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Minimal Matrix Representations of Four-Dimensional Lie Algebras

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Abstract. It is known how to find minimal dimension matrix representations for four-dimensional complex Lie algebras. The method depends on constructing left symmetric structures. In this note it is explained how to obtain the representations directly and also how to extend the results to real Lie algebras. Two different bases for the four-dimensional Lie algebras are related to each other.

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1. Introduction

Given a Lie algebra $\mathfrak{g}$ of dimension n a well known theorem due to Ado asserts that $\mathfrak{g}$ has a faithful representation as a subalgebra of $gl(p,\mathbf{R})$ for *some* p. The theorem does not give much information about the value of p but leads one to believe that p may be very large in relation to the size of n and consequently it seems to be of limited practical value. Burde [1] defines the invariant $\mu(\mathfrak{g})$ to be the minimum value of p. A little care must be exercised because there may well be inequivalent representations for which this minimum value is attained. Of course if $\mathfrak{g}$ has a trivial center then the adjoint representation furnishes a faithful representation of $\mathfrak{g}$ and in the notation used above $\mu(\mathfrak{g}) \leq n$. Nonetheless many algebras have non-trivial centers, nilpotent algebras for example, and then the adjoint representation is not faithful.

In a recent paper Kang and Bai [5] considered the problem of finding minimal dimension matrix representations for four-dimensional complex Lie algebras. Their main technique is somewhat indirect depending on a construct known as a *left symmetric structure*, an idea which has been pioneered by Burde [1]. If an n-dimensional Lie algebra admits such a structure then $\mu(\mathfrak{g}) \leq n+1$. In this note, which should properly be seen as complementing Kang and Bai [5], we shall show that one can easily obtain the results directly without the need for considering left symmetric structures. We also adapt the techniques to include real Lie algebras. Lastly although Kang and Bai do give representations, their results

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are somewhat complicated because the bases used for some of the algebras are not optimal. These bases are quoted from Burde and Steinhoff [2] who in turn refer to Patera and Zassenhaus [9]. We believe that these alternative bases were introduced for the purpose of handling algebras over finite fields. Such algebras are not considered here nor were they considered by Kang and Bai. We follow the classification of the four-dimensional real Lie algebras obtained by G. Mubarakzyanov [7] and repeated in Patera *et al.* [8]. We give in Section 3, to the extent possible, explicit isomorphisms between the two versions of the algebras and we hope that they will prove to be of interest above and beyond the question of finding minimal representations.

Kang and Bai studied both decomposable and indecomposable algebras. As regards decomposable algebras if $\mathfrak{g} = \mathfrak{g}_1 \bigoplus \mathfrak{g}_2$ obviously $\mu(\mathfrak{g}) \leq \mu(\mathfrak{g}_1) + \mu(\mathfrak{g}_2)$. However, we frequently have $\mu(\mathfrak{g}) < \mu(\mathfrak{g}_1) + \mu(\mathfrak{g}_2)$. For example $\mathfrak{sl}(2,\mathbb{C}) \bigoplus \mathbb{C} \simeq \mathfrak{gl}(2,\mathbb{C})$. In fact a summand of a solvable algebra $\mathfrak{g}$ with $\mathbb{C}$ can be obtained by adding multiples of the identity to a representation for $\mathfrak{g}$. The only slightly surprising case is that $\mu(r_2(\mathbb{C}) \bigoplus r_2(\mathbb{C})) = 3$ and not 4. We can understand this result by considering the Lie algebra of the four-dimensional matrix Lie group

$$\begin{bmatrix} e^x & 0 & y \\ 0 & e^w & z \\ 0 & 0 & 1 \end{bmatrix}.$$

Henceforth we shall assume that the Lie algebras considered are indecomposable over $\mathbb{R}$.

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2. Real and complex four-dimensional Lie algebras

First of all we make some general comments about *indecomposable* four-dimensional Lie algebras. Such algebras are of four types:

- Nilpotent, that is, four-dimensional nilradical
- Three-dimensional abelian nilradical
- Three-dimensional non-abelian nilradical isomorphic to the Heisenberg algebra
- Two-dimensional abelian nilradical

The first and fourth types here are unique up to isomorphism. In fact the *real* four-dimensional Lie algebras were classified by G. Mubarakzyanov [7]. They can be found easily in Patera *et al.* [8] and we repeat them in the Appendix for the reader's convenience giving the non-zero brackets in the basis $\{e_1, e_2, e_3, e_4\}$ in each case. We give in each case also the dimension of a minimal representation in the real $\mu_{\mathbb{R}}$ and complex contexts $\mu_{\mathbb{C}}$, respectively.

We shall not take the time to redo Mubarakzyanov's classification here except to make two remarks. First of all, algebras $4.1 - 4.6$ above can be easily understood because they have a three-dimensional abelian ideal $\imath$, which in the case of algebras $4.2 - 4.6$ is the nilradical. These algebras are classified, up to isomorphism, by the projective class of the Jordan normal form of $\mathrm{ad}(e)$ where e spans a complement to $\imath$. As regards algebras $4.7 - 4.11$ above they can be obtained by noting that there is a one-one correspondence between three-dimensional algebras that have an abelian nilradical and four-dimensional algebras that have the Heisenberg algebra as nilradical. Unfortunately such a simple correspondence does not exist in higher dimensions.

Now let us consider the *complex* four-dimensional Lie algebras. All we have to do is to start with the real algebras and use the same brackets but now with complex coefficients.

As such 4.6 is equivalent over $\mathbb{C}$ to

$$[e_1,e_4]=ae_1,[e_2+ie_3,e_4]=(b+i)(e_2+ie_3),[e_2-ie_3,e_4]=(b-i)(e_2-ie_3)$$

that is, after scaling and permuting, a special case of 4.5. Algebra 4.10 is equivalent over $\mathbb{C}$ to 4.8 via the transformation:

$$\overline{e_1}=-2ie_1,\overline{e_2}=e_2+ie_3,\overline{e_3}=e_2-ie_3,\overline{e_4}=-ie_4$$

Furthermore 4.11 is equivalent over $\mathbb{C}$ to 4.9 via

$$\overline{e_1}=\frac{i}{2a}e_1,\ \overline{e_2}=\frac{1}{2a}(e_2+ae_3),\ \overline{e_3}=\frac{-i}{2a}(e_2-ae_3),\ \overline{e_4}=(a+i)e_4$$

and putting $b=a+i$. Finally 4.12 is decomposable over $\mathbb{C}$: note that $[e_1+ie_2,e_3+ie_4]=0$. Thus over $\mathbb{C}$ we only have to consider algebras 4.1, 4.2, 4.3, 4.4, 4.5, 4.7, 4.8, 4.9. Notice that all these algebras have a three-dimensional nilradical and for 4.1, 4.2, 4.3, 4.4, 4.5 it is abelian whereas for algebras 4.7, 4.8, 4.9 the nilradical is isomorphic to the Heisenberg algebra.

We remark finally that many other authors have considered the four-dimensional Lie algebras, not least Lie himself. In [6] the Mubarakzyanov list 4.1, 4.2, 4.3, 4.4, 4.5, 4.7, $4.8-4.9$ corresponds in order to 69, 72, 68, 70, 67, 66, 62 allowing for some slightly different normalizations of the constants. It is interesting that Lie does not distinguish between real and complex coefficients.

3. An alternative description of four-dimensional Lie algebras

Another description of the four-dimensional complex Lie algebras appears in the literature. It comprises nine classes of algebra in comparison to the eight classes that follow from Mubarakzyanov's classification. We repeat it from Kang and Bai [5] as follows.

$$\begin{aligned}
\mathfrak{n}(\mathbb{C}) &: [e_1,e_2]=e_3,[e_1,e_3]=e_4\\
\mathfrak{g}_1(\alpha) &: [e_1,e_2]=e_2,[e_1,e_3]=e_3,[e_1,e_4]=\alpha e_4,\alpha\in\mathbb{C}^*\\
\mathfrak{g}_2(\alpha,\beta) &: [e_1,e_2]=e_3,[e_1,e_3]=e_4,\ [e_1,e_4]=\alpha e_2+\beta e_3+e_4,\alpha\in\mathbb{C}^*,\beta\in\mathbb{C}\text{ or }\alpha=\beta=0\\
\mathfrak{g}_3(\alpha) &: [e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=\alpha(e_2+e_3),\alpha\in\mathbb{C}^*\\
\mathfrak{g}_4 &: [e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_2\\
\mathfrak{g}_5 &: [e_1,e_2]=1/3e_2+e_3,[e_1,e_3]=1/3e_3,[e_1,e_4]=1/3e_4\\
\mathfrak{g}_6 &: [e_1,e_2]=e_2,[e_1,e_3]=e_3,[e_1,e_4]=2e_4,[e_2,e_3]=e_4\\
\mathfrak{g}_7 &: [e_1,e_2]=e_3,[e_1,e_3]=e_2,[e_2,e_3]=e_4\\
\mathfrak{g}_8 &: [e_1,e_2]=e_3,[e_1,e_3]=e_3-\alpha e_2,[e_1,e_4]=e_4,[e_2,e_3]=e_4,\ \alpha\in\mathbb{C}.
\end{aligned}$$

We now compare the list above with Mubarakzyanov's list of Section 2.

- $\mathfrak{n}_4(\mathbb{C})$ is isomorphic to Mubarakzyanov algebra 1
- $\mathfrak{g}_1(\alpha)$ is isomorphic to Mubarakzyanov 5 in the special case $\alpha=a, b=1$
- In the generic case, where the discriminant $-18\alpha\beta-4\alpha+\beta^2+4\beta^3-27\alpha^2\neq 0$, the matrix $\mathrm{ad}(e_1)$ will be diagonalizable and $\mathfrak{g}_2(\alpha,\beta)$ will be equivalent to a case of Mubarakzyanov 5. If, however, there are just three distinct eigenvalues for $\mathrm{ad}(e_1)$, including zero, then $\mathfrak{g}_2(\alpha,\beta)$ will be equivalent to a special case of algebra 2. $\mathfrak{g}_2(0,0)$ is isomorphic to Mubarakzyanov 3 via the transformation

$$e_1' = e_4, \quad e_2' = e_3 - e_4, \quad e_3' = e_2 - e_4, \quad e_4' = -e_1.$$

$\mathfrak{g}_2(1/27, -1/3)$ is isomorphic to Mubarakzyanov 4 via the transformation

$$e_1' = -\tfrac{1}{9}e_2 + \tfrac{2}{3}e_3 - e_4, \quad e_2' = \tfrac{1}{9}e_2 - \tfrac{1}{3}e_3, \quad e_3' = -\tfrac{1}{9}e_2, \quad e_4' = -3e_1.$$

- For each α except $\alpha = 27/4$, the algebra $\mathfrak{g}_3(\alpha)$ is isomorphic to an algebra of type Mubarakzyanov 5. For $\alpha = 27/4$ the algebra $\mathfrak{g}_3(27/4)$ is isomorphic to an algebra of type Mubarakzyanov 2 in the special case $a = -2$ via the transformation
$$e_1' = \tfrac{1}{9}e_2 + \tfrac{4}{27}e_3 + \tfrac{4}{81}e_4, \quad e_2' = e_2 + \tfrac{1}{3}e_3 - \tfrac{2}{9}e_4, \quad e_3' = -\tfrac{4}{3}e_2, \quad e_4' = -3e_1.$$
- $\mathfrak{g}_4$ is isomorphic to a special case of Mubarakzyanov 5: $[e_1, e_4] = e_1, [e_2, e_4] = -\overline{\omega}e_2, [e_3, e_4] = -\omega e_3$ where $\omega = 1/2(1 + \sqrt{3}i)$ via the transformation
$$e_1' = e_2 + e_3 + e_4,\ e_2' = -\omega e_2 - \overline{\omega}e_3 + e_4,\ e_3' = -\overline{\omega}e_2 - \omega e_3 + e_4,\ e_4' = -e_1.$$
- $\mathfrak{g}_5$ is isomorphic to Mubarakzyanov 2 in the special case $a = 1$, via the transformation
$$e_1' = -\tfrac{1}{3}e_4, \quad e_2' = e_3, \quad e_3' = \tfrac{1}{2}e_2, \quad e_4' = e_1.$$
- $\mathfrak{g}_6$ is isomorphic to Mubarakzyanov 9 for $b = 1$, via the transformation
$$e_1' = -e_4, \quad e_2' = e_3, \quad e_3' = e_2, \quad e_4' = -e_1.$$
- $\mathfrak{g}_7$ is isomorphic to Mubarakzyanov 8 via the transformation
$$e_1' = e_4, \quad e_2' = e_2 + e_3, \quad e_3' = \tfrac{1}{2}(e_3 - e_2), \quad e_4' = -e_1.$$
- $\mathfrak{g}_8(1/4)$ is equivalent to Mubarakzyanov 7 via the transformation
$$e_1' = -2e_4, \quad e_2' = e_2 - 2e_3, \quad e_3' = -2e_3, \quad e_4' = -2e_1.$$
$\mathfrak{g}_8(0)$ is equivalent to Mubarakzyanov $9(b = 0)$ via the transformation
$$e_1' = e_4, \quad e_2' = -e_3, \quad e_3' = e_2 - e_3, \quad e_4' = e_1.$$
For $\alpha \neq 0, 1/4$ the algebra $\mathfrak{g}_8(\alpha)$ is equivalent to Mubarakzyanov $9b$ where $b = (1 + \sqrt{1-4\alpha})/(1 - \sqrt{1-4\alpha})$ via the transformation
$$e_1' = e_4,\ e_2' = \tfrac{-\alpha}{\sqrt{1-4\alpha}}e_2 + \tfrac{1-\sqrt{1-4\alpha}}{2\sqrt{1-4\alpha}}e_3,\ e_3' = e_2 - \tfrac{1+\sqrt{1-4\alpha}}{2\alpha}e_3,\ e_4' = \tfrac{2}{\sqrt{1-4\alpha}-1}e_1.$$

4. Minimal representations for four-dimensional complex algebras

Let us consider solvable four-dimensional complex algebras that have faithful representations in $\mathfrak{gl}(3, \mathbb{C})$. Then by Lie's theorem we may assume, because the ground field is $\mathbb{C}$, that the representation is in the 3×3 upper triangular matrices. The commutator of two such matrices lies in the strictly upper triangular matrices. It follows that the derived algebra is at most three-dimensional with equality if and only if it is isomorphic to the Heisenberg algebra. Hence algebras 4.2, 4.4 and 4.5 cannot be represented in $\mathfrak{gl}(3, \mathbb{C})$. It is important to understand that one cannot assume a priori that the nilradical is represented in the strictly upper triangular matrices. For example for all values of a and b the matrix $e^{ax+by}\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$ gives a group representation whose Lie algebra is isomorphic to Heisenberg.

In the case of algebras 4.1 and 4.3 we can argue as follows: if either algebra could be represented in $\mathfrak{gl}(3, \mathbb{C})$ it could actually be represented in $\mathfrak{sl}(3, \mathbb{C})$. By Lie's theorem we may assume, because the ground field is $\mathbb{C}$, that the representation is in the 3×3 upper triangular matrices. We are at liberty to modify the matrices corresponding to e_3 and e_4 by adding multiples of the identity. Furthermore e_1 and e_2, being commutators, must be strictly upper triangular matrices and so have trace zero. Thus algebras 4.1 and 4.3 would have to be four-dimensional subalgebras of the space of trace-free upper triangular matrices which is denoted as algebra 5.36 in Patera *et al.* [8].

Now suppose that we take as a basis for algebra 5.36 part of the standard representation for $\mathfrak{sl}(3,\mathbb{C})$, that is,

$$X_1 = \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0&0&0\\0&0&1\\0&0&0 \end{bmatrix} \quad X_3 = \begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix} \quad H_1 = \begin{bmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{bmatrix} \quad H_2 = \begin{bmatrix} 0&0&0\\0&1&0\\0&0&-1 \end{bmatrix}.$$

Then clearly $\mathrm{ad}(X_1), \mathrm{ad}(X_2), \mathrm{ad}(X_3)$ are nilpotent whereas $\mathrm{ad}(H_1), \mathrm{ad}(H_2)$ are semi-simple: this fact is a consequence of lemma A page 18 in Humphreys [4] but it can easily be seen directly. It follows that there is no four-dimensional nilpotent subalgebra of algebra 5.36 and so algebra 4.1 cannot be represented in 5.36. As regards 4.3 we can only obtain a four-dimensional subalgebra of 5.36 that has a three-dimensional nilradical by taking a fixed linear combination of H_1 and H_2 as a generator which would mean that the nilradical would be Heisenberg and not abelian. Thus algebra 4.3 cannot be represented in 5.36 either. *It follows that only algebras* 4.7, 4.8 *and* 4.9 *can be represented in* $\mathfrak{gl}(3,\mathbb{C})$. Now it is known from Ghanam *et al.* [3]' for all the four-dimensional algebras $\mathfrak{g}$ that $\mu(\mathfrak{g}) \leq 4$. We shall show below explicitly for algebras 4.8 and 4.9 that actually $\mu(\mathfrak{g}) = 3$. Therefore the only case that remains in doubt is algebra 4.7.

Continuing from the beginning of this Section we note that algebra 4.7 has three-dimensional derived algebra. Such a four-dimensional solvable complex algebra that can be represented in $\mathfrak{gl}(3,\mathbb{C})$ may be assumed to have generators of the form:

$$(4.1)\; E_1 = \begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix} \quad E_3 = \begin{bmatrix} 0&0&0\\0&0&1\\0&0&0 \end{bmatrix} \quad E_4 = \begin{bmatrix} a&d&e\\0&b&f\\0&0&c \end{bmatrix}.$$

But then

$$[E_1,E_4] = (c-a)E_1, [E_2,E_4] = fE_1 + (b-a)E_2, [E_3,E_4] = -dE_1 + (c-b)E_3$$

and we see that is impossible to choose a,b,c,d,e,f so as to obtain algebra 7 and so for it $\mu = 4$. More precisely, by adding multiples of E_1 to E_2 it is possible to reduce to the case $d = f = 0$ and then $\mathrm{ad}(E_4)$ is semi-simple; however, in algebra 4.7 the matrix $\mathrm{ad}(e_4)$ is not semi-simple but has a nilpotent part that cannot be removed by change of basis. Thus algebra 4.7 cannot be represented in $\mathfrak{gl}(3,\mathbb{C})$.

To obtain representations we note that algebras $4.2-4.5$ and 4.7 have trivial center and therefore the adjoint representation suffices to give a representation for which $p = 4$ in the notation of the first paragraph of Section 1. For algebra 4.1, a filiform Lie algebra, it is well known that it arises from the matrix Lie group of the form $\begin{bmatrix} 1&w&\frac{w^2}{2}&x\\0&1&w&y\\0&0&1&z\\0&0&0&1 \end{bmatrix}$.

Finally the following matrix Lie group gives a three-dimensional representation for algebra 4.9: to obtain algebra 4.8 we need only set $b = -1$. The coordinates are taken in the order (x,y,z,w).

$$\begin{bmatrix} e^{(b+1)w} & ye^{bw} & x \\ 0 & e^{bw} & z \\ 0 & 0 & 1 \end{bmatrix}.$$

5. Minimal representations for four-dimensional real algebras

Let us now consider the minimal representations for four-dimensional real algebras. Immediately by comparing with the complex algebras we can assert for algebras 4.1, 4.2, 4.3, 4.4, 4.5 and 4.7 that $\mu = 4$ and we use the same representations as in the complex cases but with real numbers in place of complex numbers. Similarly in cases 4.8 and 4.9 we conclude that $\mu = 3$ and obtain the real representations from the complex ones. It remains to discuss the real algebras 4.6, 4.10, 4.11 and 4.12.

As regards algebra 4.6 suppose that it has a faithful representation in $\mathfrak{gl}(3,\mathbb{R})$. Then Lie's theorem is still applicable but the representing matrices may have some strictly complex entries. However, if we apply Lie's theorem just to the nilradical it can be represented in the strictly upper triangular 3×3 matrices with real entries. It follows that the nilradical is isomorphic to Heisenberg and in particular not abelian. Hence there can be no representation in $\mathfrak{gl}(3,\mathbb{R})$. Furthermore the algebra has trivial center and so $\mu = 4$ and the adjoint representation gives a representation for which $p = 4$.

If algebra 4.10 or 4.11 has a faithful representation in $\mathfrak{gl}(3,\mathbb{R})$ then just as in the preceding paragraph the nilradical can be put into real strictly upper triangular form by a real transformation. Thus without loss of generality we may assume that e_1, e_2 and e_3 are represented by E_1, E_2 and E_3 as in equation 1. Of course we have no control over the matrix E_4 that represents e_4 but it does not matter: it is impossible to satisfy either of the brackets $[e_2,e_4] = ae_2 - e_3, [e_3,e_4] = e_2 + ae_3$. In fact with E_1, E_2 and E_3 as in equation 1 and E_4 arbitrary the matrix $[E_2,E_4] - aE_2 + E_3$ has $(2,3)$-entry 1. Hence there can be no representation in $\mathfrak{gl}(3,\mathbb{R})$. In these two cases the following matrix Lie group has Lie algebra isomorphic to algebra 4.11 and for $a = 0$ algebra 4.10:

$$\begin{bmatrix} e^{2aw} & -e^{aw}(x\sin w + y\cos w) & e^{aw}(x\cos w - y\sin w) & z \\ 0 & e^{aw}\cos w & e^{aw}\sin w & x \\ 0 & -e^{aw}\sin w & e^{aw}\cos w & y \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The coordinates are taken in the order (z,x,y,w). Thus for algebras 4.10 and 4.11 we have $\mu = 4$.

Finally for algebra 4.12 it arises as the Lie algebra of the following matrix Lie group where the coordinates are taken in the order (x,y,z,w):

$$\begin{bmatrix} e^z\cos w & e^z\sin w & x \\ -e^z\sin w & e^z\cos w & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus for algebra 4.12 we have $\mu = 3$.

To summarize: over $\mathbb{C}$ for algebras $4.1 - 4.5$ and 4.7, $\mu = 4$ and for algebras 4.8 and 4.9, $\mu = 3$. Over $\mathbb{R}$ for algebras $4.1 - 4.7$, 4.10 and 4.11, $\mu = 4$ and for algebras 4.8, 4.9 and 4.12, $\mu = 3$. The results are also displayed in the following Appendix.

Appendix: The four-dimensional indecomposable real Lie algebras

algebra	non-zero brackets	$\mu_{\mathbb{R}}$	$\mu_{\mathbb{C}}$
$g_{4,1}$	$[e_2,e_4]=e_1,[e_3,e_4]=e_2$	4	4
$g_{4,2}$	$[e_1,e_4]=ae_1,[e_2,e_4]=e_2,[e_3,e_4]=e_2+e_3$	4	4
$g_{4,3}$	$[e_1,e_4]=e_1,[e_3,e_4]=e_2$	4	4
$g_{4,4}$	$[e_1,e_4]=e_1,[e_2,e_4]=e_1+e_2,[e_3,e_4]=e_2+e_3$	4	4
$g_{4,5}$	$(a\neq 0,b\neq 0)$: $[e_1,e_4]=e_1,[e_2,e_4]=ae_2,[e_3,e_4]=be_3$	4	4
$g_{4,6}$	$(a\neq 0,b\geq 0)$: $[e_1,e_4]=ae_1,[e_2,e_4]=be_2-e_3,[e_3,e_4]=e_2+be_3$	4	4
$g_{4,7}$	$[e_2,e_3]=e_1,[e_1,e_4]=2e_1,[e_2,e_4]=e_2,[e_3,e_4]=e_2+e_3$	4	4
$g_{4,8}$	$[e_2,e_3]=e_1,[e_2,e_4]=e_2,[e_3,e_4]=-e_3$	3	3
$g_{4,9}$	$(-1<b\leq 1)$: $[e_2,e_3]=e_1,[e_1,e_4]=(b+1)e_1,[e_2,e_4]=e_2,[e_3,e_4]=be_3$	3	3
$g_{4,10}$	$[e_2,e_3]=e_1,[e_2,e_4]=-e_3,[e_3,e_4]=e_2$	4	3
$g_{4,11}$	$(a>0)$: $[e_2,e_3]=e_1,[e_1,e_4]=2ae_1,[e_2,e_4]=ae_2-e_3,[e_3,e_4]=e_2+ae_3$	4	3
$g_{4,12}$	$[e_1,e_3]=e_1,[e_2,e_3]=e_2,[e_1,e_4]=-e_2,[e_2,e_4]=e_1$	3	3

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