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G-Frames and Direct Sums

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Abstract. In this paper we study g-frames on the direct sum of Hilbert spaces. We generalize some of the results about g-frames on super Hilbert spaces to the direct sum of a countable number of Hilbert spaces. Also we study the direct sum of g-frames, g-Riesz bases and g-orthonormal bases for these spaces. Moreover we consider perturbations, duals and equivalences for the direct sum of g-frames.

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1. Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer (see [10]) in 1952 to study some problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann and Meyer (see [9]). Frames are very useful in characterization of function spaces and other fields of applications such as filter bank theory (see [4]), sigma-delta quantization (see [3]), signal and image processing (see [5]) and wireless communications (see [11]). First we recall the definition of frames.

Let H be a Hilbert space and let I be a finite or countable subset of $\mathbb{Z}$. A family $\{f_i\}_{i\in I} \subseteq H$ is a *frame* for H, if there exist $0 < A \leq B < \infty$, such that for each $f \in H$,

$$A\|f\|^2 \leq \sum_{i\in I} \left|\left\langle f, f_i\right\rangle\right|^2 \leq B\|f\|^2.$$

In this case we say that $\{f_i\}_{i\in I}$ is an (A,B) frame. A and B are the lower and upper frame bounds, respectively. If only the right-hand side inequality is required, it is called a *Bessel* sequence. A frame is *tight*, if $A = B$. If $A = B = 1$, it is called a *Parseval* frame. A family $\{f_i\}_{i\in I} \subseteq H$ is *complete* if the span of $\{f_i\}_{i\in I}$ is dense in H. We say that $\{f_i\}_{i\in I}$ is a *Riesz basis* for H, if it is complete in H and there exist two constants $0 < A \leq B < \infty$, such that for each sequence of scalars $\{c_i\}_{i\in I} \in \ell^2(I)$,

$$A\sum_{i\in I} |c_i|^2 \leq \left\|\sum_{i\in I} c_i f_i\right\|^2 \leq B\sum_{i\in I} |c_i|^2,$$

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or equivalently

$$A\sum_{i\in F}|c_i|^2 \le \left\|\sum_{i\in F} c_i f_i\right\|^2 \le B\sum_{i\in F}|c_i|^2,$$

for each sequence of scalars $\{c_i\}_{i\in F}$, where F is a finite subset of I. In this case we say that $\{f_i\}_{i\in I}$ is an (A,B) Riesz basis. For more results about frames see [8].

Sun in [16] introduced g-frames as a generalization of frames. He showed that oblique frames, pseudo frames and fusion frames [2, 7] are special cases of g-frames. Let I be a finite or countable subset of $\mathbb{Z}$ and H be a Hilbert space. For each $i\in I$, let H_i be a Hilbert space and $L(H,H_i)$ be the set of all bounded, linear operators from H to H_i. We call $\Lambda=\{\Lambda_i\in L(H,H_i): i\in I\}$ a *g-frame* for H with respect to $\{H_i : i\in I\}$ if there exist two positive constants A and B such that

$$A\|f\|^2 \le \sum_{i\in I}\left\|\Lambda_i f\right\|^2 \le B\|f\|^2,$$

for each $f\in H$. In this case we say that Λ is an (A,B) g-frame. A and B are the lower and upper g-frame bounds, respectively. We call Λ an *A-tight* g-frame if $A=B$ and we call it a *Parseval* g-frame if $A=B=1$. If only the second inequality is required, we call it a *g-Bessel sequence*. If Λ is an (A,B) g-frame, then the *g-frame operator* S_Λ is defined by $S_\Lambda f=\sum_{i\in I}\Lambda_i^*\Lambda_i f$, which is a bounded, positive and invertible operator such that $A.I\le S_\Lambda\le B.I$. The *canonical dual* g-frame for Λ is defined by $\{\tilde{\Lambda}_i\in L(H,H_i): i\in I\}$, where $\tilde{\Lambda}_i=\Lambda_i S_\Lambda^{-1}$, which is an $(1/B,1/A)$ g-frame for H and for each $f\in H$, we have

$$f=\sum_{i\in I}\Lambda_i^*\tilde{\Lambda}_i f=\sum_{i\in I}\tilde{\Lambda}_i^{\,*}\Lambda_i f.$$

If Λ is a g-Bessel sequence, then the g-Bessel sequence $\{\Gamma_i\in L(H,H_i): i\in I\}$ is called an alternate dual or a dual of Λ if

$$f=\sum_{i\in I}\Gamma_i^*\Lambda_i f=\sum_{i\in I}\Lambda_i^*\Gamma_i f,$$

for each $f\in H$. Now define

$$\oplus_{i\in I}H_i=\left\{\{f_i\}_{i\in I}|f_i\in H_i, \left\|\{f_i\}_{i\in I}\right\|_2{}^2=\sum_{i\in I}\|f_i\|^2<\infty\right\}.$$

$\oplus_{i\in I}H_i$ with pointwise operations and inner product as

$$\left\langle \{f_i\}_{i\in I},\{g_i\}_{i\in I}\right\rangle=\sum_{i\in I}\left\langle f_i,g_i\right\rangle$$

is a Hilbert space.

Let $\{H_i\}_{i\in I}$ be a sequence of Hilbert spaces. Then by considering $K=\oplus_{i\in I}H_i$, we can assume that each H_i is a closed subspace of K, therefore if $f_{i_1}\in H_{i_1}$ and $f_{i_2}\in H_{i_2}$, for $i_1,i_2\in I$, then $\langle f_{i_1},f_{i_2}\rangle$ is well-defined.

We say that $\{\Lambda_i\in L(H,H_i): i\in I\}$ is *g-complete* if $\{f:\Lambda_i f=0,\forall i\in I\}=\{0\}$, and we call it a *g-orthonormal basis* for H, if

$$\left\langle \Lambda_{i_1}^* f_{i_1},\Lambda_{i_2}^* f_{i_2}\right\rangle=\delta_{i_1,i_2}\left\langle f_{i_1},f_{i_2}\right\rangle,\quad i_1,i_2\in I, f_{i_1}\in H_{i_1}, f_{i_2}\in H_{i_2},$$

and

$$\sum_{i\in I}\left\|\Lambda_i f\right\|^2=\|f\|^2,\quad \forall f\in H.$$

$\Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\}$ is a *g-Riesz basis* for H, if it is g-complete and there exist two constants $0 < A \leq B < \infty$, such that for each finite subset $F \subseteq I$ and $f_i \in H_i, i \in F$,

$$A \sum_{i \in F} \|f_i\|^2 \leq \left\| \sum_{i \in F} \Lambda_i^* f_i \right\|^2 \leq B \sum_{i \in F} \|f_i\|^2.$$

In this case we say that Λ is an (A, B) g-Riesz basis.

Let H_i and H_i' be Hilbert spaces, for each $i \in I$ and let $H = \oplus_{i \in I} H_i$ and $H' = \oplus_{i \in I} H_i'$. Recall that if $T_i \in L(H_i, H_i')$, then $T = \oplus_{i \in I} T_i$ which is defined by $T(\{h_i\}_{i \in I}) = \{T_i(h_i)\}_{i \in I}$ is a bounded operator from H to H' if and only if $\sup\{\|T_i\| : i \in I\} < \infty$. In this case $\|T\| = \sup\{\|T_i\| : i \in I\}$ and $T^* = \oplus_{i \in I} T_i^*$. If H and K are Hilbert spaces, then $H \oplus K$ is called a *super Hilbert space*.

Recently some authors were interested in g-frames on super Hilbert spaces, see [12, Proposition 2.16], [17] and [1]. In this paper we consider g-frames on the direct sum of a finite or countable number of Hilbert spaces.

In Section 2 we study g-frames, g-Riesz bases and g-orthonormal bases for the direct sum of Hilbert spaces. We also construct the direct sum of g-frames (resp. g-Riesz bases, g-orthonormal bases) for a finite or countable number of g-frames (resp. g-Riesz bases, g-orthonormal bases). In Section 3 we consider perturbations, duals and equivalences for the direct sum of g-frames.

2. The direct sum of g-frames

Throughout this note all of the Hilbert spaces are separable. I, J, K_i's, K_{ij}'s are finite or countable subsets of $\mathbb{Z}$ and H, H_i's, H_{ij}'s are Hilbert spaces. We start with the following proposition which is a generalization of [1, Proposition 2.3]:

Proposition 2.1. *Let $\{\Lambda_{ij} \in L(H, H_{ij}) : i \in I\}$ be a sequence for each $j \in J$ and $\{e_{ij,k} : k \in K_{ij}\}$ be an orthonormal basis for H_{ij}. Suppose that $\Theta_i : H \longrightarrow \oplus_{j \in J} H_{ij}$ which is defined by $\Theta_i(f) = \{\Lambda_{ij} f\}_{j \in J}$ is a bounded operator for each $i \in I$, and suppose that $\psi_{ij,k} = \Lambda_{ij}^*(e_{ij,k})$. Then $\{\psi_{ij,k} : j \in J, i \in I, k \in K_{ij}\}$ is a frame (resp. tight frame, Bessel sequence, Riesz basis, orthonormal basis) for H if and only if $\{\Theta_i \in L(H, \oplus_{j \in J} H_{ij}) : i \in I\}$ is a g-frame (resp. tight g-frame, g-Bessel sequence, g-Riesz basis, g-orthonormal basis).*

Proof. For each $f \in H$, we have

$$\sum_{i \in I} \|\Theta_i f\|^2 = \sum_{i \in I} \sum_{j \in J} \left\|\Lambda_{ij} f\right\|^2 = \sum_{j \in J} \sum_{i \in I} \sum_{k \in K_{ij}} \left|\langle f, \psi_{ij,k} \rangle\right|^2. \tag{2.1}$$

This shows that $\{\psi_{ij,k} : j \in J, i \in I, k \in K_{ij}\}$ is a frame (resp. tight frame, Bessel sequence, complete set) if and only if $\{\Theta_i\}_{i \in I}$ is a g-frame (resp. tight g-frame, g-Bessel sequence, g-complete set).

Let $\{\psi_{ij,k} : j \in J, i \in I, k \in K_{ij}\}$ be an (A, B) Riesz basis and F be a finite subset of I. Suppose that $f \in H$ and $\{f_{ij}\}_{j \in J} \in \oplus_{j \in J} H_{ij}$ for each $i \in F$. We have

$$\left\langle \Theta_i^* \left(\{f_{ij}\}_{j \in J}\right), f \right\rangle = \left\langle \{f_{ij}\}_{j \in J}, \{\Lambda_{ij} f\}_{j \in J} \right\rangle = \sum_{j \in J} \left\langle f_{ij}, \Lambda_{ij} f \right\rangle = \left\langle \sum_{j \in J} \Lambda_{ij}^* f_{ij}, f \right\rangle,$$

therefore $\Theta_i^*(\{f_{ij}\}_{j\in J}) = \sum_{j\in J}\Lambda_{ij}^* f_{ij}$, so

$$\Big\|\sum_{i\in F}\Theta_i^*\big(\{f_{ij}\}_{j\in J}\big)\Big\|^2 = \Big\|\sum_{i\in F}\sum_{j\in J}\Lambda_{ij}^* f_{ij}\Big\|^2.$$

Suppose that $f_{ij} = \sum_{k\in K_{ij}} c_{ij,k}e_{ij,k}$, thus $\Lambda_{ij}^*(f_{ij}) = \sum_{k\in K_{ij}} c_{ij,k}\psi_{ij,k}$. Hence

$$\Big\|\sum_{i\in F}\Theta_i^*\big(\{f_{ij}\}_{j\in J}\big)\Big\|^2 = \Big\|\sum_{j\in J}\sum_{i\in F}\sum_{k\in K_{ij}} c_{ij,k}\psi_{ij,k}\Big\|^2. \tag{2.2}$$

Since $f_{ij} = \sum_{k\in K_{ij}} c_{ij,k}e_{ij,k}$, then

$$\big\|\{f_{ij}\}_{j\in J}\big\|^2 = \sum_{j\in J}\|f_{ij}\|^2 = \sum_{j\in J}\sum_{k\in K_{ij}}|c_{ij,k}|^2,$$

for each $i\in F$, therefore

$$\sum_{i\in F}\big\|\{f_{ij}\}_{j\in J}\big\|^2 = \sum_{i\in F}\sum_{j\in J}\sum_{k\in K_{ij}}\big|c_{ij,k}\big|^2 = \sum_{j\in J}\sum_{i\in F}\sum_{k\in K_{ij}}|c_{ij,k}|^2. \tag{2.3}$$

Now by using (2.2) and (2.3), we have

$$A\sum_{i\in F}\big\|\{f_{ij}\}_{j\in J}\big\|^2 = A\sum_{j\in J}\sum_{i\in F}\sum_{k\in K_{ij}}\big|c_{ij,k}\big|^2 \le \Big\|\sum_{j\in J}\sum_{i\in F}\sum_{k\in K_{ij}} c_{ij,k}\psi_{ij,k}\Big\|^2 = \Big\|\sum_{i\in F}\Theta_i^*\big(\{f_{ij}\}_{j\in J}\big)\Big\|^2,$$

similarly

$$\Big\|\sum_{i\in F}\Theta_i^*\big(\{f_{ij}\}_{j\in J}\big)\Big\|^2 \le B\sum_{i\in F}\big\|\{f_{ij}\}_{j\in J}\big\|^2.$$

This means that $\{\Theta_i\}_{i\in I}$ is an (A,B) g-Riesz basis. The converse is obtained similarly by choosing a finite sequence of scalars $\{c_{ij,k}\}$, using (2.2), (2.3) and the fact that $\{\Theta_i\}_{i\in I}$ is a g-Riesz basis.

Now let $\{\psi_{ij,k} : j\in J, i\in I, k\in K_{ij}\}$ be an orthonormal basis. Suppose that $i,\ell\in I$, $\{f_{ij}\}_{j\in J}\in\oplus_{j\in J}H_{ij}$ and $\{g_{\ell j}\}_{j\in J}\in\oplus_{j\in J}H_{\ell j}$. We have $f_{ij} = \sum_{k\in K_{ij}}\langle f_{ij}, e_{ij,k}\rangle e_{ij,k}$, $g_{\ell j} = \sum_{k\in K_{\ell j}}\langle g_{\ell j}, e_{\ell j,k}\rangle e_{\ell j,k}$. Then

$$\begin{aligned}
\big\langle\Theta_i^*(\{f_{ij}\}_{j\in J}), \Theta_\ell^*(\{g_{\ell j}\}_{j\in J})\big\rangle &= \Big\langle\sum_{j\in J}\Lambda_{ij}^*(f_{ij}), \sum_{j\in J}\Lambda_{\ell j}^*(g_{\ell j})\Big\rangle\\
&= \sum_{j\in J}\sum_{r\in J}\sum_{k\in K_{ij}}\sum_{d\in K_{\ell r}}\big\langle\langle f_{ij}, e_{ij,k}\rangle\psi_{ij,k}, \langle g_{\ell r}, e_{\ell r,d}\rangle\psi_{\ell r,d}\big\rangle\\
&= \sum_{j\in J}\sum_{r\in J}\sum_{k\in K_{ij}}\sum_{d\in K_{\ell r}}\langle f_{ij}, e_{ij,k}\rangle\langle e_{\ell r,d}, g_{\ell r}\rangle\langle\psi_{ij,k}, \psi_{\ell r,d}\rangle.
\end{aligned}$$

Now if $i=\ell$, then

$$\begin{aligned}
\sum_{j\in J}\sum_{r\in J}\sum_{k\in K_{ij}}\sum_{d\in K_{\ell r}}\langle f_{ij}, e_{ij,k}\rangle\langle e_{\ell r,d}, g_{\ell r}\rangle\langle\psi_{ij,k}, \psi_{\ell r,d}\rangle &= \sum_{j\in J}\sum_{k\in K_{ij}}\langle f_{ij}, e_{ij,k}\rangle\langle e_{ij,k}, g_{ij}\rangle\\
&= \sum_{j\in J}\langle f_{ij}, g_{ij}\rangle = \big\langle\{f_{ij}\}_{j\in J}, \{g_{ij}\}_{j\in J}\big\rangle,
\end{aligned}$$

so $\big\langle\Theta_i^*(\{f_{ij}\}_{j\in J}), \Theta_i^*(\{g_{ij}\}_{j\in J})\big\rangle = \big\langle\{f_{ij}\}_{j\in J}, \{g_{ij}\}_{j\in J}\big\rangle$. If $i\ne\ell$, then $\langle\psi_{ij,k}, \psi_{\ell r,d}\rangle = 0$. Therefore $\big\langle\Theta_i^*(\{f_{ij}\}_{j\in J}), \Theta_\ell^*(\{g_{\ell j}\})\big\rangle = 0$. The second condition of g-orthonormal basis follows from (2.1). Conversely let $\{\Theta_i\}_{i\in I}$ be a g-orthonormal basis. Let $i_1, i_2 \in I$, $j_1, j_2 \in J$, $k_1 \in K_{i_1 j_1}$ and $k_2 \in K_{i_2 j_2}$. Then

$$\left\langle \psi_{i_1 j_1,k_1}, \psi_{i_2 j_2,k_2} \right\rangle = \left\langle \Lambda^*_{i_1 j_1}(e_{i_1 j_1,k_1}), \Lambda^*_{i_2 j_2}(e_{i_2 j_2,k_2}) \right\rangle = \left\langle \Theta^*_{i_1}(f_{i_1 j_1,k_1}), \Theta^*_{i_2}(f_{i_2 j_2,k_2}) \right\rangle,$$

where $f_{i_1 j_1,k_1} = \{\delta_{j_1,j} e_{i_1 j_1,k_1}\}_{j\in J}$ and $f_{i_2 j_2,k_2} = \{\delta_{j_2,j} e_{i_2 j_2,k_2}\}_{j\in J}$. Hence

$$\left\langle \psi_{i_1 j_1,k_1}, \psi_{i_2 j_2,k_2} \right\rangle = \delta_{i_1,i_2} \left\langle f_{i_1 j_1,k_1}, f_{i_2 j_2,k_2} \right\rangle = \delta_{i_1,i_2} \delta_{j_1,j_2} \delta_{k_1,k_2},$$

which shows that $\{\psi_{ij,k} : j \in J, i \in I, k \in K_{ij}\}$ is an orthonormal basis. ∎

Proposition 2.2. *Let $\{\Theta_i \in L(H, \oplus_{j\in J} H_{ij}) : i \in I\}$ be a g-frame (resp. tight g-frame, g-Bessel sequence, g-Riesz basis, g-orthonormal basis). Then there exists a g-frame (resp. tight g-frame, g-Bessel sequence, g-Riesz basis, g-orthonormal basis) $\{\Lambda_{ij} \in L(H, H_{ij}) : i \in I, j \in J\}$ such that $\Theta_i(f) = \{\Lambda_{ij} f\}_{j\in J}$, for each $i \in I$ and $f \in H$.*

Proof. Define $\pi_{ij} : \oplus_{\ell\in J} H_{i\ell} \longrightarrow H_{ij}$ by $\pi_{ij}(\{f_{i\ell}\}_{\ell\in J}) = f_{ij}$ and $\Lambda_{ij} = \pi_{ij} \circ \Theta_i$, for each $i \in I$ and $j \in J$. It is clear that $\Theta_i(f) = \{\Lambda_{ij} f\}_{j\in J}$, for each $i \in I$ and $f \in H$, so by Proposition 2.1, $\{\psi_{ij,k} = \Lambda^*_{ij}(e_{ij,k}) : j \in J, i \in I, k \in K_{ij}\}$ is a frame (resp. tight frame, Bessel sequence, Riesz basis, orthonormal basis) for H, where $\{e_{ij,k}\}_{k\in K_{ij}}$ is an orthonormal basis for H_{ij}. Now the result follows from [16, Theorem 3.1]. ∎

In the rest of this note, Φ_j and Ψ_j are $\{\Lambda_{ij} \in L(H_j, H_{ij}) : i \in I\}$ and $\{\Gamma_{ij} \in L(H_j, H_{ij}) : i \in I\}$, respectively, for each $j \in J$. We say that $\{\Phi_j\}_{j\in J}$ is an (A,B)-bounded family of g-frames (resp. g-Riesz bases), if Φ_j is an (A_j, B_j) g-frame (resp. g-Riesz basis) such that $A = inf\{A_j : j \in J\} > 0$ and $B = sup\{B_j : j \in J\} < \infty$. Also we call $\{\Phi_j\}_{j\in J}$ a B-bounded family of g-Bessel sequences, if Φ_j is a g-Bessel sequence for each $j \in J$ with upper bound B_j such that $B = sup\{B_j : j \in J\} < \infty$.

Theorem 2.1. *$\{\Phi_j\}_{j\in J}$ is an (A,B)-bounded (resp. a B-bounded) family of g-frames (resp. g-Bessel sequences) if and only if $\oplus_{j\in J}\Phi_j = \{\oplus_{j\in J}\Lambda_{ij} \in L(\oplus_{j\in J} H_j, \oplus_{j\in J} H_{ij}) : i \in I\}$ is an (A,B) g-frame (resp. a g-Bessel sequence with upper bound B) for $\oplus_{j\in J} H_j$. In this case the g-frame operator of $\oplus_{j\in J}\Phi_j$ is $\oplus_{j\in J} S_{\Phi_j}$, where S_{Φ_j} is the g-frame operator of Φ_j, for each $j \in J$.*

Proof. First suppose that $\{\Phi_j\}_{j\in J}$ is a B-bounded family of g-Bessel sequences. For each $j \in J, i \in I$ and $f_j \in H_j$, we have

$$\|\Lambda_{ij} f_j\|^2 \le \sum_{k\in I} \|\Lambda_{kj} f_j\|^2 \le B_j \|f_j\|^2 \le B\|f_j\|^2 \Longrightarrow \|\Lambda_{ij}\| \le \sqrt{B}.$$

Thus for each $i \in I$, we have $\sup\{\|\Lambda_{ij}\| : j \in J\} < \infty$. This means that for each $i \in I$, $\oplus_{j\in J}\Lambda_{ij}$ is a bounded operator from $\oplus_{j\in J} H_j$ to $\oplus_{j\in J} H_{ij}$. Now for each $f = \{f_j\}_{j\in J} \in \oplus_{j\in J} H_j$, we have

$$\sum_{i\in I} \|(\oplus_{j\in J}\Lambda_{ij}) f\|^2 = \sum_{i\in I}\sum_{j\in J} \|\Lambda_{ij}(f_j)\|^2.$$

Hence

$$\sum_{i\in I}\sum_{j\in J} \|\Lambda_{ij}(f_j)\|^2 = \sum_{j\in J}\sum_{i\in I} \|\Lambda_{ij}(f_j)\|^2 \le \sum_{j\in J} B_j \|f_j\|^2 \le B \sum_{j\in J} \|f_j\|^2 = B\|f\|^2,$$

so $\oplus_{j\in J}\Phi_j$ is a g-Bessel sequence for $\oplus_{j\in J} H_j$ with upper bound B. Conversely suppose that $\oplus_{j\in J}\Phi_j$ is a g-Bessel sequence with upper bound B. Let $j_0 \in J$ and $f_{j_0} \in H_{j_0}$. Then

$$\sum_{i\in I} \|\Lambda_{ij_0} f_{j_0}\|^2 = \sum_{i\in I} \|(\oplus_{j\in J}\Lambda_{ij})(\{\delta_{j_0,j} f_{j_0}\}_{j\in J})\|^2 \le B\|\{\delta_{j_0,j} f_{j_0}\}_{j\in J}\|^2 = B\|f_{j_0}\|^2.$$

This means that Φ_{j_0} is a g-Bessel sequence with upper bound B. Now suppose that $\{\Phi_j\}_{j\in J}$ is an (A,B)-bounded family of g-frames. For each $f=\{f_j\}_{j\in J}\in\oplus_{j\in J}H_j$, we have

$$\sum_{i\in I}\|(\oplus_{j\in J}\Lambda_{ij})f\|^2=\sum_{i\in I}\sum_{j\in J}\|\Lambda_{ij}(f_j)\|^2=\sum_{j\in J}\sum_{i\in I}\|\Lambda_{ij}(f_j)\|^2\geq\sum_{j\in J}A_j\|f_j\|^2\geq A\|f\|^2,$$

so $\oplus_{j\in J}\Phi_j$ is an (A,B) g-frame. The converse is also easy to verify.

Note that since $S_{\Phi_j}\leq B.I$, then by Theorem 2.2.5 in [14], $\|S_{\Phi_j}\|\leq B$, for each $j\in J$, so $\oplus_{j\in J}S_{\Phi_j}$ is a bounded operator. For each $f=\{f_j\}_{j\in J}\in\oplus_{j\in J}H_j$, we have

$$\begin{aligned}\left\langle S_{\oplus_{j\in J}\Phi_j}(f),f\right\rangle&=\left\langle\sum_{i\in I}(\oplus_{j\in J}\Lambda_{ij}^*)(\oplus_{j\in J}\Lambda_{ij})(\{f_j\}_{j\in J}),\{f_j\}_{j\in J}\right\rangle=\sum_{i\in I}\sum_{j\in J}\left\langle\Lambda_{ij}{}^*\Lambda_{ij}(f_j),f_j\right\rangle\\&=\sum_{i\in I}\sum_{j\in J}\|\Lambda_{ij}(f_j)\|^2=\sum_{j\in J}\sum_{i\in I}\|\Lambda_{ij}(f_j)\|^2=\sum_{j\in J}\left\langle\sum_{i\in I}\Lambda_{ij}{}^*\Lambda_{ij}(f_j),f_j\right\rangle\\&=\sum_{j\in J}\left\langle S_{\Phi_j}(f_j),f_j\right\rangle=\left\langle(\oplus_{j\in J}S_{\Phi_j})f,f\right\rangle,\end{aligned}$$

therefore $S_{\oplus_{j\in J}\Phi_j}=\oplus_{j\in J}S_{\Phi_j}$. ∎

Recall that a g-frame is called exact if it ceases to be a g-frame whenever any of its elements is removed. For more results about exact g-frames, see [13]. Now we have the following result:

Corollary 2.1. *Let $\{\Phi_j\}_{j\in J}$ be a bounded family of g-frames. If Φ_{j_0} is an exact g-frame, for some $j_0\in J$, then $\oplus_{j\in J}\Phi_j$ is exact.*

Proof. Suppose that $i_0\in I$ such that $\{\oplus_{j\in J}\Lambda_{ij}\}_{i\in I-\{i_0\}}$ is a g-frame. Then by Theorem 2.1, $\{\Lambda_{ij_0}\}_{i\in I-\{i_0\}}$ is a g-frame, which is a contradiction with the fact that Φ_{j_0} is exact. ∎

Theorem 2.2.

(a) *$\{\Phi_j\}_{j\in J}$ is an (A,B)-bounded family of g-Riesz bases if and only if $\oplus_{j\in J}\Phi_j$ is an (A,B) g-Riesz basis.*

(b) *Φ_j is a g-orthonormal basis, for each $j\in J$ if and only if $\oplus_{j\in J}\Phi_j$ is a g-orthonormal basis.*

Proof. (a) First let $\{\Phi_j\}_{j\in J}$ be an (A,B)-bounded family of g-Riesz bases. By [16, Corollary 3.2], each Φ_j is a g-Bessel sequence with upper bound B and therefore by Theorem 2.1, $\oplus_{j\in J}\Phi_j$ is a g-Bessel sequence and it is easy to see that $\oplus_{j\in J}\Phi_j$ is g-complete. Let F be a finite subset of I and let $\{g_{ij}\}_{j\in J}\in\oplus_{j\in J}H_{ij}$, for each $i\in F$. For proving that $\oplus_{j\in J}\Phi_j$ is an (A,B) g-Riesz basis, we must show that

$$A\sum_{i\in F}\left\|\{g_{ij}\}_{j\in J}\right\|^2\leq\left\|\sum_{i\in F}(\oplus_{j\in J}\Lambda_{ij}^*)(\{g_{ij}\}_{j\in J})\right\|^2\leq B\sum_{i\in F}\left\|\{g_{ij}\}_{j\in J}\right\|^2,$$

or equivalently

$$A\sum_{i\in F}\sum_{j\in J}\|g_{ij}\|^2\leq\sum_{j\in J}\left\|\sum_{i\in F}\Lambda_{ij}^*(g_{ij})\right\|^2\leq B\sum_{i\in F}\sum_{j\in J}\|g_{ij}\|^2.$$

Now since each Φ_j is an (A,B) g-Riesz basis, then we have

$$A\sum_{i\in F}\sum_{j\in J}\|g_{ij}\|^2=\sum_{j\in J}A\sum_{i\in F}\|g_{ij}\|^2\leq\sum_{j\in J}\left\|\sum_{i\in F}\Lambda_{ij}^*(g_{ij})\right\|^2,$$

and

$$B\sum_{i\in F}\sum_{j\in J}\|g_{ij}\|^2=\sum_{j\in J}B\sum_{i\in F}\|g_{ij}\|^2\geq\sum_{j\in J}\left\|\sum_{i\in F}\Lambda_{ij}^*(g_{ij})\right\|^2.$$

Conversely suppose that $\oplus_{j\in J}\Phi_j$ is an (A,B) g-Riesz basis and $j_0\in J$. It is easy to see that Φ_{j_0} is g-complete. Now let F be a finite subset of I and $f_{ij_0}\in H_{ij_0}$, for each $i\in F$. Then

$$\begin{aligned}A\sum_{i\in F}\left\|f_{ij_0}\right\|^2&=A\sum_{i\in F}\left\|\{\delta_{j_0,j}f_{ij_0}\}_{j\in J}\right\|^2\\&\leq\left\|\sum_{i\in F}(\oplus_{j\in J}\Lambda_{ij}^*)(\{\delta_{j_0,j}f_{ij_0}\}_{j\in J})\right\|^2=\left\|\sum_{i\in F}\Lambda_{ij_0}^*(f_{ij_0})\right\|^2,\end{aligned}$$

and

$$\begin{aligned}\left\|\sum_{i\in F}\Lambda_{ij_0}^*(f_{ij_0})\right\|^2&=\left\|\sum_{i\in F}(\oplus_{j\in J}\Lambda_{ij}^*)(\{\delta_{j_0,j}f_{ij_0}\}_{j\in J})\right\|^2\\&\leq B\sum_{i\in F}\left\|\{\delta_{j_0,j}f_{ij_0}\}_{j\in J}\right\|^2=B\sum_{i\in F}\left\|f_{ij_0}\right\|^2.\end{aligned}$$

This means that Φ_{j_0} is an (A,B) g-Riesz basis.

(b) It follows from Theorem 2.1 that Φ_j is a Parseval g-frame for each $j\in J$ if and only if $\oplus_{j\in J}\Phi_j$ is a Parseval g-frame. Now suppose that Φ_j is a g-orthonormal basis, for each $j\in J$. Let $i,\ell\in I$, $\{f_{ij}\}_{j\in J}\in\oplus_{j\in J}H_{ij}$ and $\{g_{\ell j}\}_{j\in J}\in\oplus_{j\in J}H_{\ell j}$. Then

$$\left\langle(\oplus_{j\in J}\Lambda_{ij}^*)(\{f_{ij}\}_{j\in J}),(\oplus_{j\in J}\Lambda_{\ell j}^*)(\{g_{\ell j}\}_{j\in J})\right\rangle=\sum_{j\in J}\left\langle\Lambda_{ij}^*(f_{ij}),\Lambda_{\ell j}^*(g_{\ell j})\right\rangle.$$

If $i\neq\ell$, then $\sum_{j\in J}\left\langle\Lambda_{ij}^*(f_{ij}),\Lambda_{\ell j}^*(g_{\ell j})\right\rangle=0$, and therefore

$$\left\langle(\oplus_{j\in J}\Lambda_{ij}^*)(\{f_{ij}\}_{j\in J}),(\oplus_{j\in J}\Lambda_{\ell j}^*)(\{g_{\ell j}\}_{j\in J})\right\rangle=0.$$

If $i=\ell$, then

$$\left\langle(\oplus_{j\in J}\Lambda_{ij}^*)(\{f_{ij}\}_{j\in J}),(\oplus_{j\in J}\Lambda_{\ell j}^*)(\{g_{\ell j}\}_{j\in J})\right\rangle=\sum_{j\in J}\left\langle f_{ij},g_{ij}\right\rangle=\left\langle\{f_{ij}\}_{j\in J},\{g_{ij}\}_{j\in J}\right\rangle,$$

so $\oplus_{j\in J}\Phi_j$ is a g-orthonormal basis. The converse is easy to verify. ∎

Note that [12, Proposition 2.16] and [1, Proposition 2.6] are special cases of Theorems 2.1 and 2.2.

3. Perturbations, duals and equivalences

we recall the following definitions from [6] and [12]:

Definition 3.1. *Let* $\Lambda=\{\Lambda_i\in L(H,H_i):i\in I\}$ *and* $\Gamma=\{\Gamma_i\in L(H,H_i):i\in I\}$ *be two sequences and* $0\leq\lambda_1,\lambda_2<1$.

(i) *Let* $\varepsilon>0$. *We say that* Γ *is a* $(\lambda_1,\lambda_2,\varepsilon)$*-perturbation of* Λ *if for each* $i\in I$ *and* $f\in H$, *we have*

$$\|\Lambda_i f-\Gamma_i f\|\leq\lambda_1\|\Lambda_i f\|+\lambda_2\|\Gamma_i f\|+\varepsilon\|f\|.$$

(ii) *Let* $\{c_i\}_{i\in I}$ *be a sequence of positive numbers such that* $\sum_{i\in I}c_i^2<\infty$. *We say that* Γ *is a* $(\lambda_1,\lambda_2,\{c_i\}_{i\in I})$*-perturbation of* Λ *if for each* $i\in I$ *and* $f\in H$, *we have*

$$\|\Lambda_i f-\Gamma_i f\|\leq\lambda_1\|\Lambda_i f\|+\lambda_2\|\Gamma_i f\|+c_i\|f\|.$$

Proposition 3.1. *Let $\{\Phi_j\}_{j\in J}$ and $\{\Psi_j\}_{j\in J}$ be bounded families of g-Bessel sequences. Then Ψ_j is a $(\lambda_1,\lambda_2,\varepsilon)$-perturbation of Φ_j, for each $j\in J$ if and only if $\oplus_{j\in J}\Psi_j$ is a $(\lambda_1,\lambda_2,\varepsilon)$-perturbation of $\oplus_{j\in J}\Phi_j$.*

Proof. First suppose that Ψ_j is a $(\lambda_1,\lambda_2,\varepsilon)$-perturbation of Φ_j, for each $j\in J$ and suppose that $f=\{f_j\}_{j\in J}\in\oplus_{j\in J}H_j$. Let F be a finite subset of J. Then for each $i\in I$, we have

$$\begin{aligned}\big\|\{(\Lambda_{ij}-\Gamma_{ij})f_j\}_{j\in F}\big\|_2 &\le \big\|\big\{\lambda_1\|\Lambda_{ij}f_j\|+\lambda_2\|\Gamma_{ij}f_j\|+\varepsilon\|f_j\|\big\}_{j\in F}\big\|_2\\ &\le \big\|\big\{\lambda_1\|\Lambda_{ij}f_j\|\big\}_{j\in F}\big\|_2+\big\|\big\{\lambda_2\|\Gamma_{ij}f_j\|\big\}_{j\in F}\big\|_2+\big\|\big\{\varepsilon\|f_j\|\big\}_{j\in F}\big\|_2\\ &\le \lambda_1\Big(\sum_{j\in J}\|\Lambda_{ij}f_j\|^2\Big)^{\frac12}+\lambda_2\Big(\sum_{j\in J}\|\Gamma_{ij}f_j\|^2\Big)^{\frac12}+\varepsilon\Big(\sum_{j\in J}\|f_j\|^2\Big)^{\frac12}\\ &=\lambda_1\big\|\oplus_{j\in J}\Lambda_{ij}f\big\|+\lambda_2\big\|\oplus_{j\in J}\Gamma_{ij}f\big\|+\varepsilon\|f\|.\end{aligned}$$

Since the above inequality holds for each finite subset of J, then we have

$$\begin{aligned}\big\|\oplus_{j\in J}\Lambda_{ij}f-\oplus_{j\in J}\Gamma_{ij}f\big\| &= \big\|\{(\Lambda_{ij}-\Gamma_{ij})f_j\}_{j\in J}\big\|_2\\ &\le \lambda_1\big\|\oplus_{j\in J}\Lambda_{ij}f\big\|+\lambda_2\big\|\oplus_{j\in J}\Gamma_{ij}f\big\|+\varepsilon\|f\|.\end{aligned}$$

This means that $\oplus_{j\in J}\Psi_j$ is a $(\lambda_1,\lambda_2,\varepsilon)$-perturbation of $\oplus_{j\in J}\Phi_j$.

For the converse it is enough to note that for each $i\in I$, $j_0\in J$ and $f_{j_0}\in H_{j_0}$ we can write

$$\begin{aligned}&\|\Lambda_{ij_0}f_{j_0}-\Gamma_{ij_0}f_{j_0}\|\\ &=\|(\oplus_{j\in J}\Lambda_{ij})(\{\delta_{j_0,j}f_{j_0}\}_{j\in J})-(\oplus_{j\in J}\Gamma_{ij})(\{\delta_{j_0,j}f_{j_0}\}_{j\in J})\|\\ &\le\lambda_1\|\oplus_{j\in J}\Lambda_{ij}(\{\delta_{j_0,j}f_{j_0}\}_{j\in J})\|+\lambda_2\|\oplus_{j\in J}\Gamma_{ij}(\{\delta_{j_0,j}f_{j_0}\}_{j\in J})\|+\varepsilon\|\{\delta_{j_0,j}f_{j_0}\}_{j\in J}\|\\ &=\lambda_1\|\Lambda_{ij_0}f_{j_0}\|+\lambda_2\|\Gamma_{ij_0}f_{j_0}\|+\varepsilon\|f_{j_0}\|,\end{aligned}$$

and the result follows. ∎

Corollary 3.1. *Let $\{\Phi_j\}_{j\in J}$ be a B-bounded (resp. an (A,B)-bounded, with $(1-\lambda_1)\sqrt{A}>(\sum_{i\in I}c_i^2)^{1/2}$) family of g-Bessel sequences (resp. g-frames) and Ψ_j be a $(\lambda_1,\lambda_2,\{c_i\}_{i\in I})$-perturbation of Φ_j,for each $j\in J$. Then $\oplus_{j\in J}\Psi_j$ and Ψ_j, for each $j\in J$, are g-Bessel sequences (resp. g-frames) and $\oplus_{j\in J}\Psi_j$ is a $(\lambda_1,\lambda_2,\{c_i\}_{i\in I})$-perturbation of $\oplus_{j\in J}\Phi_j$.*

Conversely if $\oplus_{j\in J}\Psi_j$ is a g-Bessel sequence and a $(\lambda_1,\lambda_2,\{c_i\}_{i\in I})$-perturbation of $\oplus_{j\in J}\Phi_j$, then Ψ_j is a $(\lambda_1,\lambda_2,\{c_i\}_{i\in I})$-perturbation of Φ_j, for each $j\in J$.

Proof. First let Ψ_j be a $(\lambda_1,\lambda_2,\{c_i\}_{i\in I})$-perturbation of Φ_j, for each $j\in J$. Then by [12, Proposition 4.3], Ψ_j is a g-Bessel sequence with upper bound $\big(((1+\lambda_1)\sqrt{B}+(\sum_{i\in I}c_i^2)^{1/2})/(1-\lambda_2)\big)^2$, for each $j\in J$. Therefore by Theorem 2.1, $\oplus_{j\in J}\Psi_j$ is a g-Bessel sequence. If $\{\Phi_j\}_{j\in J}$ is an (A,B)-bounded family of g-frames with $(1-\lambda_1)\sqrt{A}>(\sum_{i\in I}c_i^2)^{1/2}$, then by [12, Proposition 4.3], $\big(((1-\lambda_1)\sqrt{A}-(\sum_{i\in I}c_i^2)^{1/2})/(1+\lambda_2)\big)^2$ is a lower bound for Ψ_j, for each $j\in J$. Hence by Theorem 2.1, $\oplus_{j\in J}\Psi_j$ is a g-frame. Now the rest of the proof can be obtained similar to the proof of Proposition 3.1 by using c_i instead of ε, for each $i\in I$. ∎

It was shown in [12, Definition 2.10] that if $\{\Lambda_i\in L(H,H_i):i\in I\}$ and $\{\Gamma_i\in L(H,H_i):i\in I\}$ are g-Bessel sequences with upper bounds B and D, respectively, then $\sum_{i\in I}\Gamma_i^*\Lambda_i(f)$ converges and $\|\sum_{i\in I}\Gamma_i^*\Lambda_i(f)\|\le\sqrt{BD}\|f\|$, for each $f\in H$. Therefore if $\{\Phi_j\}_{j\in J}$ and $\{\Psi_j\}_{j\in J}$ are bounded families of g-Bessel sequences, then the operator $\sum_{i\in I}(\oplus_{j\in J}\Gamma_{ij}^*)(\oplus_{j\in J}\Lambda_{ij})$ is bounded on $\oplus_{j\in J}H_j$.

Proposition 3.2. *Let $\{\Phi_j\}_{j\in J}$ and $\{\Psi_j\}_{j\in J}$ be B and D-bounded families of g-Bessel sequences, respectively. Then Ψ_j is a dual of Φ_j, for each $j \in J$ if and only if $\oplus_{j\in J}\Psi_j$ is a dual of $\oplus_{j\in J}\Phi_j$.*

Proof. Let Ψ_j be a dual of Φ_j for each $j \in J$, $f = \{f_j\}_{j\in J} \in \oplus_{j\in J}H_j$ and $j \in J$. Then

$$\sum_{i\in I}\left|\left\langle \Lambda_{ij}f_j, \Gamma_{ij}f_j\right\rangle\right| \le \left(\sum_{i\in I}\left\|\Lambda_{ij}f_j\right\|^2\right)^{\frac{1}{2}}\left(\sum_{i\in I}\left\|\Gamma_{ij}f_j\right\|^2\right)^{\frac{1}{2}} \le \sqrt{BD}\|f_j\|^2,$$

so $\sum_{i\in I}\left|\left\langle \Lambda_{ij}f_j, \Gamma_{ij}f_j\right\rangle\right|$ converges, for each $j \in J$. Also

$$\sum_{j\in J}\sum_{i\in I}\left|\left\langle \Lambda_{ij}f_j, \Gamma_{ij}f_j\right\rangle\right| \le \sqrt{BD}\sum_{j\in J}\|f_j\|^2 = \sqrt{BD}\|f\|^2,$$

therefore $\sum_{j\in J}\sum_{i\in I}\left|\left\langle \Lambda_{ij}f_j, \Gamma_{ij}f_j\right\rangle\right|$ converges. Hence

$$\sum_{j\in J}\sum_{i\in I}\left\langle \Lambda_{ij}f_j, \Gamma_{ij}f_j\right\rangle = \sum_{i\in I}\sum_{j\in J}\left\langle \Lambda_{ij}f_j, \Gamma_{ij}f_j\right\rangle.$$

Now we have

$$\begin{aligned}
&\left\langle \sum_{i\in I}(\oplus_{j\in J}\Gamma^*_{ij})(\oplus_{j\in J}\Lambda_{ij})(\{f_j\}_{j\in J}), \{f_j\}_{j\in J}\right\rangle \\
&= \sum_{i\in I}\left\langle \{\Gamma^*_{ij}\Lambda_{ij}f_j\}_{j\in J}, \{f_j\}_{j\in J}\right\rangle = \sum_{i\in I}\sum_{j\in J}\left\langle \Lambda_{ij}f_j, \Gamma_{ij}f_j\right\rangle = \sum_{j\in J}\sum_{i\in I}\left\langle \Lambda_{ij}f_j, \Gamma_{ij}f_j\right\rangle \\
&= \sum_{j\in J}\left\langle \sum_{i\in I}\Gamma^*_{ij}\Lambda_{ij}f_j, f_j\right\rangle = \sum_{j\in J}\left\langle f_j, f_j\right\rangle = \left\langle \{f_j\}_{j\in J}, \{f_j\}_{j\in J}\right\rangle,
\end{aligned}$$

therefore $\sum_{i\in I}(\oplus_{j\in J}\Gamma^*_{ij})(\oplus_{j\in J}\Lambda_{ij})f = f$, for each $f \in \oplus_{j\in J}H_j$, and this means that $\oplus_{j\in J}\Psi_j$ is a dual of $\oplus_{j\in J}\Phi_j$. Conversely suppose that $\oplus_{j\in J}\Psi_j$ is a dual of $\oplus_{j\in J}\Phi_j$. Let $j_0 \in J$ and $f_{j_0} \in H_{j_0}$. Now we have

$$\begin{aligned}
\left\langle \sum_{i\in I}\Gamma^*_{ij_0}\Lambda_{ij_0}f_{j_0}, f_{j_0}\right\rangle &= \left\langle \sum_{i\in I}\left(\oplus_{j\in J}\Gamma^*_{ij}\right)\left(\oplus_{j\in J}\Lambda_{ij}\right)\left(\{\delta_{j_0,j}f_{j_0}\}_{j\in J}\right), \left\{\delta_{j_0,j}f_{j_0}\right\}_{j\in J}\right\rangle \\
&= \left\langle \left\{\delta_{j_0,j}f_{j_0}\right\}_{j\in J}, \left\{\delta_{j_0,j}f_{j_0}\right\}_{j\in J}\right\rangle = \left\langle f_{j_0}, f_{j_0}\right\rangle,
\end{aligned}$$

therefore $\sum_{i\in I}\Gamma^*_{ij_0}\Lambda_{ij_0}f_{j_0} = f_{j_0}$. This means that Ψ_{j_0} is a dual of Φ_{j_0}. ∎

Now we have the following result for canonical duals.

Proposition 3.3. *Let $\{\Phi_j\}_{j\in J}$ be an (A,B)-bounded family of g-frames. Then $\oplus_{j\in J}\widetilde{\Phi_j}$ is a g-frame and $\widetilde{\oplus_{j\in J}\Phi_j} = \oplus_{j\in J}\widetilde{\Phi_j}$.*

Proof. Since $\widetilde{\Phi_j}$ is an $(1/B_j, 1/A_j)$ g-frame, for each $j \in J$ and $\inf\{1/B_j : j \in J\} = 1/B > 0$, $\sup\{1/A_j : j \in J\} = 1/A < \infty$, then $\oplus_{j\in J}\widetilde{\Phi_j}$ is an $(1/B, 1/A)$ g-frame, by Theorem 2.1. Moreover as a consequence of Theorem 2.1, we can see that $\widetilde{\oplus_{j\in J}\Phi_j} = \{\oplus_{j\in J}\Lambda_{ij}(\oplus_{j\in J} S_{\Phi_j})^{-1} : i \in I\}$. Now by using the definition of canonical duals, it is clear that $\oplus_{j\in J}\widetilde{\Phi_j} = \{\oplus_{j\in J}\Lambda_{ij}S^{-1}_{\Phi_j} \in L(\oplus_{j\in J}H_j, \oplus_{j\in J}H_{ij}) : i \in I\}$. Thus it is enough to show that $\oplus_{j\in J}\Lambda_{ij}(\oplus_{j\in J} S_{\Phi_j})^{-1} = \oplus_{j\in J}\Lambda_{ij}S^{-1}_{\Phi_j}$, for each $i \in I$. Since $A \cdot Id_{H_j} \le S_{\Phi_j} \le B \cdot Id_{H_j}$, for each $j \in J$, then by [14, Theorem 2.2.5], we have $(1/B)\cdot Id_{H_j} \le S^{-1}_{\Phi_j} \le (1/A)\cdot Id_{H_j}$ and therefore $\|S^{-1}_{\Phi_j}\| \le$

$1/A$, for each $j \in J$. Thus $\oplus_{j\in J} S_{\Phi_j}^{-1}$ is a bounded operator. Now it is easy to see that $(\oplus_{j\in J} S_{\Phi_j})^{-1} = \oplus_{j\in J} S_{\Phi_j}^{-1}$, so for each $\{f_j\}_{j\in J} \in \oplus_{j\in J} H_j$, we have

$$\oplus_{j\in J}\Lambda_{ij}\big(\oplus_{j\in J} S_{\Phi_j}\big)^{-1}\big(\{f_j\}_{j\in J}\big) = \big\{\Lambda_{ij} S_{\Phi_j}^{-1}(f_j)\big\}_{j\in J} = \oplus_{j\in J}\Lambda_{ij} S_{\Phi_j}^{-1}\big(\{f_j\}_{j\in J}\big),$$

which completes the proof. ∎

Now we recall some definitions for g-frames from [15].

Definition 3.2. *Let $\Lambda = \{\Lambda_i \in L(H,H_i) : i \in I\}$ and $\Gamma = \{\Gamma_i \in L(H,H_i) : i \in I\}$ be two g-frames.*

(i) *We say that Λ and Γ are unitarily equivalent if there is a unitary linear operator $T : H \longrightarrow H$ such that $\Gamma_i = \Lambda_i T$, for each $i \in I$.*

(ii) *We say that Λ is isometrically related to Γ if there is an isometric linear operator $T : H \longrightarrow H$ such that $\Gamma_i = \Lambda_i T$, for each $i \in I$.*

Proposition 3.4. *Let $\{\Phi_j\}_{j\in J}$ and $\{\Psi_j\}_{j\in J}$ be bounded families of g-frames. Then*

(i) *If Φ_j and Ψ_j are unitarily equivalent, for each $j \in J$, then $\oplus_{j\in J}\Phi_j$ and $\oplus_{j\in J}\Psi_j$ are unitarily equivalent.*

(ii) *If Φ_j is isometrically related to Ψ_j, for each $j \in J$, then $\oplus_{j\in J}\Phi_j$ is isometrically related to $\oplus_{j\in J}\Psi_j$.*

Proof. (i) Suppose that Φ_j and Ψ_j are unitarily equivalent, for each $j \in J$ and $T_j : H_j \longrightarrow H_j$ is a unitary operator such that $\Gamma_{ij} = \Lambda_{ij} T_j$, for each $i \in I$. Define $T : \oplus_{j\in J} H_j \longrightarrow \oplus_{j\in J} H_j$ by $T = \oplus_{j\in J} T_j$. Since $\|T\| = sup\{\|T_j\| : j \in J\} = 1$, then T is bounded. Now it is easy to see that T is unitary and $\oplus_{j\in J}\Gamma_{ij} = (\oplus_{j\in J}\Lambda_{ij})T$, for each $i \in I$.

(ii) Suppose that Φ_j is isometrically related to Ψ_j, for each $j \in J$ and $T_j : H_j \longrightarrow H_j$ is an isometric operator such that $\Gamma_{ij} = \Lambda_{ij} T_j$, for each $i \in I$. Define $T : \oplus_{j\in J} H_j \longrightarrow \oplus_{j\in J} H_j$ by $T = \oplus_{j\in J} T_j$. Since $\|T\| = sup\{\|T_j\| : j \in J\} = 1$, then T is bounded. Now for each $f = \{f_j\}_{j\in J} \in \oplus_{j\in J} H_j$, we have

$$\|Tf\| = \left(\sum_{j\in J} \left\|T_j f_j\right\|^2\right)^{\frac{1}{2}} = \left(\sum_{j\in J} \left\|f_j\right\|^2\right)^{\frac{1}{2}} = \|f\|,$$

so T is an isometry. It is also easy to see that $\oplus_{j\in J}\Gamma_{ij} = (\oplus_{j\in J}\Lambda_{ij})T$, for each $i \in I$. ∎

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