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The Linear Arboricity of Planar Graphs without 5-Cycles with Chords

[1]HONG-YU CHEN, [2]XIANG TAN AND [3]JIAN-LIANG WU
[1]School of Sciences, Shanghai Institute of Technology, Shanghai, 201418, China
[2]School of Statistics and Mathematics, Shandong University of Finance, Jinan, Shandong, 250014, China
[3]School of Mathematics, Shandong University, Jinan, Shandong, 250100, China
[1]hongyuchen86@126.com, [2]xtandw@126.com, [3]jlwu@sdu.edu.cn

Abstract. The linear arboricity $la(G)$ of a graph G is the minimum number of linear forests which partition the edges of G. In this paper, it is proved that for a planar graph G with maximum degree $\Delta(G) \geq 7$, $la(G) = \lceil(\Delta(G))/2\rceil$ if G has no 5-cycles with chords.

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1. Introduction

In this paper, all graphs are finite, simple and undirected. For a real number x, $\lceil x \rceil$ is the least integer not less than x and $\lfloor x \rfloor$ is the largest integer not larger than x. Let G be a graph. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set, respectively. If $uv \in E(G)$, then u is said to be a *neighbor* of v, and $N_G(v)$ is the set of neighbors of v. The *degree* $d(v)$ of a vertex v is $|N_G(v)|$, $\delta(G)$ is the minimum degree of G and $\Delta(G)$ is the maximum degree of G. A k-, k^+- or k^-- vertex is a vertex of degree k, at least k, or at most k, respectively. A k- cycle is a cycle of length k. Two cycles are said to be *adjacent* if they are incident with a common edge. All undefined notations and definitions follow that of Bondy and Murty [3].

A *linear forest* is a graph in which each component is a path. A map φ from $E(G)$ to $\{1,2,\dots,t\}$ is called a *t-linear coloring* if the induced subgraph of edges having the same color α is a linear forest for $1 \leq \alpha \leq t$. The *linear arboricity* $la(G)$ of a graph G defined by Harary [7] is the minimum number t for which G has a t-linear coloring.

Akiyama, Exoo and Harary [1] conjectured that $la(G) = \lceil(\Delta(G)+1)/2\rceil$ for any regular graph G. The conjecture is equivalent to the following conjecture.

Conjecture 1.1. *For any graph G,* $\lceil(\Delta(G))/2\rceil \leq la(G) \leq \lceil(\Delta(G)+1)/2\rceil$.

The linear arboricity has been determined for complete bipartite graphs [1], complete regular multipartite graphs [13], Halin graphs [9], series-parallel graphs [11] and regular graphs with $\Delta = 3,4$ [1, 2], 5,6,8 [5], and 10 [6]. For planar graphs, more results are obtained. Conjecture 1.1 has already been proved to be true for all planar graphs, see [10]

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and [14]. Wu [10] proved that for a planar graph G with girth g and maximum degree Δ, $la(G) = \lceil(\Delta(G))/2\rceil$ if $\Delta(G) \geq 13$, or $\Delta(G) \geq 7$ and $g \geq 4$, or $\Delta(G) \geq 5$ and $g \geq 5$, or $\Delta(G) \geq 3$ and $g \geq 6$. In [12], it is proved that if G is a planar graph with $\Delta(G) \geq 7$ and without i-cycles for some $i \in \{4,5\}$, then $la(G) = \lceil(\Delta(G))/2\rceil$. Tan [8] proved that for a planar graph G with $\Delta(G) \geq 5$ and any 4-cycle is not adjacent to an i-cycle for any $i \in \{3,4,5\}$ or G has no intersecting 4-cycles and intersecting i-cycles for some $i \in \{3,6\}$, $la(G) = \lceil(\Delta(G))/2\rceil$. In this paper, we'll prove that for a planar graph G with maximum degree $\Delta(G) \geq 7$, $la(G) = \lceil(\Delta(G))/2\rceil$ if G has no 5-cycles with chords.

2. Main results and their proofs

In the section, we always assume that a planar graph G has been embedded in the plane. Let G be a planar graph and $F(G)$ be the face set of G. For $f \in F(G)$, the *degree* of f, denoted by $d(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A k-, k^+- or k^-- face is a face of degree k, at least k, or at most k, respectively.

Given a t-linear coloring φ and a vertex v of G, we denote by $C^i_\varphi(v)$ the set of colors appears i times at v, where $i = 0,1,2$. Then $|C^0_\varphi(v)| + |C^1_\varphi(v)| + |C^2_\varphi(v)| = t$ and $|C^1_\varphi(v)| + 2|C^2_\varphi(v)| = d(v)$. Let x be a vertex of G, denote $\varphi(x) = (\varphi(xy_1),\ldots,\varphi(xy_k))$, where vertices $y_1,\ldots,y_k$ are distinct neighbors of x. For any two vertices u and v, let $C_\varphi(u,v) = C^2_\varphi(u) \cup C^2_\varphi(v) \cup (C^1_\varphi(u) \cap C^1_\varphi(v))$, that is , $C_\varphi(u,v)$ is the set of colors that appear at least two times at u and v. A *monochromatic path* is a path of whose edges receive the same color. For two different edges e_1 and e_2 of G, they are said to be in the *same color component*, denoted by $e_1 \leftrightarrow e_2$ if there is a monochromatic path of G connecting them. Furthermore, if two ends of e_i are known, that is, $e_i = x_iy_i$ $(i = 1,2)$, then $x_1y_1 \leftrightarrow x_2y_2$ denotes more accurately that there is a monochromatic path from x_1 to y_2 passing through the edges x_1y_1 and x_2y_2 in G (that is, y_1 and x_2 are internal vertices in the path). Otherwise, we use $x_1y_1 \not\leftrightarrow x_2y_2$ (or $e_1 \not\leftrightarrow e_2$) to denote that such monochromatic path passing through them does not exist. Note that $x_1y_1 \leftrightarrow x_2y_2$ and $x_1y_1 \leftrightarrow y_2x_2$ are different. Let xy be an edge of G. Denote by $xy \leftrightarrow (v,i)$ that x and v have a monochromatic path of color i between them through y. Denote by $(u,i) \leftrightarrow (v,i)$ that u and v have a monochromatic path of color i between them.

Theorem 2.1. *Let G be a planar graph with $\Delta(G) \geq 7$ and without 5-cycles with chords. Then $la(G) = \lceil(\Delta(G))/2\rceil$.*

Proof. It suffices to prove that for an integer $m \geq 4$, a planar graph G without 5-cycles with chords has an m-linear coloring if $\Delta(G) \leq 2m$. Let $G = (V,E)$ be a minimal counterexample to the theorem and uv be an edge of G. Then $G - uv$ has an m-linear coloring φ. We have the two fundamental properties.

Property 2.1. $|C_\varphi(u,v)| = m$.

Proof. Suppose that $|C_\varphi(u,v)| < m$. We may modify φ to an m-linear coloring of G by setting $\varphi(uv) \in \{1,2,\ldots,m\} \backslash C_\varphi(u,v)$, a contradiction. ∎

Property 2.2. If there is a color i such that $i \in C^1_\varphi(u) \cap C^1_\varphi(v)$, then $(u,i) \leftrightarrow (v,i)$.

Proof. Suppose $(u,i) \not\leftrightarrow (v,i)$. We may modify φ to an m-linear coloring of G by setting $\varphi(uv) = i$, a contradiction. ∎

First, we prove some Lemmas for G.

Lemma 2.1. *For any $uv \in E(G)$, $d_G(u) + d_G(v) \geq 2m+2$.*

Proof. Suppose that G has an edge uv with $d_G(u)+d_G(v) \leq 2m+1$. Then $G' = G - uv$ has an m-linear coloring φ by the minimality of G. Since $d_{G'}(u)+d_{G'}(v) = d_G(u)+d_G(v)-2 \leq 2m-1$, $|C_\varphi(u,v)| = |C^2_\varphi(u) \cup C^2_\varphi(v) \cup (C^1_\varphi(u) \cap C^1_\varphi(v))| < m$. Let $\varphi(u,v) \in \{1,2,\ldots,m\} \setminus C_\varphi(u,v)$. Thus φ is extendable to an m-linear coloring of G, a contradiction. Hence the Lemma holds. ∎

By Lemma 2.1, we have

(a) $\delta(G) \geq 2$, and
(b) any two 4^--vertices are not adjacent, and
(c) any 3-face is incident with three 5^+-vertices, or at least two 6^+-vertices, and
(d) any 7-vertex has no neighbors of degree 2.

Lemma 2.2. *Every vertex is adjacent to at most two 2-vertices. Moreover, suppose that a vertex v is adjacent to two 2-vertices x, y, let x' and y' be the other neighbors of x, y, respectively. Then $x'v, y'v \notin E(G)$.*

The proof of Lemma 2.2 can be found in [4].

A k-face with consecutive vertices $v_1, v_2, \ldots, v_k$ along its boundary in some direction will be said to be a $(d(v_1), d(v_2), \ldots, d(v_k))$-face.

Lemma 2.3. *G has no $(4,6,6)$-face.*

Proof. Suppose that G has a $(4,6,6)$-face $f = u_1u_2u_3$ with $d(u_1) = 4$, $d(u_2) = d(u_3) = 6$. By the minimality of G, $G' = G - u_1u_2$ has an m-linear coloring φ.

Suppose $C^2_\varphi(u_1) = \emptyset$. Without loss of generality, let $\varphi(u_1) = (1,2,3)$ and $\varphi(u_1u_3) = 3$. Then, by properties 1 and 2, $\varphi(u_2) = (1,2,3,4,4)$ and $(u_1,i) \leftrightarrow (u_2,i)$ for $i = 1,2,3$. If $1 \notin C^2_\varphi(u_3)$, then recolor u_1u_3 with 1, and color u_1u_2 with 3. Thus G is m-linear colorable, a contradiction. So we can assume $1 \in C^2_\varphi(u_3)$. Similarly, $2 \in C^2_\varphi(u_3)$. Since $\varphi(u_1u_3) = 3$, $(u_1,3) \leftrightarrow (u_2,3)$, we have $3 \in C^2_\varphi(u_3)$. That is, $\varphi(u_3) = (1,1,2,2,3,3)$. Then we can recolor u_1u_3 with 4, and color u_1u_2 with 3. So φ is extendable to an m-linear coloring of G, a contradiction.

Suppose $C^2_\varphi(u_1) \neq \emptyset$. Without loss of generality, let $\varphi(u_1) = (1,1,2)$. Then $\varphi(u_2) = (2,3,3,4,4)$ and $(u_1,2) \leftrightarrow (u_2,2)$ by properties 1 and 2. We consider two cases.

Case 1: $\varphi(u_1u_3) = 2$. Then $2 \in C^2_\varphi(u_3)$. If $3 \notin C^2_\varphi(u_3)$, then recolor u_1u_3 with 3, and color u_1u_2 with 2. So $3 \in C^2_\varphi(u_3)$. Similarly, $4 \in C^2_\varphi(u_3)$. Thus $\varphi(u_3) = (2,2,3,3,4,4)$. Then we can recolor u_2u_3 with 1, and color u_1u_2 with $\varphi(u_2u_3)$. So φ is extendable to an m-linear coloring of G, a contradiction.

Case 2: $\varphi(u_1u_3) = 1$. If $2 \notin C^2_\varphi(u_3)$, then recolor u_1u_3 with 2, and color u_1u_2 with 1. So $2 \in C^2_\varphi(u_3)$. Similarly, $3 \in C^2_\varphi(u_3)$, $4 \in C^2_\varphi(u_3)$. But it is impossible since $d(u_3) = 6$.

Hence G has no $(4,6,6)$-face. ∎

Lemma 2.4. *Suppose that a planar graph G contains no 5-cycles with chords and $\delta(G) \geq 2$. Then all of the following results hold.*

(a) *Any vertex v is incident with at most $\lfloor (2d(v))/3 \rfloor$ 3-faces.*
(b) *A 3-face is adjacent to a 4-face if and only if the two faces are incident with a common 2-vertex.*

(c) *If a face is adjacent to two nonadjacent 3-faces, then the face must be* 5^+*-face.*
(d) *If a* $d(\geq 7)$*-vertex* v *is incident with a 3-face, then* v *is incident with at least two* 5^+*-faces.*

Proof. Since if there are three 3-faces f_1, f_2, f_3 such that they are incident with a common vertex and f_2 is adjacent to f_1 and f_3, then vertices incident with them form a 5-cycle with chords, so (a) holds. If a 3-face f is adjacent to a 4-face, then all three vertices incident with the 3-face f must be incident with the 4-face, too. So there is a vertex just incident with these two faces and it follows that the vertex is a 2-vertex. Hence (b) holds. For (c), suppose that a face f is adjacent to two nonadjacent 3-faces. As in part (a), f is not a 3-face, for otherwise a 5-cycle with chords exists. By (b), f is not a 4-face. So f must be a 5^+-face and (c) holds.

For (d), suppose that a $d(\geq 7)$-vertex v is incident with a 3-face. If v is a cut vertex, then (d) is obvious. So assume that v is not a cut vertex. Then all the faces incident with v are distinct. Let $f_1, f_2, \ldots, f_d$ be faces incident with v clockwise, and let $v_1, v_2, \ldots, v_d$ be vertices adjacent to v clockwise, let v_i be incident with $f_i, i = 1, 2, \ldots, d$ and let v_d be incident with f_d and f_1. Assume that f_1 is the 3-face. Then by (a), f_2 or f_d is not a 3-face. Without loss of generality, assume that f_d is not a 3-face.

If f_d is a 4-face, then $d(v_d) = 2$ by (b). Thus f_2 must be a 3-face or a 5^+-face. If f_2 is a 3-face, then f_3 or f_4 must be a 5^+-face. So one of f_2, f_3, f_4 is a 5^+-face. Similarly, if f_{d-1} is a 4-face, then $d(v_{d-1}) = 2$. Thus one of f_{d-1} or f_{d-2}, giving two 5^+-faces since $d \geq 7$.

Suppose that f_d is a 5^+-face. If f_2 is a 3-face, then f_3 must be a 4-face or 5^+-face. If f_3 is a 4-face, then f_4 or f_5 must be a 5^+-face. So one of the faces in $\{f_3, f_4, f_5\}$ is a 5^+-face. If f_2 is a 4-face, then f_3 or f_4 must be a 5^+-face. Thus we prove (d). ∎

By Euler's formula $|V| - |E| + |F| = 2$, we have

$$\sum_{v\in V}(2d(v) - 6) + \sum_{f\in F}(d(f) - 6) = -6(|V| - |E| + |F|) = -12 < 0.$$

We define h to be the initial charge. Let $h(v) = 2d(v) - 6$ for each $v \in V(G)$ and $h(f) = d(f) - 6$ for each $f \in F(G)$. In the following, we will reassign a new charge denoted by $h'(x)$ to each $x \in V(G) \cup F(G)$ according to the discharging rules. Since our rules only move charges around, and do not affect the sum, we have

$$\sum_{x\in V(G)\cup F(G)} h'(x) = \sum_{x\in V(G)\cup F(G)} h(x) = -12 \tag{2.1}$$

In the following, we will show that $h'(x) \geq 0$ for each $x \in V(G) \cup F(G)$, a contradiction to (2.1), completing the proof.

Now, let us introduce the needed discharging rules as follows:

R1. Each 8^+-vertex sends 1 to each of its adjacent 2-vertices.
R2. Each 7^+-vertex sends $3/2$ to each of its incident 3-faces;
1 to each of its incident 4-faces;
$1/3$ to each of its incident 5-faces.
R3. Each 6-vertex sends 1 to each of its incident faces.
R4. Each 5-vertex sends 1 to each of its incident 3-faces;
$1/2$ to each of its incident 4-faces;
$1/4$ to each of its incident 5-faces.
R5. Each 4-vertex sends $1/2$ to each of its incident faces.

Note that our discharging rules are just designed such that charge only flows from vertices to faces apart from the 8^+-vertices giving charge to 2-vertices.

Let f be a face of G. Throughout we will use Lemma 2.1 given the vertices around f are necessarily adjacent to their neighbors around f. If $d(f) \geq 6$, then $h'(f) \geq h(f) = d(f) - 6 \geq 0$.

Suppose $d(f) = 5$. Then f is incident with at most two 3^--vertices by Lemma 2.1. If f is incident with two 3^--vertices, then the other three vertices must be 7^+-vertices. It follows that $h'(f) \geq h(f) + 1/3 \times 3 = 0$ by R2. If f is incident with one 3^--vertex, then f is incident with at least two 7^+-vertices. It follows that $h'(f) \geq h(f) + 1/3 \times 2 + 1/4 \times 2 = 1/6 > 0$ by R2, R3, R4 and R5. If f is not incident with any 3^--vertices, then f receives at least $1/4$ from each of its incident vertices by R2, R3, R4 and R5. Hence $h'(f) \geq h(f) + 1/4 \times 5 = 1/4 > 0$.

Suppose $d(f) = 4$. Then f is incident with at most two 3^--vertices by Lemma 2.1. If f is incident with at least one 3^--vertex, then f is incident with at least two 7^+-vertices, and it follows that $h'(f) \geq h(f) + 1 \times 2 = 0$ by R2. If f is not incident with any 3^--vertices, then f receives at least $1/2$ from each of its incident vertices by R2, R3, R4 and R5. Hence $h'(f) \geq h(f) + 1/2 \times 4 = 0$.

Suppose $d(f) = 3$. Then f is incident with at most one 3^--vertex by Lemma 2.1. If f is incident with one 3^--vertex, then the other two vertices must be 7^+-vertices. It follows that $h'(f) \geq h(f) + 3/2 \times 2 = 0$ by R2. Otherwise, f is incident with at most one 4-vertex. If f is incident with one 4-vertex, then f is incident with at least one 7^+-vertex by Lemma 2.3 and the other vertex must be a 6^+-vertex. Hence $h'(f) \geq h(f) + 1/2 + 3/2 + 1 = 0$ by R5, R2 and R3. If f is not incident with any 4-vertices, then f receives at least 1 from each of its incident vertices by R2, R3 and R4. Hence $h'(f) \geq h(f) + 1 \times 3 = 0$.

Let v be a vertex of G. If $d(v) = 2$, then $h'(v) = h(v) + 2 = 0$ by R1. If $d(v) = 3$, then $h'(v) = h(v) = 0$. If $d(v) = 4$, then $h'(v) \geq h(v) - 1/2 \times 4 = 0$ by R5. If $d(v) = 5$, then by Lemma 2.4(a), v is incident with at most three 3-faces, so $h'(v) \geq h(v) - 1 \times 3 - 1/2 \times 2 = 0$ by R4. If $d(v) = 6$, then $h'(v) \geq h(v) - 1 \times 6 = 0$ by R3.

Suppose $d(v) = 7$. By Lemma 2.4(a), v is incident with at most four 3-faces. If v is incident with four 3-faces, then the other three faces incident with v must be 5^+-faces by Lemma 2.1(d) and Lemma 2.4(b). So $h'(v) \geq h(v) - 3/2 \times 4 - 1/3 \times 3 = 1$ by R2. If v is incident with three 3-faces, then v is incident with at most one 4-face by Lemma 2.1(d) and Lemma 2.4(b). It follows that $h'(v) \geq h(v) - 3/2 \times 3 - 1 - 1/3 \times 3 = 3/2$. If v is incident with at most two 3-faces, then $h'(v) \geq h(v) - 3/2 \times 2 - 1 \times 5 = 0$.

Suppose $d(v) = 8$. By Lemma 2.2, v is adjacent to at most two 2-vertices. Let t be the number of 3-faces incident with v.

If v is adjacent to two 2-vertices, then $t \leq 4$ by Lemma 2.2 and Lemma 2.4(a). If $t = 0$, then $h'(v) \geq h(v) - 2 - 8 \times 1 = 0$. So we can assume $1 \leq t \leq 4$. If $1 \leq t \leq 2$, then v is incident with at least two 5^+-faces by Lemma 2.4(d). So we have $h'(v) \geq h(v) - 2 - 2 \times 1/3 - (6-t) \times 1 - 3/2 \times t = 4/3 - t/2 > 0$. If $3 \leq t \leq 4$, then v is incident with at least three 5^+-faces by Lemma 2.2 and Lemma 2.4(b). It follows that $h'(v) \geq h(v) - 2 - 3 \times 1/3 - (5-t) \times 1 - 3/2 \times t = 2 - t/2 \geq 0$.

Suppose v is adjacent to at most one 2-vertex.

Clearly, $t \leq 5$ by Lemma 2.4(a). If $t \leq 2$, then $h'(v) \geq h(v) - 1 - (8-t) \times 1 - 3/2 \times t = 1 - t/2 \geq 0$. If $t \geq 3$, then v is incident with at least three 5^+-faces by Lemma 2.4(b)(c). Therefore, $h'(v) \geq h(v) - 1 - 3 \times 1/3 - (5-t) \times 1 - 3/2 \times t = 3 - t/2 > 0$.

Suppose $d(v) \geq 9$. By Lemma 2.2, v is adjacent to at most two 2-vertices. Similarly, let t be the number of 3-faces incident with v.

If v is adjacent to two 2-vertices, then $t \leq \lfloor (2(d(v)-2))/3 \rfloor$ by Lemma 2.2 and the proof of Lemma 2.4(a). If $t \geq 1$, then v is incident with at least two 5^+-faces by Lemma 2.4(d). Hence, $h'(v) \geq h(v) - 2 - 2 \times 1/3 - 3/2 \times t - (d(v) - 2 - t) \times 1 = d(v) - t/2 - 20/3 \geq 0$ by R1 and R2. If $t = 0$, then $h'(v) \geq h(v) - 2 - d(v) \times 1 = d(v) - 8 > 0$ by R1 and R2.

If v is adjacent to at most one 2-vertex, then $t \leq \lfloor (2d(v))/3 \rfloor$ by Lemma 2.4(a). If $t \geq 1$, then v is incident with at least two 5^+-faces by Lemma 2.4(d). Hence, $h'(v) \geq h(v) - 1 - 2 \times 1/3 - 3/2 \times t - (d(v) - 2 - t) \times 1 = d(v) - t/2 - 17/3 \geq 0$ by R1 and R2. If $t = 0$, then $h'(v) \geq h(v) - 1 - d(v) \times 1 = d(v) - 7 > 0$ by R1 and R2.

Hence we complete the proof of the theorem. ∎

3. Conclusions

For any planar graph G with maximum degree Δ, it is known that $la(G) = \lceil \Delta/2 \rceil$, if one of the following conditions holds.

(a) $\Delta \geq 13$;
(b) $\Delta \geq 7$ and $g \geq 4$ or $\Delta \geq 5$ and $g \geq 5$ or $\Delta \geq 3$ and $g \geq 6$, where g is the girth of G;
(c) $\Delta \geq 7$ and without 4- or 5-cycles;
(d) $\Delta \geq 7$ and without 5-cycles with chords

The case when $\Delta \geq 7$ and any two cycles of length i and j $(3 \leq i \leq j \leq 5)$, respectively, are not adjacent is proved to be true in the author's another paper [4]. Combining all the above results, we conjecture that for any planar graph G with $\Delta \geq 5$, $la(G) = \lceil \Delta/2 \rceil$.

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