b_n)SP_C(y_n-\lambda Ay_n), & n\geq 1,\end{cases} \tag{1.9}$$

where $\lambda\in(0,2\alpha),\mu\in(0,2\beta)$ and $\{a_n\},\{b_n\}\subset[0,1]$. Then they proved the sequence $\{x_n\}$ converges strongly to a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of problem (1.7) under some control conditions.

Recently, Wangkeeree [20] suggested and analyzed a new iterative scheme for finding a common element of the fixed point set of common fixed points of a countable family of nonexpansive mappings and the set of solutions of the variational inequality problem for an inverse-strongly monotone mapping in a real Hilbert space. More precisely, they studied the following iterative algorithm

$$
\begin{cases} x_1 = x \in C, \\ y_n = P_C(x_n - \lambda_n B x_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A) S_n P_C(y_n - \lambda_n B y_n), \quad n \geq 1, \end{cases} \tag{1.10}
$$

and proved a strong convergence under some suitable conditions.

Very recently, Qin and Kang [14] proposed an explicit viscosity approximation method for finding a common element of the set of fixed points of strict pseudo-contractions and the set of solutions of variational inequalities with inverse-strongly monotone mappings. They introduced the following iterative algorithm

$$
\begin{cases} x_1 \in C, \\ z_n = P_C(x_n - \mu_n B x_n), \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \left[\delta_{(1,n)} S x_n + \delta_{(2,n)} y_n + \delta_{(3,n)} z_n\right], \quad n \geq 1, \end{cases} \tag{1.11}
$$

and obtained a strong convergence theorem.

On the other hand, Yao *et al.* [24] introduced the following system of general variational inequlities in Banach spaces. Let C be a nonempty closed convex subset of a real Banach space E. For given two operators $A, B : C \to E$, they considered the problem of finding $(x^*, y^*) \in C \times C$ such that

$$
\begin{cases} \langle Ay^* + x^* - y^*, j(x - x^*) \rangle \geq 0, \quad \forall x \in C, \\ \langle Bx^* + y^* - x^*, j(x - y^*) \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.12}
$$

which is called the system of general variational inequalities in a real Banach space. Under some suitable conditions they proved a strong convergence theorem by using the following iterative algorithm:

$$
\begin{cases} x_0 \in C, \\ y_n = Q_C(x_n - B x_n), \\ x_{n+1} = a_n u + b_n x_n + c_n Q_C(y_n - A y_n), \quad n \geq 0, \end{cases} \tag{1.13}
$$

where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three sequences in $(0,1)$ and $u \in C$.

In this paper, motivated and inspired by the above facts, we introduce the following system of variational inequalities in a Banach space: Let C be a nonempty closed convex subset of a real Banach space E. Let $\{A_i\}_{i=1}^M : C \to E$ be a family of mappings. First we consider the following problem of finding $(x_1^*, x_2^*, \ldots, x_M^*) \in C \times C \ldots \times C$ such that

$$
\begin{cases} \langle \mu_M A_M x_M^* + x_1^* - x_M^*, j(x - x_1^*) \rangle \geq 0, & \forall x \in C, \\ \left\langle \mu_{M-1} A_{M-1} x_{M-1}^* + x_M^* - x_{M-1}^*, j(x - x_M^*) \right\rangle \geq 0, & \forall x \in C, \\ \quad \vdots \\ \left\langle \mu_2 A_2 x_2^* + x_3^* - x_2^*, j(x - x_3^*) \right\rangle \geq 0, & \forall x \in C, \\ \langle \mu_1 A_1 x_1^* + x_2^* - x_1^*, j(x - x_2^*) \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.14}
$$

which is called a more general system of variational inequalities in Banach spaces, where $\mu_i > 0$ for all $i \in \{1,2,\ldots,M\}$. The set of solutions to (1.14) is denoted by Ω. In particular, if $M = 2, A_1 = B, A_2 = A, \mu_1 = \mu_2 = 1, x_1^* = x^*, x_2^* = y^*$, then problem (1.14) reduces to problem (1.12). Subsequently, we introduce a new iterative algorithm for finding a common element of the set of solutions of a general variational inequality problem (1.14) for finite inverse-strongly accretive mappings and the set of common fixed points of a countable family of strict pseudocontractive mappings in a Banach space. The results presented in this paper improve and extend the corresponding results announced by Qin and Kang [14], Wangkeeree [20], Yao *et al.* [24], Ceng *et al.* [4] and many others in the literature.

2. Preliminaries

A Banach space E is said to be strictly convex, if whenever x and y are not collinear, then: $\|x+y\| < \|x\| + \|y\|$. The modulus of convexity of E is defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|(x+y)\| : \|x\|, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\},$$

for all $\varepsilon \in [0,2]$. E is said to be uniformly convex if $\delta_E(0) = 0$, and $\delta_E(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$. Hilbert space H is 2-uniformly convex, while L^p is $\max\{p,2\}$-uniformly convex for every $p > 1$.

Let $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm of E is said to be Gâteaux differentiable if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S(E)$. In this case, E is said to be smooth. The norm of E is said to be uniformly Gâteaux differentiable, if for each $y \in S(E)$, the limit (2.1) is attained uniformly for $x \in S(E)$. The norm of the E is said to be Frêchet differentiable, if for each $x \in S(E)$, the limit (2.1) is attained uniformly for $y \in S(E)$. The norm of E is called uniformly Frêchet differentiable, if the limit (2.1) is attained uniformly for $x, y \in S(E)$. It is well-known that (uniform) Frêchet differentiability of the norm E implies (uniform) Gâteaux differentiability of norm E.

Let $\rho_E : [0,\infty) \to [0,\infty)$ be the modulus of smoothness of E defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x \in S(E), \|y\| \leq t\right\}.$$

A Banach space E is said to be uniformly smooth if $(\rho_E(t))/t \to 0$ as $t \to 0$. A Banach space E is said to be q-uniformly smooth, if there exists a fixed constant $c > 0$ such that $\rho_E(t) \leq ct^q$. It is well-known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable. If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth, and hence the norm of E is uniformly Fréchet differentiable, in particular, the norm of E is Fréchet differentiable. Typical example of uniformly smooth Banach spaces is L^p, where $p > 1$. More precisely, L^p is $\min\{p,2\}$-uniformly smooth for every $p > 1$.

Recall that, if C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a mapping $P : C \to D$ is sunny [18] provided

$$P(x + t(x - P(x))) = P(x) \quad \text{for all}\, x \in C \,\text{and} \;\; t \geq 0,$$

whenever $x+t(x-P(x)) \in C$. A mapping $P: C \to D$ is called a retraction if $Px = x$ for all $x \in D$. Furthermore, P is a sunny nonexpansive retraction from C onto D if P is retraction from C onto D which is also sunny and nonexpansive.

A subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D. The following propositions concern the sunny nonexpansive retraction.

Proposition 2.1. [18] *Let C be a closed convex subset of a smooth Banach space E. Let D be a nonempty subset of C. Let $P: C \to D$ be a retraction and let J be the normalized duality mapping on E. Then the following are equivalent:*

(a) *P is sunny and nonexpansive.*
(b) $\|Px-Py\|^2 \leq \langle x-y, J(Px-Py)\rangle$, $\forall x,y \in C$.
(c) $\langle x-Px, J(y-Px)\rangle \leq 0$, $\forall x \in C, y \in D$.

Proposition 2.2. [11] *If E is strictly convex and uniformly smooth and if $T: C \to C$ is a nonexpansive mapping having a nonempty fixed point set $F(T)$, then the set $F(T)$ is a sunny nonexpansive retraction of C.*

In order to prove our main results, we need the following lemmas.

Lemma 2.1. [10] *Let E be a real smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g: [0,2r] \to \mathbb{R}$ such that $g(0) = 0$ and $g(\|x-y\|) \leq \|x\|^2 - 2\langle x, jy\rangle + \|y\|^2$, for all $x,y \in B_r$.*

Lemma 2.2. [19] *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ which satisfies the following condition: $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n x_n + (1-\beta_n) z_n$, $n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.3. [22] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\alpha_n)a_n + \delta_n$, $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n/\alpha_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. [21] *Let E be a real q-uniformly smooth Banach space, then there exists a constant $C_q > 0$ such that*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q x\rangle + C_q \|y\|^q,$$

for all $x,y \in E$. In particular, if E is real 2-uniformly smooth Banach space, then there exists a best smooth constant $K > 0$ such that

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, jx\rangle + 2\|Ky\|^2,$$

for all $x,y \in E$.

Lemma 2.5. [1] *Let C be a nonempty closed convex subset of a Banach space E. Let $S_1, S_2, \cdots$ be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^{\infty} \sup \|S_{n+1}x - S_n x\| : x \in C < \infty$. Then for each $y \in C$, $\{S_n y\}$ converges strongly to some point of C. Moreover, let S be a mapping of C into itself defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in C$. Then* $\lim_{n\to\infty} \sup\{\|Sx - S_n x\| : x \in C\} = 0$.

Lemma 2.6. [3] *Let C be a closed convex subset of a strictly convex Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C. Suppose $\cap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by $Sx = \sum_{n=1}^{\infty} \lambda_n T_n x$ for $x \in C$ is well defined, non-expansive and $F(S) = \cap_{n=1}^{\infty} F(T_n)$ holds.*

Lemma 2.7. [2] *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let T be nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence of C such that $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then x is a fixed point of T.*

Lemma 2.8. [22] *Let E be a uniformly smooth Banach space, C be a closed convex subset of E, $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let $f \in \Pi_C$. Then the sequence $\{x_t\}$ define by*

$$x_t = tf(x_t) + (1-t)Tx_t$$

converges strongly to a point in $F(T)$. If we define a mapping $Q : \Pi_C \to F(T)$ by

$$Q(f) := \lim_{t \to 0} x_t, \quad \forall f \in \Pi_C.$$

Then $Q(f)$ solves the following variational inequality:

$$\langle (I-f)Q(f), j(Q(f)-p) \rangle \leq 0, \quad \forall f \in \Pi_C, p \in F(T).$$

Lemma 2.9. [5] *In a Banach space E, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y) \rangle, \quad \forall x, y \in E,$$

where $j(x+y) \in J(x+y)$.

Lemma 2.10. [27] *Let C be a nonempty convex subset of a real 2-uniformly smooth Banach space E and $T : C \to C$ be a λ-strict pseudo-contraction. For $\alpha \in (0,1)$, we define $T_\alpha x = (1-\alpha)x + \alpha Tx$. Then, as $\alpha \in (0, \lambda/K^2]$, $T_\alpha : C \to C$ is nonexpansive such that $F(T_\alpha) = F(T)$.*

Lemma 2.11. *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Let the mapping $A : C \to E$ be α-inverse-strongly accretive. Then, we have*

$$\|(I-\mu A)x - (I-\mu A)y\|^2 \leq \|x-y\|^2 + 2\mu(\mu K^2 - \alpha)\|Ax - Ay\|^2,$$

where $\mu > 0$. In particular, if $\mu \leq \alpha/K^2$, then $I - \mu A$ is nonexpansive.

Proof. Indeed, for all $x, y \in C$, it follows from Lemma 2.4 that

$$\begin{aligned}
\|(I-\mu A)x - (I-\mu A)y\|^2 &= \|x-y-\mu(Ax-Ay)\|^2 \\
&\leq \|x-y\|^2 - 2\mu\langle Ax-Ay, j(x-y)\rangle + 2\mu^2K^2\|Ax-Ay\|^2 \\
&\leq \|x-y\|^2 - 2\mu\alpha\|Ax-Ay\|^2 + 2\mu^2K^2\|Ax-Ay\|^2 \\
&= \|x-y\|^2 + 2\mu(\mu K^2 - \alpha)\|Ax-Ay\|^2.
\end{aligned}$$

It is clear that if $0 < \mu \leq \alpha/K^2$, then $I - \mu A$ is nonexpansive. This completes the proof. ∎

Lemma 2.12. *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. Let $A_i : C \to E$*

be an α_i-inverse-strongly accretive mapping, where $i \in \{1,2,\ldots,M\}$. Let $G : C \to C$ be a mapping defined by

$$G(x) = Q_C(I-\mu_M A_M)Q_C(I-\mu_{M-1}A_{M-1})\ldots Q_C(I-\mu_2 A_2)Q_C(I-\mu_1 A_1)x, \forall x \in C.$$

If $0 < \mu_i \le \alpha_i/K^2$, $i = 1,2,\ldots,M$, then $G : C \to C$ is nonexpansive.

Proof. Put $\Theta^i = Q_C(I-\mu_i A_i)Q_C(I-\mu_{i-1}A_{i-1})\ldots Q_C(I-\mu_2 A_2)Q_C(I-\mu_1 A_1)$, $i = 1,2,\ldots,M$ and $\Theta^0 = I$, where I is identity mapping. Then $G = \Theta^M$. For all $x,y \in C$, it follows from Lemma 2.11 that

$$\begin{aligned}\|Gx-Gy\| &= \left\|\Theta^M x - \Theta^M y\right\| \\ &= \left\|Q_C(I-\mu_M A_M)\Theta^{M-1}x - Q_C(I-\mu_M A_M)\Theta^{M-1}y\right\| \\ &\le \left\|(I-\mu_M A_M)\Theta^{M-1}x - (I-\mu_M A_M)\Theta^{M-1}y\right\| \\ &\le \left\|\Theta^{M-1}x - \Theta^{M-1}y\right\| \\ &\ \ \vdots \\ &\le \left\|\Theta^0 x - \Theta^0 y\right\|, \\ &= \|x-y\|\end{aligned}$$

which implies G is nonexpansive. This completes the proof. ∎

Lemma 2.13. *Let C be a nonempty closed convex subset of a real smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. Let $A_i : C \to E$ be nonlinear mapping, where $i = 1,2,\ldots,M$. For given $x_i^* \in C$, $i = 1,2,\ldots,M$, $(x_1^*, x_2^*, \ldots, x_M^*)$ is a solution of problem* (1.14) *if and only if*

$$x_i^* = Q_C(I-\mu_{i-1}A_{i-1})x_{i-1}^*, x_1^* = Q_C(I-\mu_M A_M)x_M^*, \quad i = 2,\ldots,M.$$

That is

$$x_1^* = Q_C(I-\mu_M A_M)Q_C(I-\mu_{M-1}A_{M-1})\ldots Q_C(I-\mu_2 A_2)Q_C(I-\mu_1 A_1)x_1^*.$$

Proof. We can rewrite (1.14) as

$$(2.2)\qquad \begin{cases} \langle x_1^* - (x_M^* - \mu_M A_M x_M^*), j(x-x_1^*)\rangle \ge 0, & \forall x \in C, \\ \left\langle x_M^* - (x_{M-1}^* - \mu_{M-1}A_{M-1}x_{M-1}^*), j(x-x_M^*)\right\rangle \ge 0, & \forall x \in C, \\ \quad\vdots \\ \left\langle x_3^* - (x_2^* - \mu_2 A_2 x_2^*), j(x-x_3^*)\right\rangle \ge 0, & \forall x \in C. \\ \langle x_2^* - (x_1^* - \mu_1 A_1 x_1^*), j(x-x_2^*)\rangle \ge 0, & \forall x \in C. \end{cases}$$

From Proposition 2.1, we deduce that (2.2) is equivalent to

$$x_i^* = Q_C(I-\mu_{i-1}A_{i-1})x_{i-1}^*, x_1^* = Q_C(I-\mu_M A_M)x_M^*, \quad i = 2,\ldots,M.$$

Therefore we have

$$x_1^* = Q_C(I-\mu_M A_M)Q_C(I-\mu_{M-1}A_{M-1})\ldots Q_C(I-\mu_2 A_2)Q_C(I-\mu_1 A_1)x_1^*.$$

∎

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E to C. Let the mapping $A_i : C \to E$ be η_i-inverse-strongly accretive, where $i \in \{1,2,\dots,M\}$. Let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of λ_i-strict pseudocontractive mappings of C into itself such that $F = \cap_{i=1}^{\infty} F(T_i) \cap \Omega \neq \emptyset$. Let $\lambda = \inf\{\lambda_i : i \in \mathbb{N}\} > 0$, $L = \sup_{i\geq 1}(1+\lambda_i)/\lambda_i$. Let $\{x_n\}$ be a sequence generated by the following manner: $x_1 \in C$,*

$$\begin{cases} z_n = Q_C(I-\mu_M A_M)\dots Q_C(I-\mu_2 A_2)Q_C(I-\mu_1 A_1)x_n, \\ y_n = (1-\delta_n)z_n + \delta_n T_n z_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\beta_n-\alpha_n)y_n, \end{cases} \tag{3.1}$$

where $0 < \mu_i < \eta_i/K^2, i \in \{1,2,\dots,M\}$. Suppose that $\{\alpha_n\},\{\beta_n\}$ and $\{\delta_n\}$ be sequences in $[0,1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $0 < a \leq \delta_n \leq \lambda/K^2$, $\lim_{n\to\infty} |\delta_{n+1} - \delta_n| = 0$.

Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \|T_{n+1}x - T_n x\| < \infty$ for any bounded subset D of C and let T be a mapping of C into itself defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$ and suppose that $F(T) = \cap_{i=1}^{\infty} F(T_i)$. Then $\{x_n\}$ converges strongly to $q \in F$, which solves the following variational inequality:

$$\langle q - f(q), j(q-p)\rangle \leq 0 \quad \forall\, p \in F.$$

Proof. We divide the proof into five steps.

Step 1: We show that $\{x_n\}$ is bounded. Put $\Theta^i = Q_C(I-\mu_i A_i)\dots Q_C(I-\mu_2 A_2)Q_C(I-\mu_1 A_1)$ and $\Theta^0 = I$, where I is identity mapping and $i \in \{1,2,\dots,M\}$. Then $z_n = \Theta^M x_n$. Take $x^* \in F$, by Lemma 2.13, we have $x^* = \Theta^M x^*$, it follows from Lemma 2.12 that

$$\|z_n - x^*\| = \left\|\Theta^M x_n - \Theta^M x^*\right\| \leq \|x_n - x^*\|. \tag{3.2}$$

Put $S_n = (1-\delta_n)I + \delta_n T_n$, it follows from Lemma 2.10 and (3.2) that

$$\|y_n - x^*\| = \|S_n z_n - S_n x^*\| \leq \|z_n - x^*\| \leq \|x_n - x^*\|. \tag{3.3}$$

By (3.3), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\alpha_n f(x_n) + \beta_n x_n + (1-\beta_n-\alpha_n)y_n - x^*\| \\ &= \|\alpha_n(f(x_n)-x^*) + \beta_n(x_n-x^*) + (1-\beta_n-\alpha_n)(y_n-x^*)\| \\ &\leq (1-\beta_n-\alpha_n)\|x_n-x^*\| + \beta_n\|x_n-x^*\| + \alpha_n\|f(x_n)-x^*\| \\ &\leq (1-\alpha_n)\|x_n-x^*\| + \alpha_n\alpha\|x_n-x^*\| + \alpha_n\|f(x^*)-x^*\| \\ &= (1-\alpha_n(1-\alpha))\|x_n-x^*\| + \alpha_n(1-\alpha)\frac{\|f(x^*)-x^*\|}{1-\alpha}. \end{aligned}$$

By induction, we have

$$\|x_n - x^*\| \leq \max\left\{\|x_1-x^*\|, \frac{\|f(x^*)-x^*\|}{1-\alpha}\right\}, \quad \forall n \geq 2,$$

which implies that the sequence$\{x_n\}$ is bounded. By (3.2) and (3.3), we have that $\{y_n\}$ and $\{z_n\}$ are also bounded.

Step 2: We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. By Lemma 2.12, we have

$$\|z_{n+1}-z_n\| = \left\|\Theta^M x_{n+1} - \Theta^M x_n\right\| \le \|x_{n+1}-x_n\|, \tag{3.4}$$

and

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \|S_{n+1}z_{n+1}-S_nz_n\| \\
&\le \|S_{n+1}z_{n+1}-S_{n+1}z_n\| + \|S_{n+1}z_n - S_nz_n\| \\
&\le \|z_{n+1}-z_n\| + \|[(1-\delta_{n+1})z_n+\delta_{n+1}T_{n+1}z_n]-[(1-\delta_n)z_n+\delta_nT_nz_n]\| \\
&= \|x_{n+1}-x_n\| + \|(\delta_{n+1}-\delta_n)(T_{n+1}z_n-z_n)+\delta_n(T_{n+1}z_n-T_nz_n)\| \\
&\le \|x_{n+1}-x_n\| + |\delta_{n+1}-\delta_n|\,\|T_{n+1}z_n-z_n\| + \delta_n\,\|T_{n+1}z_n-T_nz_n\| \\
&\le \|x_{n+1}-x_n\| + |\delta_{n+1}-\delta_n|M_1 + \|T_{n+1}z_n-T_nz_n\|, \qquad (3.5)
\end{aligned}$$

where $M_1=\sup_{n\ge1}\|T_{n+1}z_n-z_n\|$.

Put $l_n=(x_{n+1}-\beta_nx_n)/(1-\beta_n)$, for all $n\ge1$, that is,

$$x_{n+1}=(1-\beta_n)l_n+\beta_nx_n, \quad \forall n\ge1. \tag{3.6}$$

We observe that

$$\begin{aligned}
l_{n+1}-l_n &= \frac{\alpha_{n+1}(f(x_{n+1})-y_{n+1})+(1-\beta_{n+1})y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n(f(x_n)-y_n)+(1-\beta_n)y_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}(f(x_{n+1})-y_{n+1})}{1-\beta_{n+1}} - \frac{\alpha_n(f(x_n)-y_n)}{1-\beta_n} + y_{n+1}-y_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|l_{n+1}-l_n\| \le{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|f(x_{n+1})\|+\|y_{n+1}\|) \\
&+\frac{\alpha_n}{1-\beta_n}(\|f(x_n)\|+\|y_n\|)+\|y_{n+1}-y_n\|.
\end{aligned} \tag{3.7}$$

Substituting (3.5) into (3.7), we have

$$\begin{aligned}
\|l_{n+1}-l_n\|-\|x_{n+1}-x_n\| \le{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|f(x_{n+1})\|+\|y_{n+1}\|)+\frac{\alpha_n}{1-\beta_n}(\|f(x_n)\|+\|y_n\|) \\
&+|\delta_{n+1}-\delta_n|M_1+\|T_{n+1}z_n-T_nz_n\|.
\end{aligned}$$

By conditions (i)–(iii) and the assumption on $\{T_n\}$, we obtain

$$\limsup_{n\to\infty}(\|l_{n+1}-l_n\|-\|x_{n+1}-x_n\|)\le 0.$$

From Lemma 2.2, we have $\lim_{n\to\infty}\|l_n-x_n\|=0$. It follows from (3.6) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\,\|l_n-x_n\|=0. \tag{3.8}$$

We note that

$$\begin{aligned}
\|x_n-y_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-y_n\| \\
&= \|x_n-x_{n+1}\|+\|\alpha_n(f(x_n)-y_n)+\beta_n(x_n-y_n)\| \\
&\le \|x_n-x_{n+1}\|+\alpha_n\,\|f(x_n)-y_n\|+\beta_n\,\|x_n-y_n\|,
\end{aligned}$$

which implies

$$\|x_n - y_n\| \le \frac{1}{1-\beta_n}(\|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - y_n\|).$$

By (3.8) and conditions (i), (ii), we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.9}$$

Step 3: We show that $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

From Lemma 2.9 and (3.3), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n(f(x_n) - y_n) + \beta_n(x_n - x^*) + (1-\beta_n)(y_n - x^*)\|^2 \\
&\le \|\beta_n(x_n - x^*) + (1-\beta_n)(y_n - x^*)\|^2 + 2\alpha_n \langle f(x_n) - y_n, j(x_{n+1} - x^*)\rangle \\
&\le \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 + 2\alpha_n \|f(x_n) - y_n\| \|x_{n+1} - x^*\| \\
&\le \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\|z_n - x^*\|^2 + \alpha_n M_2 \\
&= \alpha_n M_2 + \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\left\|\Theta^M x_n - x^*\right\|^2, \qquad (3.10)
\end{aligned}$$

where $M_2 = \sup_{n\ge 1}\{2\|f(x_n) - y_n\| \|x_{n+1} - x^*\|\}$.

On the other hand, it follows from Lemma 2.11 that

$$\begin{aligned}
&\left\|\Theta^M x_n - x^*\right\|^2 \\
&= \left\|\Theta^M x_n - \Theta^M x^*\right\|^2 \\
&= \left\|Q_C(I - \mu_M A_M)\Theta^{M-1}x_n - Q_C(I - \mu_M A_M)\Theta^{M-1}x^*\right\|^2 \\
&\le \left\|(I - \mu_M A_M)\Theta^{M-1}x_n - (I - \mu_M A_M)\Theta^{M-1}x^*\right\|^2 \\
&\le \left\|\Theta^{M-1}x_n - \Theta^{M-1}x^*\right\|^2 - 2\mu_M(\eta_M - K^2\mu_M)\left\|A_M\Theta^{M-1}x_n - A_M\Theta^{M-1}x^*\right\|^2.
\end{aligned}$$

By induction, we have

$$\begin{aligned}
\left\|\Theta^M x_n - x^*\right\|^2 &\le \left\|\Theta^0 x_n - \Theta^0 x^*\right\|^2 - 2\mu_1(\eta_1 - K^2\mu_1)\left\|A_1\Theta^0 x_n - A_1\Theta^0 x^*\right\|^2 \\
&\quad - \ldots - 2\mu_M(\eta_M - K^2\mu_M)\left\|A_M\Theta^{M-1}x_n - A_M\Theta^{M-1}x^*\right\|^2. \\
&= \|x_n - x^*\|^2 - \sum_{i=1}^{M} 2\mu_i(\eta_i - K^2\mu_i)\left\|A_i\Theta^{i-1}x_n - A_i\Theta^{i-1}x^*\right\|^2. \qquad (3.11)
\end{aligned}$$

Substituting (3.11) into (3.10), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \alpha_n M_2 + \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\|x_n - x^*\|^2 \\
&\quad - \sum_{i=1}^{M} 2\mu_i(1-\beta_n)(\eta_i - K^2\mu_i)\left\|A_i\Theta^{i-1}x_n - A_i\Theta^{i-1}x^*\right\|^2,
\end{aligned}$$

which implies

$$\begin{aligned}
&\sum_{i=1}^{M} 2\mu_i(1-\beta_n)(\eta_i - K^2\mu_i)\left\|A_i\Theta^{i-1}x_n - A_i\Theta^{i-1}x^*\right\|^2 \\
&\le \alpha_n M_2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\le \alpha_n M_2 + \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|).
\end{aligned}$$

Since $0 < \mu_i < \eta_i/K^2$, $\liminf_{n\to\infty}(1-\beta_n) > 0$, $\lim_{n\to\infty}\alpha_n = 0$ and (3.8), we have

$$\lim_{n\to\infty}\left\|A_i\Theta^{i-1}x_n - A_i\Theta^{i-1}x^*\right\| = 0, i \in \{1,2,\ldots,M\}. \tag{3.12}$$

From Proposition 2.1 and Lemma 2.1, we have

$$\begin{aligned}
&\left\|\Theta^M x_n - x^*\right\|^2 \\
&= \left\|\Theta^M x_n - \Theta^M x^*\right\|^2 \\
&= \left\|Q_C(I-\mu_M A_M)\Theta^{M-1}x_n - Q_C(I-\mu_M A_M)\Theta^{M-1}x^*\right\|^2 \\
&\le \left\langle \Theta^{M-1}x_n - \Theta^{M-1}x^* - \mu_M(A_M\Theta^{M-1}x_n - A_M\Theta^{M-1}x^*), j(\Theta^M x_n - \Theta^M x^*)\right\rangle \\
&= \left\langle \Theta^{M-1}x_n - \Theta^{M-1}x^*, j(\Theta^M x_n - \Theta^M x^*)\right\rangle - \mu_M\left\langle A_M\Theta^{M-1}x_n - A_M\Theta^{M-1}x^*, j(\Theta^M x_n - \Theta^M x^*)\right\rangle \\
&\le \frac{1}{2}(\left\|\Theta^{M-1}x_n - \Theta^{M-1}x^*\right\|^2 + \left\|\Theta^M x_n - \Theta^M x^*\right\|^2 - g_M(\left\|\Theta^{M-1}x_n - \Theta^M x_n + \Theta^M x^* - \Theta^{M-1}x^*\right\|)) \\
&\quad + \mu_M\left\|A_M\Theta^{M-1}x_n - A_M\Theta^{M-1}x^*\right\|\left\|\Theta^M x_n - \Theta^M x^*\right\|,
\end{aligned}$$

which implies

$$\begin{aligned}
\left\|\Theta^M x_n - x^*\right\|^2 \le{}& \left\|\Theta^{M-1}x_n - \Theta^{M-1}x^*\right\|^2 - g_M(\left\|\Theta^{M-1}x_n - \Theta^M x_n + \Theta^M x^* - \Theta^{M-1}x^*\right\|)) \\
&+ 2\mu_M\left\|A_M\Theta^{M-1}x_n - A_M\Theta^{M-1}x^*\right\|\left\|\Theta^M x_n - \Theta^M x^*\right\|.
\end{aligned}$$

By induction, we have

$$\begin{aligned}
\left\|\Theta^M x_n - x^*\right\|^2 &\le \left\|\Theta^0 x_n - \Theta^0 x^*\right\|^2 - \sum_{i=1}^M g_i(\left\|\Theta^{i-1}x_n - \Theta^i x_n + \Theta^i x^* - \Theta^{i-1}x^*\right\|) \\
&\quad + \sum_{i=1}^M 2\mu_i\left\|A_i\Theta^{i-1}x_n - A_i\Theta^{i-1}x^*\right\|\left\|\Theta^i x_n - \Theta^i x^*\right\| \\
&= \|x_n - x^*\|^2 - \sum_{i=1}^M g_i(\left\|\Theta^{i-1}x_n - \Theta^i x_n + \Theta^i x^* - \Theta^{i-1}x^*\right\|) \\
&\quad + \sum_{i=1}^M 2\mu_i\left\|A_i\Theta^{i-1}x_n - A_i\Theta^{i-1}x^*\right\|\left\|\Theta^i x_n - \Theta^i x^*\right\|.
\end{aligned} \tag{3.13}$$

Substituting (3.13) into (3.10), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \le{}& \alpha_n M_2 + \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|x_n - x^*\|^2 \\
&- \sum_{i=1}^M (1-\beta_n)g_i(\left\|\Theta^{i-1}x_n - \Theta^i x_n + \Theta^i x^* - \Theta^{i-1}x^*\right\|) \\
&+ \sum_{i=1}^M 2(1-\beta_n)\mu_i\left\|A_i\Theta^{i-1}x_n - A_i\Theta^{i-1}x^*\right\|\left\|\Theta^i x_n - \Theta^i x^*\right\|,
\end{aligned}$$

which implies

$$\begin{aligned}
&\sum_{i=1}^M (1-\beta_n)g_i(\left\|\Theta^{i-1}x_n - \Theta^i x_n + \Theta^i x^* - \Theta^{i-1}x^*\right\|) \\
&\le \alpha_n M_2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2
\end{aligned}$$

$$+\sum_{i=1}^{M}2(1-\beta_n)\mu_i\left\|A_i\Theta^{i-1}x_n-A_i\Theta^{i-1}x^*\right\|\left\|\Theta^i x_n-\Theta^i x^*\right\|$$
$$\le \alpha_n M_2+\|x_n-x_{n+1}\|\left(\|x_n-x^*\|+\|x_{n+1}-x^*\|\right)$$
$$+\sum_{i=1}^{M}2(1-\beta_n)\mu_i\left\|A_i\Theta^{i-1}x_n-A_i\Theta^{i-1}x^*\right\|\left\|\Theta^i x_n-\Theta^i x^*\right\|.$$

Since $\liminf_{n\to\infty}(1-\beta_n)>0$, $\lim_{n\to\infty}\alpha_n=0$, (3.8) and (3.12), we have

$$\lim_{n\to\infty} g_i\left(\left\|\Theta^{i-1}x_n-\Theta^i x_n+\Theta^i x^*-\Theta^{i-1}x^*\right\|\right)=0,\quad \forall i\in\{1,2,\dots,M\}.$$

It follows from the properties of g_i that

$$\lim_{n\to\infty}\left\|\Theta^{i-1}x_n-\Theta^i x_n+\Theta^i x^*-\Theta^{i-1}x^*\right\|=0,\quad \forall i\in\{1,2,\dots,M\}. \tag{3.14}$$

From (3.14), we have

$$\|x_n-z_n\|=\left\|\Theta^0 x_n-\Theta^M x_n\right\|\le\sum_{i=1}^{M}\left\|\Theta^{i-1}x_n-\Theta^i x_n+\Theta^i x^*-\Theta^{i-1}x^*\right\|\to 0\quad\text{as}\quad n\to\infty. \tag{3.15}$$

From (3.9) and (3.15), we obtain

$$\|y_n-z_n\|\le\|y_n-x_n\|+\|x_n-z_n\|\to 0\quad\text{as}\quad n\to\infty. \tag{3.16}$$

We note $y_n-z_n=\delta_n(T_nz_n-z_n)$. It follows from (3.16) and $\delta_n>a>0$, we have

$$\lim_{n\to\infty}\|T_nz_n-z_n\|=0. \tag{3.17}$$

By (3.15) and (3.17), we get

$$\begin{aligned}\|x_n-T_nx_n\|&\le\|x_n-z_n\|+\|z_n-T_nz_n\|+\|T_nz_n-T_nx_n\|\\&\le(1+L)\|x_n-z_n\|+\|z_n-T_nz_n\|\\&\to 0 \text{ as } n\to\infty.\end{aligned} \tag{3.18}$$

From (3.9) and (3.15), we have

$$\begin{aligned}\|x_n-S_nx_n\|&\le\|x_n-S_nz_n\|+\|S_nz_n-S_nx_n\|\\&\le\|x_n-S_nz_n\|+\|z_n-x_n\|\\&=\|x_n-y_n\|+\|z_n-x_n\|\\&\to 0 \text{ as } n\to\infty.\end{aligned} \tag{3.19}$$

Define a mapping $Sx=(1-\delta)x+\delta Tx$, where $\delta\in(0,\lambda/K^2)$ is a constant. Then by Lemma 2.10, we have $F(S)=F(T)=\cap_{i=1}^{\infty}F(T_i)$. We note that

$$\begin{aligned}\|x_n-Sx_n\|&\le\|x_n-S_nx_n\|+\|S_nx_n-Sx_n\|\\&=\|x_n-S_nx_n\|+\|(1-\delta_n)x_n+\delta_nT_nx_n-(1-\delta)x_n-\delta Tx_n\|\\&=\|x_n-S_nx_n\|+\|(\delta_n-\delta)(T_nx_n-x_n)+\delta(T_nx_n-Tx_n)\|\\&\le\|x_n-S_nx_n\|+|\delta_n-\delta|\,\|T_nx_n-x_n\|+\delta\,\|T_nx_n-Tx_n\|.\end{aligned}$$

By (3.18), (3.19) and Lemma 2.5, we have

$$\lim_{n\to\infty}\|x_n-Sx_n\|=0. \tag{3.20}$$

Define a mapping $Wx = (1-\theta)Sx + \theta Gx$, where G is defined by Lemma 2.12, $\theta \in (0,1)$ is a constant. Then by Lemma 2.6, we have that $F(W) = F(S) \cap F(G) = F(S) \cap \Omega = F$. We observe that

$$\begin{aligned}\|x_n - Wx_n\| &= \|(1-\theta)(x_n - Sx_n) + \theta(x_n - Gx_n)\| \\ &\le (1-\theta)\|x_n - Sx_n\| + \theta\|x_n - Gx_n\| \\ &= (1-\theta)\|x_n - Sx_n\| + \theta\|x_n - z_n\|.\end{aligned}$$

By (3.15) and (3.20), we obtain

$$\lim_{n\to\infty}\|x_n - Wx_n\| = 0. \tag{3.21}$$

Step 4: We claim that

$$\limsup_{n\to\infty}\langle f(q) - q, j(x_n - q)\rangle \le 0, \tag{3.22}$$

where $q = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction

$$x \mapsto tf(x) + (1-t)Wx.$$

Then x_t solves the fixed point equation $x_t = tf(x_t) + (1-t)Wx_t$. Thus we have

$$\|x_t - x_n\| = \|(1-t)(Wx_t - x_n) + t(f(x_t) - x_n)\|.$$

It follows from Lemma 2.9 that

$$\begin{aligned}\|x_t - x_n\|^2 &= \|(1-t)(Wx_t - x_n) + t(f(x_t) - x_n)\|^2 \\ &\le (1-t)^2\|Wx_t - x_n\|^2 + 2t\langle f(x_t) - x_n, j(x_t - x_n)\rangle \\ &\le (1-t)^2(\|Wx_t - Wx_n\| + \|Wx_n - x_n\|)^2 + 2t\langle f(x_t) - x_n, j(x_t - x_n)\rangle \\ &\le (1-t)^2(\|x_t - x_n\| + \|Wx_n - x_n\|)^2 + 2t\langle f(x_t) - x_n, j(x_t - x_n)\rangle \\ &= (1-t)^2\left[\|x_t - x_n\|^2 + 2\|x_t - x_n\|\|Wx_n - x_n\| + \|Wx_n - x_n\|^2\right] \\ &\quad + 2t\langle f(x_t) - x_t, j(x_t - x_n)\rangle + 2t\langle x_t - x_n, j(x_t - x_n)\rangle \\ &= (1-2t+t^2)\|x_t - x_n\|^2 + f_n(t) + 2t\langle f(x_t) - x_t, j(x_t - x_n)\rangle + 2t\|x_t - x_n\|^2,\end{aligned} \tag{3.23}$$

where

$$f_n(t) = (1-t)^2(2\|x_t - x_n\| + \|x_n - Wx_n\|)\|x_n - Wx_n\| \to 0, \quad \text{as} \quad n \to \infty. \tag{3.24}$$

It follows from (3.23) that

$$\langle x_t - f(x_t), j(x_t - x_n)\rangle \le \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}f_n(t). \tag{3.25}$$

Let $n \to \infty$ in (3.25) and note that (3.24) yields

$$\limsup_{n\to\infty}\langle x_t - f(x_t), j(x_t - x_n)\rangle \le \frac{t}{2}M_3, \tag{3.26}$$

where $M_3 > 0$ is a constant such that $M_3 \ge \|x_t - x_n\|^2$ for all $t \in (0,1)$ and $n \ge 1$. Taking $t \to 0$ from (3.26), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t - f(x_t), j(x_t - x_n)\rangle \le 0. \tag{3.27}$$

On the other hand, we have

$$\begin{aligned}\langle f(q)-q, j(x_n-q)\rangle &= \langle f(q)-q, j(x_n-q)\rangle - \langle f(q)-q, j(x_n-x_t)\rangle \\ &\quad + \langle f(q)-q, j(x_n-x_t)\rangle - \langle f(q)-x_t, j(x_n-x_t)\rangle \\ &\quad + \langle f(q)-x_t, j(x_n-x_t)\rangle - \langle f(x_t)-x_t, j(x_n-x_t)\rangle \\ &\quad + \langle f(x_t)-x_t, j(x_n-x_t)\rangle \\ &= \langle f(q)-q, j(x_n-q)-j(x_n-x_t)\rangle + \langle x_t-q, j(x_n-x_t)\rangle \\ &\quad + \langle f(q)-f(x_t), j(x_n-x_t)\rangle + \langle f(x_t)-x_t, j(x_n-x_t)\rangle .\end{aligned}$$

It follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(q)-q, j(x_n-q)\rangle &\le \limsup_{n\to\infty}\langle f(q)-q, j(x_n-q)-j(x_n-x_t)\rangle \\ &\quad + \|x_t-q\|\limsup_{n\to\infty}\|x_n-x_t\| + \alpha\|q-x_t\|\limsup_{n\to\infty}\|x_n-x_t\| \\ &\quad + \limsup_{n\to\infty}\langle f(x_t)-x_t, j(x_n-x_t)\rangle .\end{aligned}$$

Noticing that j is norm-to-norm uniformly continuous on bounded subsets of C, it follows from (3.27), we have

$$\limsup_{n\to\infty}\langle f(q)-q, j(x_n-q)\rangle = \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(q)-q, j(x_n-q)\rangle \le 0.$$

Hence, (3.22) holds.

Step 5: Finally we prove that $x_n \to q$ as $n \to \infty$.

From (3.3), we have

$$\begin{aligned}&\|x_{n+1}-q\|^2 \\ &= \alpha_n\langle f(x_n)-q, j(x_{n+1}-q)\rangle + \beta_n\langle x_n-q, j(x_{n+1}-q)\rangle \\ &\quad + \langle (1-\beta_n-\alpha_n)(y_n-q), j(x_{n+1}-q)\rangle \\ &\le (1-\beta_n-\alpha_n)\|x_n-q\|\,\|x_{n+1}-q\| + \beta_n\|x_n-q\|\,\|x_{n+1}-q\| \\ &\quad + \alpha_n\langle f(x_n)-f(q), j(x_{n+1}-q)\rangle + \alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle \\ &\le (1-\alpha_n)\|x_n-q\|\,\|x_{n+1}-q\| + \alpha_n\alpha\|x_n-q\|\,\|x_{n+1}-q\| + \alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle \\ &= [1-\alpha_n(1-\alpha)]\|x_n-q\|\,\|x_{n+1}-q\| + \alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle \\ &\le \frac{1-\alpha_n(1-\alpha)}{2}(\|x_n-q\|^2 + \|x_{n+1}-q\|^2) + \alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle \\ &\le \frac{1-\alpha_n(1-\alpha)}{2}\|x_n-q\|^2 + \frac{1}{2}\|x_{n+1}-q\|^2 + \alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle ,\end{aligned}$$

which implies

$$\|x_{n+1}-q\|^2 \le [1-\alpha_n(1-\alpha)]\|x_n-q\|^2 + \alpha_n(1-\alpha)\frac{2\langle f(q)-q, j(x_{n+1}-q)\rangle}{1-\alpha}. \tag{3.28}$$

Apply Lemma 2.3 to (3.28), we obtain that $x_n \to q$ as $n \to \infty$. This completes the proof. ∎

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