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Modular and Distributive Congruence Lattices of Monounary Algebras

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Abstract. In this paper, we describe all modular and distributive lattices which are isomorphic to the congruence lattices of monounary algebras.

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1. Introduction

Monounary algebras are algebras with one unary operation. Monounary and partial monounary algebras play a significant role in the study of algebraic structures. Moreover, there exists a close connection between monounary algebras and some types of automata. The advantage of monounary algebras is their relatively simple visualization. They can be represented by a graph, which is always planar, hence easy to draw. There are several monographs in which unary and monounary algebras are intensively studied, cf., e.g., [6, 8] or an expository paper [7].

In this paper, we deal with congruence lattices of monounary algebras. Congruences of monounary algebras were investigated by Berman [1], Egorova and Skornyakov [2, 3], Jakubíková-Studenovská [4, 5]. Egorova [2] has characterized all monounary algebras which are congruence modular and congruence distributive, in term of their graphs.

Let A be a finite set with the cardinality more than one and let $\operatorname{Im} f = \{f(a) \mid a \in A\}$ be the image of a unary operation f on A. The least non-negative integer m such that $\operatorname{Im} f^m = \operatorname{Im} f^{m+1}$ is called the *stabilizer* of f. Ratanaprasert and Denecke [9] described all congruence relations on monounary algebra $(A; f)$ in the cases that stabilizers of f are $|A|-1$ and $|A|-2$, thus we can describe their congruence lattices.

In this paper, we are interested in describing all modular and distributive lattices which are isomorphic to the congruence lattice of monounary algebras.

2. Preliminaries

In this section we introduce several notions (cf. also [6]) due to present basic results from [2], which will be applied below.

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Let $(A;f)$ be a monounary algebra. An element $c \in A$ is called *cyclic* if $f^k(c) = c$ for some positive integer k. The set of all cyclic elements of some connected component of $(A;f)$ is called a *cycle* of $(A;f)$. A *length* of a cycle is its number of elements. A cycle with one element is called a *loop*. If $(A;f)$ is connected and contains no cycle, then it is said to be a *line* if for each $a \in A$ there exists a unique $b \in A$ with $f(b) = a$. A proper subalgebra of a line is called a *ray*.

A connected monounary algebra $(A;f)$ is said to be a *cycle C with a tail T*, if C is a cycle, $A = C \cup T$, $C \cap T = \emptyset$ and there is $c \in C$ such that $f(x) \in T \cup \{c\}$ for every $x \in T$. We say that T is *branched* if $|f^{-1}(t) \setminus C| \leq 2$ for each $t \in T \cup \{c\}$, and there is a unique $b \in T \cup \{c\}$ with $|f^{-1}(b) \setminus C| = 2$. If $b \neq c$, then T is *properly* branched. If T is branched, then it will be called *shortly branched* whenever $f^{-1}(x) \setminus C = \emptyset$ for $x \in f^{-1}(b)$. In the case when $|f^{-1}(t) \setminus C| \leq 1$ for each $t \in T \cup \{c\}$, the tail T is *non-branched*.

Next, if $(A;f)$ consists of a ray C and an element $b \in A \setminus C$ such that $f^{-2}(f(b)) = \emptyset$, then is said to be a *shortly branched ray*.

Egorova [2] has characterized all monounary algebras which are congruence modular and congruence distributive, in term of their graphs. We will state those results in the following two theorems.

Theorem 2.1. [2] *Let $(A;f)$ be a monounary algebra. Then* $\mathrm{Con}(A;f)$ *is distributive if and only if $(A;f)$ belongs to one of the following types:*

(a) *a cycle, possibly with a non-branched tail;*
(b) *a line or a ray;*
(c) *the union of a line or a ray and a loop;*
(d) *the union of two cycles of relatively prime lengths;*
(e) *the union of a cycle and a cycle with a non-branched tail such that the lengths of the cycles are relatively prime.*

Theorem 2.2. [2] *Let $(A;f)$ be a monounary algebra. Then* $\mathrm{Con}(A;f)$ *is modular if and only if $(A;f)$ belongs to one of the following types:*

(a) *a cycle, possibly with a non-branched or a shortly branched tail;*
(b) *a loop with a properly branched tail;*
(c) *a line;*
(d) *a ray, possibly shortly branched;*
(e) *the union of one or two loops with a connected component satisfying one of the conditions* (*a*)–(*d*)*;*
(f) *the union of at most three cycles of relatively prime lengths, while at most one can have a tail, either non-branched or shortly branched;*
(g) *the union of at most three cycles of relatively prime lengths, and, if one of them is a loop, it can have a properly branched tail.*

3. All lattices which are isomorphic to congruence distributive monounary algebras

In this section we will describe all distributive lattices which are isomorphic to the congruence lattices of monounary algebras $(A;f)$ of all types stated in Theorem 2.1 via the following propositions.

Recall that the notation $\theta(B)$ stands for the least congruence on an algebra $\underline{A}$ for which $B \subseteq A$ is contained in one congruence class.

Let $(B;f|_B)$ be a subalgebra of a monounary algebra $(A;f)$. We will denote the restriction $\theta|_B$ of $\theta \subseteq A\times A$ on B by θ_B; and if $\theta \subseteq B\times B$, we will denote the relation $\theta \cup \{(x,x)|x\in A\}$ by θ^A. Then $\theta_B \in \mathrm{Con}(B;f|_B)$ for all $\theta\in\mathrm{Con}(A;f)$ and $\theta^A\in\mathrm{Con}(A;f)$ for all $\theta\in\mathrm{Con}(B;f|_B)$.

In the sequel of this paper, we will shortly write 'a product of finitely many finite chains' by 'a product of chains'; denote $\underline{n}$ as an n-element chain for a positive integer n; and also denote $\underline{N}$ and $\underline{Z}$ as the chains of all natural numbers and of all integers with the dual usual order, respectively.

Proposition 3.1. *If a monounary algebra $(A;f)$ is of the type (a)-(b) of Theorem 2.1, then* $\mathrm{Con}(A;f)$ *is a product of chains $\underline{P}$, $\underline{N}\times\underline{P}$, $\underline{N}\times\underline{2}^N$ or $\underline{Z}\times\underline{2}^N$.*

Proof. Let $(A;f)$ be a cycle with $|A|=n$ and let denote $\downarrow n$ the lattice of all factors of n ordered by the division of integers. For each $m\in\downarrow n$, let $f^{[j]_m}(a)=\{f^s(a)\equiv j(mod\ m)\}$. Then $\wp_m=\{f^{[j]_m}(a)|j=0,1,2,\dots,m-1\}$ is a partition of A which corresponds to the congruence θ_m modulo m restriction to A; that is, $\theta_m=\{(x,y)|x,y\in f^{[j]_m}(a)$ for $j\in\{0,1,2,\dots,m-1\}\}$. Hence, the map $\alpha:\downarrow n\longrightarrow\mathrm{Con}(A;f)$ defined by $\alpha(m)=\theta_m$ for all $m\in\downarrow n$ is clearly an order-isomorphism; so, $\mathrm{Con}(A;f)$ is dually isomorphic to $\downarrow n$ which is a product of chains.

Now, let $(A;f)$ be a cycle C with a non-branched tail T. Then either $T=\{f^{-j}(c)|j=1,\dots,n-1\}$ for some $n\in N$ or $T=\{f^{-j}(c)|j\in N\}$. Hence, the mapping $\alpha:\theta\longmapsto(j,\theta_C)$ for all $\theta\in\mathrm{Con}(A;f)$ and $j\in N$ is clearly an order-isomorphism. Also, the result in the case (a) shows that $\mathrm{Con}(C;f|_C)$ is a product of chains. Therefore, either $\mathrm{Con}(A;f)\cong\underline{n}\times\underline{P}$ for some product of chains $\underline{P}$ which is also a product of chains; or $\mathrm{Con}(A;f)\cong\underline{N}\times\underline{P}$.

Let $(A;f)$ be a ray. We may assume that $\{0,1,2,\dots\}$ with $f(i)=i+1$ for all $i\in N$. One can prove similarly as above that there exists a sublattice A^0 of $\mathrm{Con}(A;f)$ which is dually isomorphic to the lattice of all natural numbers ordered by the division of integers which is dually isomorphic to the lattice of the power set of all primes; therefore, A^0 is isomorphic to the product $\underline{2}^N$. But also for each $k\in N$, we can apply the same proof that the congruence lattice of the ray $\{k,k+1,k+2,\dots\}$ is a sublattice A^k of $\mathrm{Con}(A;f)$ which is $\underline{2}^N$; hence, $\mathrm{Con}(A;f)$ is isomorphic to the product $\underline{N}\times\underline{2}^N$.

Since a ray $\{a,f(a),f^2(a),\dots,f^k(a)\dots\}$ is a subalgebra of a line for each element a in the line, the congruence lattice of a line is isomorphic to the product $\underline{Z}\times\underline{2}^N$. ∎

Remark 3.1. Let $(A;f)$ be a monounary algebra and assume that $a,b\in A$ belong to cycles of length p and q, respectively. Let $\theta\in\mathrm{Con}(A;f)$.

(i) If $(a,f^r(a))\in\theta$ for some $0<r\le p-1$, then $(a,f^{kr}(a))\in\theta$ for each non-negative integer k.

(ii) If r and p are relatively prime and $(a,f^r(a))\in\theta$ then $\{a,f(a),\dots,f^{p-1}(a)\}$ is contained in a block of the quotient algebra A/θ.

(iii) If $(p,q)=1$ and $(a,b)\in\theta$, then $\{a,f(a),\dots,f^{p-1}(a),b,f(b),\dots,f^{q-1}(b)\}$ is contained in a block of A/θ.

Recall that a linear sum of an ordered set $\underline{P}$ with a one-element chain $\underline{1}$ is an ordered set $\underline{P}\oplus\underline{1}$ which represents $\underline{P}$ with a new top element added.

Proposition 3.2. *If a monounary algebra $(A;f)$ is of the type (d) or (e) in Theorem 2.1, then* $\mathrm{Con}(A;f)$ *is either $\underline{P}\oplus\underline{1}$ or $(\underline{N}\times\underline{P})\oplus\underline{1}$ for some product of chains $\underline{P}$.*

Proof. Let $(A;f)$ be a monounary algebra of the type (d) or (e) in Theorem 2.1. Then A is the disjoint union $B_1\cup B_2$ where $(B_i;f|_{B_i})$ is a cycle on the set B_i for each $i\in\{1,2\}$ and one of them can be a cycle with a non-branched tail such that the lengths of the cycles are relatively prime. By the results of Proposition 3.1, $\mathrm{Con}(B_i;f|_{B_i})$ is a product of chains for $i\in\{1,2\}$. Since $\{B_1,B_2\}$ is a partition on A, $\theta_1^A\cup\theta_2^A$ is an element in $\mathrm{Con}(A;f)$ for all $\theta_i\in\mathrm{Con}(B_i;f|_{B_i})$ and $i\in\{1,2\}$; hence, $\mathrm{Con}(A;f)$ is a sublattice of the power set of $A\times A$; so, the map $\beta:(\theta_1,\theta_2)\longrightarrow\theta_1^A\vee\theta_2^A$ is an order embedding from $\mathrm{Con}(B_1;f|_{B_1})\times\mathrm{Con}(B_2;f|_{B_2})$ into $\mathrm{Con}(A;f)$.

Let $\theta_i\in\mathrm{Con}(B_i;f|_{B_i})$ for $i\in\{1,2\}$. Then $(x,y)\notin\theta_1^A\vee\theta_2^A$ whenever $x\in B_1$ and $y\in B_2$; so, $\theta_1^A\vee\theta_2^A\neq A\times A$; hence, $A\times A\notin\mathrm{Im}\,\beta$. Now, if $\theta\in\mathrm{Con}(A;f)\setminus\{A\times A\}$ then $\theta_{B_1}^A\cup\theta_{B_2}^A\in\mathrm{Con}(A;f)$ where $\theta_{B_1}^A\cup\theta_{B_2}^A\subseteq\theta$. If $(a,b)\in\theta$ with $a\in B_1$ and $b\in B_2$, the result (iii) in Remark 3.1 implies that $A=B_1\cup B_2$ is a subset of a block of A/θ since the lengths of $f|_{B_1}$ and $f|_{B_2}$ are relatively prime; so, $\theta=A\times A$, a contradiction. So, if $(a,b)\in\theta$ then $\{a,b\}\subseteq B_i$ for some $i\in\{1,2\}$; hence, $(a,b)\in\theta_{B_i}\subseteq\theta_{B_1}^A\cup\theta_{B_2}^A$. Thus, $\theta\in\mathrm{Im}\,\beta$. Therefore, $\mathrm{Con}(A;f)\setminus\{A\times A\}=\mathrm{Im}\,\beta\cong\mathrm{Con}(B_1;f|_{B_1})\times\mathrm{Con}(B_2;f|_{B_2})$. Hence, $\mathrm{Con}(A;f)$ is either $\underline{P}\oplus\underline{1}$ or $(\underline{N}\times\underline{P})\oplus\underline{1}$ for some product of chains $\underline{P}$. ∎

Proposition 3.3. *If a monounary algebra $(A;f)$ is the union of a line or a ray and a loop, then* $\mathrm{Con}(A;f)$ *is* $\underline{Z}\times(\underline{2}^N\oplus\underline{1})$ *or* $\underline{N}\times(\underline{2}^N\oplus\underline{1})$.

Proof. Let $(A;f)$ be the union of a line or a ray and a loop, then $A=B\cup\{c\}$ is a disjoint union of a line or a ray $(B;f|_B)$ and a loop $\{c\}$. Hence, $\mathrm{Con}(B;f|_B)$ is the product $\underline{Z}\times\underline{2}^N$ or $\underline{N}\times\underline{2}^N$. If c is joined with an element $d\in B$, then the set $\{c,d,f(d),\dots\}$ is the block of a congruence which contains $D\times D\in\mathrm{Con}(D;f|_D)$ where D is the ray $\{d,f(d),f^2(d),\dots\}$. Hence, $\mathrm{Con}(A;f)$ is $\underline{Z}\times(\underline{2}^N\oplus\underline{1})$ or $\underline{N}\times(\underline{2}^N\oplus\underline{1})$. ∎

Corollary 3.1. *A distributive lattice is (up to isomorphism) a congruence lattice of some monounary algebra if and only if it is one of the lattices from Propositions 3.1–3.3.*

Proof. From the characterization of the congruence lattices of monounary algebras fulfilling the conditions (a)-(e) of Theorem 2.1 it is easy to show that to each lattice L of the mentioned lattices there exists an algebra (A,f) with $\mathrm{Con}(A,f)\cong L$. ∎

4. All lattices which are isomorphic to congruence modular monounary algebras

In this section we will describe all modular lattices which are isomorphic to the congruence lattices of monounary algebras $(A;f)$ of all types stated in Theorem 2.2. Since all distributive lattices are modular, it is enough to deal only with the cases in the following propositions.

Proposition 4.1. *If a monounary algebra $(A;f)$ is a cycle with a shortly branched tail, then* $\mathrm{Con}(A;f)$ *is* $\underline{M}_3\times\underline{P}$ *for some product of chains* $\underline{P}$.

Proof. Let $C=\{c,f(c),\dots f^{k-1}(c)\}$ for some positive integer k be the cycle of $(A;f)$ and let $A=\{a,b\}\cup C$ with $f(a)=f(b)=f^k(c)=c$. Then the set

$$\bar{M}_3=\left\{\Delta_A,\theta(a,b),\theta\left(a,f^{k-1}(c)\right),\theta\left(b,f^{k-1}(c)\right),\theta\left(\left\{a,b,f^{k-1}(c)\right\}\right)\right\}$$

forms a sublattice of $\mathrm{Con}(A;f)$ which is isomorphic to $\underline{M}_3$. Note that for each $\theta\in\mathrm{Con}(C;f_C)$ the relation $\bar{\theta}=\theta\cup\{(x,x)|x\in\{a,b\}\}\in\mathrm{Con}(A;f)$. Now, let define $\alpha:\bar{M}_3\times\mathrm{Con}(C;f_C)\longrightarrow\mathrm{Con}(A;f)$ and $\beta:\mathrm{Con}(A;f)\longrightarrow\bar{M}_3\times\mathrm{Con}(C;f_C)$ respectively by $\alpha(\phi,\theta)=\phi\vee\bar{\theta}$ for all $(\phi,\theta)\in\bar{M}_3\times\mathrm{Con}(C;f_C)$ and $\beta(\theta)=(\theta|_{\{a,b,f^{k-1}(c)\}},\theta|_C)$ for all $\theta\in\mathrm{Con}(A;f)$. One can prove that α and β are embeddings which $\alpha \circ \beta = id_{\mathrm{Con}(A;f)}$ and $\beta \circ \alpha = id_{\bar{M}_3 \times \mathrm{Con}(C;f_C)}$. Therefore, $\mathrm{Con}(A;f) \cong \bar{M}_3 \times \mathrm{Con}(C;f_C)$. By Proposition 3.1, $\mathrm{Con}(C;f_C)$ is a product of chains; so, $\mathrm{Con}(A;f)$ is $\underline{M}_3 \times \underline{P}$ where $\underline{P}$ is a product of chains. ∎

A lattice $\underline{L}$ is called an $\underline{M}_3$*-rectangle* if

(i) there is a sublattice $\underline{B}$ of $\underline{L}$ which is either $\underline{n} \times \underline{k}$, $\underline{N} \times \underline{k}$, $\underline{n} \times \underline{N}$ or $\underline{N} \times \underline{N}$ for some positive integers n and k,

(ii) for each $i, j \in N$, there exists an element $c_{i,j} \in L \backslash B$ such that $\{a_{i,j} = a_{i+1,j} \wedge c_{i,j} \wedge a_{i,j+1},\ a_{i+1,j},\ c_{i,j},\ a_{i,j+1},\ a_{i+1,j+1} = a_{i+1,j} \vee c_{i,j} \vee a_{i,j+1}\}$ forms an $\underline{M}_3$-sublattice of $\underline{L}$, and

(iii) no other comparabilities in $\underline{L}$ other than in (i) and (ii).

If the lattice $\underline{B}$ of $\underline{L}$ is $\underline{2} \times \underline{N}$, then $\underline{L}$ is called a simple $\underline{M}_3$*-rectangle*. A lattice $\underline{L}$ is called an $\underline{M}_3$*-Flag* if there are $m_1, m_2, m_3 \in L$ such that $L = \downarrow m_1 \cup \uparrow m_2 \cup \uparrow m_3$, $\downarrow m_1 = \underline{2} \times \underline{k}$ for some positive integer k and $\uparrow m_i$ is an $\underline{M}_3$*-rectangle* for each $i \in \{2,3\}$. If $\downarrow m_1 = \underline{2} \times \underline{N} \times \underline{2}^N$ and $\uparrow m_2 \cup \uparrow m_3 = \underline{K} \times \underline{N} \times \underline{2}^N$ where $\underline{K}$ is a simple $\underline{M}_3$*-rectangle*, we will call $\underline{L}$, an $\underline{M}_3$*-Flag power*.

Proposition 4.2. *If a monounary algebra* $(A;f)$ *is a loop with a properly branched tail or a shortly branched ray, then* $\mathrm{Con}(A;f)$ *is an* $\underline{M}_3$*-Flag or an* $\underline{M}_3$*-Flag power.*

Proof. Let A = $\{a_0, a_1, \ldots, a_k, \ldots, b_{k+1}, b_{k+2}, \ldots\}$ where $f(a_i) = a_{i-1}, f(b_j) = b_{j-1}$ for all $i \in N$ and $j \in N \backslash \{1, \ldots, k+1\}$, $f(b_{k+1}) = a_k$ and $f(a_0) = a_0$. Let denote for each $t \in N$ and $s > k$, $\downarrow a_t = \{a_0, \ldots, a_t\}$ and $\downarrow b_s = \{a_0, \ldots, a_k, b_{k+1}, \ldots, b_s\}$. Then $A_1 = \downarrow a_k, A_2 = A_1 \cup \{a_{k+1}, a_{k+2} \ldots\}$ and $A_3 = A_1 \cup \{b_{k+1}, b_{k+2} \ldots\}$ are subalgebras of $(A;f)$ each of which is a loop with a non-branched tail. By applying Proposition 3.1 or the result from [9], each $Con(A_i; f|_{A_i})$ is a chain $\underline{C}_i$ for all $i \in \{1,2,3\}$ where

$$\underline{C}_1 = \{\Delta_A \subseteq \theta(\downarrow a_1)\} \subseteq \cdots \subseteq \theta(\downarrow a_k)\},$$
$$\underline{C}_2 = \{\Delta_A \subseteq \theta(\downarrow a_1) \subseteq \cdots \subseteq \theta(\downarrow a_k) \subseteq \theta(\downarrow a_{k+1}) \subseteq \ldots\}, \text{ and}$$
$$\underline{C}_3 = \{\Delta_A \subseteq \theta(\downarrow a_1) \subseteq \cdots \subseteq \theta(\downarrow a_k) \subseteq \theta(\downarrow b_{k+1}) \subseteq \ldots\}.$$

Since $\theta \cup \theta(a_{k+1}, b_{k+1}) \in Con(A;f)$ for all $\theta \in \underline{C}_1$, the product $\underline{2} \times \underline{C}_1$ is a sublattice of $Con(A;f)$. Now, let m_1 be the greatest element of $\underline{2} \times \underline{C}_1, m_2 := \theta(\downarrow a_{k+1})$ and $m_3 := \theta(\downarrow b_{k+1})$. Then $\downarrow m_1 = \underline{2} \times \underline{k}$. Since $\theta(\downarrow a_t) \cup \theta(b_{k+1}, a_{t+1}) \in Con(A;f)$ for all $t \geq k+1$, it is clear that $\{\theta(\downarrow a_t),\ \theta(b_{k+1},\ a_{t+1}),\ \theta(\downarrow a_{t+1}),\ \theta(\downarrow a_t \cup \downarrow b_{k+1}),\ \theta(\downarrow a_{t+1} \cup \downarrow b_{k+1})\}$ forms a sublattice of $Con(A;f)$ which is isomorphic to $\underline{M}_3$. Besides for each $t \geq k$ and $s \geq k$, the sublattice $\{\theta(\downarrow a_t \cup \downarrow b_s),\ \theta(\downarrow a_t \cup \downarrow b_s) \cup \theta(a_{t+1}, b_s),\ \theta(\downarrow a_t \cup \downarrow b_{s+1}),\ \theta(\downarrow a_{t+1} \cup \downarrow b_s),\ \theta(\downarrow a_{t+1} \cup \downarrow b_{s+1})\}$ of $Con(A;f)$ is isomorphic to $\underline{M}_3$. Therefore, $\uparrow m_2$ is an $\underline{M}_3$*-rectangle*. A similarly proof shows that $\uparrow m_3$ is also an $\underline{M}_3$*-rectangle*. Hence, $Con(A;f) = \downarrow m_1 \cup \uparrow m_2 \cup \uparrow m_3$ is an $\underline{M}_3$*-Flag*.

Now, let $(A;f)$ be a shortly branched ray. Then, $A = \{a_1, \ldots, a_k, \ldots, b_{k+1}\}$ where $f(a_i) = a_{i+1}$ for all $i \in N$ and $f(b_{k+1}) = a_k$. Let $A_1 = \{a_k, a_{k+1}, \ldots\}$ and $A_2 = \{b_{k-1}, a_k, a_{k+1}, \ldots\}$. Then A_1 and A_2 are ray subalgebras of $\underline{A}$. By applying Proposition 3.1 and the above proof, $\downarrow m_1 = \underline{2} \times Con(A_1;f) \cong \underline{2} \times \underline{N} \times \underline{2}^N$ and $\uparrow m_2 \cup \uparrow m_3$ is $\underline{L} \times \underline{N} \times \underline{2}^N$ where $\underline{L}$ is a simple $\underline{M}_3$*-rectangle*. Therefore, $Con(A;f)$ is an $\underline{M}_3$*-Flag power*. ∎

We note that a cycle in Proposition 3.1 and 3.2 can be a loop and if the element c of a loop is joined with an element a of the connected component then the set $\{c, a, f(a), \ldots\}$ is contained in the same block of each congruence relation. We have the following corollary.

Corollary 4.1. *If a monounary algebra $(A;f)$ is the union of a loop and a connected component satisfying of the condition (a) in Theorem 2.1, then* $\mathrm{Con}(A;f)$ *is* $P\oplus\underline{1}$, $\underline{n}\times(\underline{P}\oplus\underline{1})$ *or* $\underline{N}\times(\underline{P}\oplus\underline{1})$.

Proposition 4.3. *If a monounary algebra $(A;f)$ is the union of a loop and a connected component satisfying one of the conditions (a)–(d) in Theorem 2.2, then* $\mathrm{Con}(A;f)$ *is either* $\underline{P}\oplus\underline{1}$, $\underline{n}\times(\underline{P}\oplus\underline{1})$, $\underline{N}\times(\underline{P}\oplus\underline{1})$, $\underline{M}_3\times(\underline{P}\oplus\underline{1})$, $\underline{2}\times\underline{L}$, $\underline{N}\times(\underline{2}^N\oplus\underline{1})$, $\underline{Z}\times(\underline{2}^N\oplus\underline{1})$, *where $\underline{P}$ is a product of chains and $\underline{L}$ is an $\underline{M}_3$-Flag power.*

Proof. By the results from Section 3 and Corollary 4.1, it is enough to prove only the following cases.

Let $(A;f)$ be the union of a loop $\{d\}$ and a cycle B with a shortly branched tail. Then $\mathrm{Con}(B;f|_B)$ is $\underline{M}_3\times\underline{P}$ where $\underline{P}$ is a product of chains. If d is related to an element of B with respect to a $\theta\in\mathrm{Con}(A;f)$, then A/θ is either $\{\{a\},\{b\},C\cup\{d\}\},\{\{b\},C\cup\{a,d\}\},\{\{a\},C\cup\{b,d\}\},\{\{a,b\},C\cup\{d\}\}$ or $\{A\}$; hence, $\mathrm{Con}(A;f)$ is $\underline{M}_3\times(\underline{P}\oplus\underline{1})$

Next, let $(A;f)$ be the union of a loop d and a loop with a properly branched tail B. Then Proposition 4.2 shows that $\mathrm{Con}(B;f|_B)$ is an $\underline{M}_3$*-Flag* and the result from [9] shows that each $\theta\in\mathrm{Con}(B;f|_B)$ either contains only one block whose cardinality is more than one, or contains only two blocks whose cardinality is more than one. Hence, each $\theta\in\mathrm{Con}(A;f)$ is either $\theta_B\cup\{(d,d)\}$ or add d in a block whose cardinality is more than one for each element in $\mathrm{Con}(B;f|_B)$. So $\mathrm{Con}(A;f)$ is $\underline{2}\times\mathrm{Con}(B;f|_B)$. Therefore $\mathrm{Con}(A;f)$ is $\underline{2}\times\underline{L}$ where $\underline{L}$ is an $\underline{M}_3$*-Flag*.

By the same arguing as in the above paragraph, one can see that if $(A;f)$ is the union of a loop and a line or a ray with a shortly branched tail, then $\mathrm{Con}(A;f)$ is $\underline{N}\times(\underline{2}^N\oplus\underline{1})$, $\underline{Z}\times(\underline{2}^N\oplus\underline{1})$ or $\underline{2}\times\underline{L}$ where $\underline{L}$ is an $\underline{M}_3$*-Flag power.* ∎

A lattice $\underline{L}$ with the greatest element 1 is said to be an $\underline{M}_3$*-head lattice* if

(i) $\underline{L}$ contains exactly three co-atoms m_1,m_2 and m_3 where $\downarrow m_1$ is $\underline{P}$, or $\underline{P}\oplus\underline{1}$ and $\downarrow m_2$ and $\downarrow m_3$ are either $\underline{P}$, $\underline{P}\oplus\underline{1}$, $\underline{n}\times(\underline{P}\oplus\underline{1})$, $\underline{N}\times(\underline{P}\oplus\underline{1})$, $\underline{N}\times(\underline{2}^N\oplus\underline{1})$, $\underline{Z}\times(\underline{2}^N\oplus\underline{1})$, $\underline{M}_3\times(\underline{P}\oplus\underline{1})$, $\underline{2}\times\underline{L}$, $\underline{L}\times(\underline{P}\oplus\underline{1})$ or $\underline{L}\times(\underline{N}\times\underline{2}^N)$ where $\underline{P}$ is a product of chains and $\underline{L}$ is an $\underline{M}_3$*-Flag*, and

(ii) the set $\{m,m_1,m_2,m_3,1\}$ forms a sublattice of $\underline{L}$ which is isomorphic to $\underline{M}_3$ where m is the greatest element of $\bigcap_{i=1}^{3}\downarrow m_i$.

Proposition 4.4. *If a monounary algebra $(A;f)$ is the union of at most three cycles of relatively prime lengths, while at most one can have a tail, either non-branched or shortly branched or a loop with a properly branched tail, then* $\mathrm{Con}(A;f)$ *is* $\underline{L}$, $\underline{N}\times\underline{L}$, $\underline{n}\times\underline{N}\times\underline{L}$, *or* $\underline{N}\times\underline{N}\times\underline{L}$ *where $\underline{L}$ is an M_3-head lattice.*

Proof. Let A be the disjoint union $B_1\cup B_2\cup B_3$ where $(B_i;f|_{B_i})$ is a cycle for each $i\in\{1,2\}$ and the cardinalities of B_1,B_2 and the cycle in B_3 are relatively prime lengths (each of the three cycles can be a loop). Then, $\mathrm{Con}(B_i;f|_{B_i})$ is a chain or a product of chains for each $i\in\{1,2\}$. By Proposition 3.2, $\mathrm{Con}(B_1\cup B_2;f|_{B_1\cup B_2})$ is $\underline{P}$ or $\underline{P}\oplus\underline{1}$ for some product of chains $\underline{P}$. A similar proof of Proposition 3.2 shows that $\mathrm{Con}(B_i\cup B_3;f|_{B_i\cup B_3})$ is $(\mathrm{Con}(B_i;f_{B_i})\times\mathrm{Con}(B_3;f_{B_3}))\oplus\underline{1}$ for each $i\in\{1,2\}$. And also, $\mathrm{Con}(B_i\cup B_j;f_{B_i\cup B_j})\times\mathrm{Con}(B_k;f_{B_k})$ can be embedded as a sublattice of $\mathrm{Con}(A;f)$ for each $1\le i\neq j\le 3$ and $k\notin\{i,j\}$.

Let $\sigma=(123)$ be a permutation on $\{1,2,3\}$ and for each $i\in\{1,2,3\}$, let $C_i:=\mathrm{Con}(B_i\cup B_{\sigma(i)};f|_{B_i\cup B_{\sigma(i)}})\times\mathrm{Con}(B_k;f|_{B_k})$ and let m_i be the greatest element of C_i. We can see that $(x,y)\notin m_i$ for all $x\in B_i$ and $y\in B_{\sigma^2(i)}$ which implies that $m_i\neq A\times A$ for all i. Now, let

$i \in \{1,2,3\}$ and $m_i \subset \theta \in \mathrm{Con}(A;f)$. Then there exists $(a,b) \in \theta$ and $(a,b) \notin m_i$. So, $\{a,b\} \nsubseteq B_i \cup B_{\sigma(i)}$ and $\{a,b\} \nsubseteq B_{\sigma^2(i)}$. Let $x,y \in A$. If $\{x,y\} \subseteq B_i \cup B_{\sigma(i)}$ or $\{x,y\} \subseteq B_{\sigma^2(i)}$ then $(x,y) \in m_i \subset \theta$. If not, we may assume that $a,x \in B_i \cup B_{\sigma(i)}$ and $y,b \in B_{\sigma^2(i)}$; so $x,a \in B_i \cup B_{\sigma(i)}$ implies that $(x,a) \in m_i \subset \theta$ and $(b,y) \in m_i \subset \theta$; hence, $(x,y) \in m_i \subset \theta$. Therefore, $\theta = A \times A$. Altogether, m_i is a co-atom of $\mathrm{Con}(A;f)$. A similar argument implies that m_1, m_2 and m_3 are the only co-atoms of $\mathrm{Con}(A;f)$. So, $m_i \vee m_{\sigma(i)} = A \times A$ for all $i = \{1,2,3\}$.

Let $m = \theta_{B_1} \cup \theta_{B_2} \cup \theta_{B_3}$. The relatively prime of the cardinalities of B_1, B_2 and the cycle in B_3 implies that m is the greatest lower bound of $\{m_1, m_2, m_3\}$ and has no other congruences between m and m_i for each $i \in \{1,2,3\}$. Therefore, $\{m, m_1, m_2, m_3, A \times A\}$ forms a sublattice of $\mathrm{Con}(A;f)$ which is isomorphic to $\underline{M}_3$ where m is the greatest element of $\bigcap_{i=1}^{3} \downarrow m_i$.

Assume that B_i is a cycle for each $i \in \{1,2\}$. We first assume that B_3 is a cycle with a shortly branched tail. As we stated above that $\mathrm{Con}(B_i \cup B_3; f|_{B_i \cup B_3})$ is $[\mathrm{Con}(B_i; f|_{B_i}) \times \mathrm{Con}(B_3; f|_{B_3})] \oplus \underline{1}$ and since a product of two product of chains is a product of chains, one can see that $\downarrow m_2$ and $\downarrow m_3$ are $\underline{M}_3 \times (\underline{P} \oplus \underline{1})$ where $\underline{P}$ is a product of chains. Therefore, $\mathrm{Con}(A;f)$ is an M_3-*head lattice*. Now, assume that B_3 is a cycle with a non-branched tail. For each $a \in B_3 \backslash C$, $\downarrow a$ is a cycle with a non-branched tail; so, the above proof shows that $\mathrm{Con}(B_1 \cup B_2 \cup \downarrow a; f|_{B_1 \cup B_2 \cup \downarrow a})$ is an $\underline{M}_3$-*head* lattice. Therefore, $\mathrm{Con}(A;f)$ is either $\underline{L}$ or $\underline{N} \times \underline{L}$ where $\underline{L}$ is an $\underline{M}_3$-*head lattice*.

Finally, if B_3 is a loop with a properly branched tail then $\mathrm{Con}(B_3'; f|_{B_3'})$ is an $\underline{M}_3$-*Flag* for each subset B_3' of B_3; so, $\mathrm{Con}(B_1 \cup B_2 \cup B_3'; f|_{B_1 \cup B_2 \cup B_3'})$ is an $\underline{M}_3$-*head* lattice. Therefore, $\mathrm{Con}(A;f)$ is either $\underline{L}$, $\underline{N} \times \underline{L}$, $\underline{N} \times \underline{k} \times \underline{L}$ or $\underline{N} \times \underline{N} \times \underline{L}$ where $\underline{L}$ is an $\underline{M}_3$-*head* lattice. ∎

Arguing similarly as in Corollary 3.1 we obtain

Corollary 4.2. *A modular lattice is (up to isomorphism) a congrunce lattice of some monounary algebra if and only if it is one of the lattices from Proposition 3.1–3.3 and Proposition 4.1–4.4.*

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