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# New Results on the Stability and Boundedness of Nonlinear Differential Equations of Fifth Order with Multiple Deviating Arguments

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**Abstract.** We address differential equations of fifth order with multiple deviating arguments and investigate the stability and boundedness of solutions of the equation considered by functional Liapunov approach.

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## 1. Introduction

Consider nonlinear differential equations of fifth order with multiple deviating arguments, $\tau_i$, of the form:

$$x^{(5)}(t)+\sum_{i=1}^{n}\varphi_i(x^{(4)}(t-\tau_i))x^{(4)}(t)+\sum_{i=1}^{n}f_i(x'''(t-\tau_i))+\alpha_3x''(t)+\alpha_4x'(t)+\alpha_5x(t)$$

$$=p(t,x(t),x(t-\tau_1),...,x(t-\tau_n),...,x^{(4)}(t),...,x^{(4)}(t-\tau_n)). \tag{1.1}$$

Writing (1.1) as a system of first order differential equations, we get

$$x'(t)=y(t),\quad y'(t)=z(t),\quad z'(t)=w(t),\quad w'(t)=u(t),$$

$$u'(t)=-\sum_{i=1}^{n}\varphi_i(u(t-\tau_i))u(t)-\sum_{i=1}^{n}f_i(w(t))-\alpha_3z(t)-\alpha_4y(t)-\alpha_5x(t)$$

$$+\sum_{i=1}^{n}\int_{t-\tau_i}^{t}f_i'(w(s))u(s)ds$$

$$+p(t,x(t),x(t-\tau_1),...,x(t-\tau_n),...,u(t),u(t-\tau_1),...,u(t-\tau_n)), \tag{1.2}$$

where $\tau_i$, ($i=1, 2,...,n$), are positive constants, that is, fixed multiple deviating arguments; $\varphi_i, f_i$ and $p$ are continuous functions for the arguments displayed explicitly in (1.1); $\alpha_3,\alpha_4$ and $\alpha_5$ are some positive constants. It is assumed that $f_i(0)=0$ and the derivatives $\frac{\partial}{\partial w}f_i(w)$

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exist and are continuous for all $w$, and all solutions considered are also assumed to be real valued.

The qualitative theory of nonlinear differential equations of higher order has wide applications in science and technology (see for example the papers of Chlouverakis and Sprott [8] and Linz [16]). In particular, by now using the Liapunov's second method [21], the stability, instability and boundedness of solutions of certain fifth order nonlinear differential equations with and without delay have received and are still receiving intense attentions by authors. For a comprehensive treatment of the subject on the topic we refer the reader to the papers of Abou-El Ela and Sadek [1, 2], Adesina and Ukpera [3–5], Burganskaja [6], Chukwu [9], Hong [17], Ogundare [23], Sinha [24], Tejumola and Afuwape [25], Tunç [26–40] and the references cited therein for some works performed on the topic, which include some nonlinear differential equations of fifth order with and without a deviating argument. Meanwhile, especially, since 1960s many good books, most of them are in Russian literature, have also been published on the qualitative theory of differential equations with deviating arguments (see for example the books of Burton [7], Èl'sgol'ts [10], Èl'sgol'ts and Norkin [11], Gopalsamy [12], Hale [13], Hale and Verduyn Lunel [15], Kolmanovskii and Myshkis [18], Kolmanovskii and Nosov [19], Krasovskii [20], Makay [22], Yoshizawa [40] and the references listed in these books).

It should be noted that some works performed on the stability and boundedness of solutions of nonlinear differential equations of the fifth order with and without a deviating argument can be summarized in detail as follows:

In 1971, Burganskaja [6] and Sinha [24] considered certain classes of fifth order nonlinear equations without delay using the Liapunov second method. Burganskaja established sufficient conditions for the stability in the large of the zero solution while Sinha investigated the stability of the critical points.

In 1976, Chukwu [9] also discussed the stability and boundedness of certain class of fifth order.

Later, in 1990 and 1995, Hong [17] and Abou-El-Ela and Sadek [2] studied the stability and boundedness of solutions for a class of non-linear differential equations of the fifth order without delay.

In 1995, 1996, 2002 and 2007, Tunç [26–29] introduced some Liapunov functions to study certain classes of equations of the fifth order without delay and established conditions for the asymptotic stability in the large of the trivial solutions as well as boundedness of all solutions of the corresponding non-homogeneous equations considered.

On equations with delay, Tunç [30–33] used functional Liapunov approach to establish results on boundedness of solutions for the classes of delay differential equations of the fifth order. Also recently in [34] and [35] sufficient conditions for the asymptotic stability of the trivial solutions of certain classes of nonlinear fifth order differential equations with delay considered were established.

However, it is also worth mentioning that, to the best of our knowledge, up to now, the stability and boundedness of solutions for nonlinear differential equations of fifth order with multiple deviating arguments have not been discussed in the literature. The basic reason for the lack of any paper for these type differential equations is due to the difficulty of construction or definition of appropriate Liapunov functions or functionals for higher order nonlinear differential equations. The construction or definition of Liapunov functions and functionals also remain as general problem in the literature. Our aim is to define an appropriate Liapunov functional for studying the stability and boundedness of solutions of (1.1). Motivated by the above mentioned papers and books, we establish some sufficient conditions which guarantee the stability and boundedness of solutions for (1.1). Our work is a continuation of the stability and boundedness results related to the nonlinear differential equations of fifth order with and without a deviating argument that were mentioned above. In fact, when we take into consideration the differential equations of the fifth order discussed in the literature, it can be seen that all of the equations studied do not include any deviating argument or only include a deviating argument. Further, the works related to the stability and boundedness of solutions are also very important in the theory and applications of differential equations, and the investigation of the stability and boundedness of solutions for nonlinear differential equations of fifth order with multiple deviating arguments takes an important place for the researchers working in these areas.

## 2. Preliminaries

In this work we use the following notions: $\Re^n$ is the space of $n$- vectors. For a given number $r \geqslant 0, C^n$ denotes the space of continuous functions mapping the interval $[-r, 0]$ into $\Re^n$ and for $\phi \in C^n, \|\phi\| = \sup_{-r\leqslant\phi\leqslant 0} \|\phi(\theta)\|$. $C_H^n$ will denote the set of $\phi$ in $C^n$ for which $\|\phi\| < H$. For any continuous function $x(u)$ defined on $-r \leqslant u \leqslant A, A > 0$, any fixed $t, 0 \leqslant t \leqslant A$, the symbol $x_t$ will denote the function $x(t+\theta), -r \leqslant \theta \leqslant 0$.

If $F(\phi)$ is a functional defined for every $\phi$ in $C_H^n$ and $\dot{x}(t)$ is the right side derivative of $x(t)$, we consider the autonomous functional differential equation:

$$\dot{x}(t) = F(x_t), \quad t \geqslant 0. \tag{2.1}$$

We say $x(\phi)$is a solution of (2.1) with the initial condition $\phi$ in $C_H^n$ at $t = 0$ if there is an $A > 0$ such that $x(\phi)$is a function from $[-r, A)$ into $\Re^n$ such that $x_t(\phi)$is in $C_H^n$for $0 \leqslant t < A, \quad x_0(\phi) = \phi$ and $x(\phi)(t)$ satisfies (2.1) for $0 \leqslant t < A$.

**Definition 2.1.** [14] *Let $V$ be a continuous scalar functional in $C_H^n$. The derivative of $V$ along the solutions of* (2.1) *will be defined by*

$$\dot{V}(\phi) = \limsup_{h\to 0^+} \frac{V(x_h(\phi)) - V(\phi))}{h}.$$

**Lemma 2.1.** [14] *Suppose $F(0) = 0$. Let $V$ be a continuous functional defined on $C_H^n$ with $V(0) = 0$ and let $u(s)$ be a function, non-negative and continuous for $0 \leqslant s < \infty, \quad u(s) \to \infty$ as $s \to \infty$ with $u(0) = 0$. If for all $\phi$ in $C_H^n, u(\|\phi(0)\|) \leqslant V(\phi), \quad \dot{V}(\phi) \leqslant 0$, then the solution $x(t) = 0$ of* (2.1) *is stable.*

*Let $R \subset C_H^n$ be a set of all functions $\phi \in C_H^n$ where $\dot{V}(\phi) = 0$. If $\{0\}$is the largest invariant set in $R$, then the solution $x(t) = 0$ of* (2.1) *is asymptotically stable.*

## 3. Main results

Let $p(t, x(t), .x(t-\tau_1), \ldots, x(t-\tau_n), \ldots, u(t), u(t-\tau_1), \ldots, u(t-\tau_n)) \equiv 0$.

Our first main result is the following theorem.

**Theorem 3.1.** *We assume that there are continuous functions $\varphi_i, f_i$ and $p$ such that $f_i(.)$ are differentiable and positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \varepsilon, \varepsilon_0, \delta_i, \lambda_i, L_i$ and $\tau_i$ such that the following conditions hold for every $x, y, z, w$ and $u$ :*

(i)

$$\alpha_1 > 0, \quad \alpha_1\alpha_2 - \alpha_3 > 0, \quad (\alpha_1\alpha_2 - \alpha_3)\alpha_3 - (\alpha_1\alpha_4 - \alpha_5)\alpha_1 > 0,$$
$$\delta_0 := (\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3) - (\alpha_1\alpha_4 - \alpha_5)^2 > 0, \quad \alpha_5 > 0,$$
$$\Delta_1 := \frac{(\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - (\alpha_1\alpha_4 - \alpha_5) > 2\varepsilon\alpha_2,$$
$$\Delta_2 := \frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} - \frac{\alpha_1\alpha_4 - \alpha_5}{\alpha_1\alpha_2 - \alpha_3} - \frac{\varepsilon}{\alpha_1} > 0.$$

(ii)

$$2\varepsilon_0 \leqslant \sum_{i=1}^{n} \varphi_i(u(t-\tau_i)) - \alpha_1 \leqslant \min\left\{\frac{\varepsilon\alpha_2(\alpha_1\alpha_4 - \alpha_5)^2}{4\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)^2}, \frac{\varepsilon}{4\alpha_1^2}, \frac{\varepsilon\alpha_4}{4\delta^2}\right\}.$$

(iii) $f_i(0) = 0, f_i(w) \neq 0, (w \neq 0), \frac{f_i(w)}{w} \geqslant \delta_i, \sum\limits_{i=1}^{n} \delta_i = \alpha_2$ *and*

$$\left(\sum_{i=1}^{n} \frac{f_i(w)}{w} - \alpha_2\right)^2 \leqslant \min\left\{\frac{\varepsilon^2\alpha_2(\alpha_1\alpha_4 - \alpha_5)^2}{4\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)^2}, \frac{\varepsilon^2\alpha_2}{4\delta^2}\right\}, (w \neq 0),$$

*and*

$$\left|f_i'(w)\right| \leqslant L_i, (i = 1, 2, ..., n), \textit{for all} w \in \Re.$$

*Then, the zero solution of* (1.1) *is asymptotically stable provided that*

$$\tau < \min\left\{\frac{\varepsilon\alpha_4}{\delta L}, \frac{\varepsilon\alpha_2(\alpha_1\alpha_4 - \alpha_5)}{\alpha_4(\alpha_1\alpha_2 - \alpha_3)L}, \frac{\varepsilon}{2\alpha_1 L}, \frac{\varepsilon_0}{L + 2\lambda}\right\},$$

*where* $L = \sum\limits_{i=1}^{n} L_i$, $\lambda = \sum\limits_{i=1}^{n} \lambda_i$ *and* $\tau = \max\{\tau_1, \tau_2, ..., \tau_n\}$.

*Proof.* We define a Liapunov functional $V$=$V(x_t, y_t, z_t, w_t, u_t)$ by

$$\begin{aligned}
2V =& u^2 + 2\alpha_1 uw + \frac{2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} uz + 2\delta uy + 2\sum_{i=1}^{n} \int_0^w f_i(\xi)d\xi \\
& + \left[\alpha_1^2 - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\right] w^2 + 2\left[\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta\right] wz \\
& + 2\alpha_1\delta wy + 2\alpha_4 wy + 2\alpha_5 wx + \alpha_1\alpha_3 z^2 \\
& + \left[\frac{\alpha_2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \alpha_4 - \alpha_1\delta\right] z^2 + 2\delta\alpha_2 yz + 2\alpha_1\alpha_4 zy - 2\alpha_5 zy + 2\alpha_1\alpha_5 zx \\
& + \frac{\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} y^2 + (\delta\alpha_3 - \alpha_1\alpha_5) y^2 + \frac{2\alpha_4\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} yx + \delta\alpha_5 x^2 \\
& + 2\sum_{i=1}^{n} \lambda_i \int\limits_{-\tau_i}^{0} \int\limits_{t+s}^{t} u^2(\theta)d\theta ds,
\end{aligned} \tag{3.1}$$

where $s$ is a real variable such that the integral $\int\limits_{-\tau_i}^{0} \int\limits_{t+s}^{t} u^2(\theta)d\theta ds$ is non-negative, $\lambda_i$ are some positive constants which will be determined later in the proof and $\delta$ is a positive constant defined by

$$\delta := \frac{\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}+\varepsilon. \tag{3.2}$$

▮

It is clear that $V(0,0,0,0,0)=0$ and $2\int_0^w f_i(\xi)d\xi = 2\int_0^w \frac{f_i(\xi)}{\xi}\xi d\xi \geqslant 2\int_0^w \delta_i \xi d\xi = \delta_i w^2$. Hence, $2\sum_{i=1}^n \int_0^w f_i(\xi)d\xi \geqslant \sum_{i=1}^n \delta_i w^2 = \alpha_2 w^2$ since $\sum_{i=1}^n \delta_i = \alpha_2$. Then, the Liapunov functional $V = V(x_t,y_t,z_t,w_t,u_t)$ can be revised as follows

$$\begin{aligned} 2V \geqslant & \left[u+\alpha_1 w+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\right]^2+\frac{\alpha_4\delta_0}{(\alpha_1\alpha_4-\alpha_5)^2}\left(z+\frac{\alpha_5}{\alpha_4}y\right)^2 \\ & +\frac{\alpha_1\alpha_4-\alpha_5}{\alpha_1\alpha_2-\alpha_3}\left[\frac{\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}x+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{(\alpha_1\alpha_4-\alpha_5)}y+\alpha_1 z+w\right]^2 \\ & +\Delta_2(w+\alpha_1 z)^2+\frac{\varepsilon}{\alpha_1}w^2+\frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4-\alpha_5)}y^2+\varepsilon\alpha_5 x^2 \\ & +2\varepsilon\left(\frac{\alpha_3\alpha_4-\alpha_2\alpha_5}{\alpha_1\alpha_4-\alpha_5}\right)yz+2\sum_{i=1}^n \lambda_i \int_{-\tau_i}^0\int_{t+s}^t u^2(\theta)d\theta ds \end{aligned} \tag{3.3}$$

provided that

$$\frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4-\alpha_5)} \geqslant \varepsilon\left[\varepsilon+\frac{2\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-\alpha_3\right],$$

which we now assume.

Clearly, it follows from the first seven terms included in (3.3) that there exist sufficiently small positive constants $D_i$, $(i=1,\ 2,3,\ 4,\ 5)$, such that

$$\begin{aligned} 2V \geqslant & D_1x^2+D_2y^2+D_3z^2+D_4w^2+D_5u^2+2\varepsilon\left(\frac{\alpha_3\alpha_4-\alpha_2\alpha_5}{\alpha_1\alpha_4-\alpha_5}\right)yz \\ & +2\sum_{i=1}^n \lambda_i \int_{-\tau_i}^0\int_{t+s}^t u^2(\theta)d\theta ds. \end{aligned} \tag{3.4}$$

Consider the terms

$$V_3 =: \frac{D_2}{2}y^2+2\varepsilon\left(\frac{\alpha_3\alpha_4-\alpha_2\alpha_5}{\alpha_1\alpha_4-\alpha_5}\right)yz+\frac{D_3}{2}z^2,$$

which are contained in (3.4).

It can be easily seen that $V_3$ is positive semi-definite provided that

$$\varepsilon^2 \leqslant \left(\frac{\alpha_1\alpha_4-\alpha_5}{\alpha_3\alpha_4-\alpha_2\alpha_5}\right)^2\frac{D_2D_3}{4} = D_6, D_6 > 0.$$

By using the above estimate, we get from (3.4) that

$$2V \geqslant D_1x^2+\frac{D_2}{2}y^2+\frac{D_3}{2}z^2+D_4w^2+D_5u^2+2\sum_{i=1}^n \lambda_i \int_{-\tau_i}^0\int_{t+s}^t u^2(\theta)d\theta ds.$$

On the other hand, since the integral $\lambda_i \int\limits_{-\tau_i}^{0} \int\limits_{t+s}^{t} u^2(\theta)d\theta ds$ is non-negative, it is obvious that there exists a positive constant $D_7$ which satisfies the inequality

$$x^2+y^2+z^2+w^2+u^2 \leqslant D_7^{-1} V(x_t,y_t,z_t,w_t,u_t),$$

where $D_7 = \frac{1}{2}\min\{D_1, 2^{-1}D_2, 2^{-1}D_3, D_4, D_5\}$.

The time derivative of $V$ in (3.1) with respect to (1.2) leads that

$$\begin{aligned}
\frac{d}{dt}V(x_t,y_t,z_t,w_t,u_t) = &-\left[\sum_{i=1}^{n}\varphi_i(u(t-\tau_i)) - \alpha_1\right]u^2 \\
&-\left\{\alpha_1\sum_{i=1}^{n}\frac{f_i(w)}{w} - \left[\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \delta\right]\right\}w^2 \\
&-\left\{\frac{\alpha_3\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - [\delta\alpha_2 + (\alpha_1\alpha_4-\alpha_5)]\right\}z^2 \\
&-\left[\delta\alpha_4 - \frac{\alpha_4\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\right]y^2 \\
&-\alpha_1\left[\sum_{i=1}^{n}\varphi_i(u(t-\tau_i)) - \alpha_1\right]wu \\
&-\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\left[\sum_{i=1}^{n}\varphi_i(u(t-\tau_i)) - \alpha_1\right]zu \\
&-\delta\left[\sum_{i=1}^{n}\varphi_i(u(t-\tau_i)) - \alpha_1\right]yu \\
&-\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\left[\sum_{i=1}^{n}\frac{f_i(w)}{w} - \alpha_2\right]wz \\
&-\delta\left[\sum_{i=1}^{n}\frac{f_i(w)}{w} - \alpha_2\right]wy + \sum_{i=1}^{n}u\int\limits_{t-\tau_i}^{t} f_i'(w(s))u(s)ds \\
&+\sum_{i=1}^{n}\alpha_1 w\int\limits_{t-\tau_i}^{t} f_i'(w(s))u(s)ds + \sum_{i=1}^{n}\delta y\int\limits_{t-\tau_i}^{t} f_i'(w(s))u(s)ds \\
&+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\sum_{i=1}^{n}z\int\limits_{t-\tau_i}^{t} f_i'(w(s))u(s)ds \\
&+\sum_{i=1}^{n}(\lambda_i\tau_i)u^2 - \sum_{i=1}^{n}\lambda_i\int\limits_{t-\tau_i}^{t} u^2(s)ds.
\end{aligned} \tag{3.5}$$

In view of the assumptions of Theorem 3.1 and (3.2), we get

$$\left[\sum_{i=1}^{n}\varphi_i(u(t-\tau_i))-\alpha_1\right]\geqslant 2\varepsilon_0,$$
$$\alpha_1\sum_{i=1}^{n}\frac{f_i(w)}{w}-\left[\alpha_3+\frac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-\delta\right]\geqslant\varepsilon,$$
$$\frac{\alpha_3\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-[\delta\alpha_2+(\alpha_1\alpha_4-\alpha_5)]>\varepsilon\alpha_2,$$
$$\delta\alpha_4-\frac{\alpha_4\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}=\varepsilon\alpha_4.$$

From the assumption $|f_i'(w)|\leqslant L_i$ and estimate $2|ab|\leqslant a^2+b^2$ we get

$$\sum_{i=1}^{n}u\int_{t-\tau_i}^{t}f_i'(w(s))u(s)ds\leqslant\frac{1}{2}\sum_{i=1}^{n}(L_i\tau_i)u^2+\frac{1}{2}\sum_{i=1}^{n}L_i\int_{t-\tau_i}^{t}u^2(s)ds,$$

$$\sum_{i=1}^{n}\alpha_1 w\int_{t-\tau_i}^{t}f_i'(w(s))u(s)ds\leqslant\frac{1}{2}\sum_{i=1}^{n}(\alpha_1L_i\tau_i)w^2+\frac{\alpha_1}{2}\sum_{i=1}^{n}L_i\int_{t-\tau_i}^{t}u^2(s)ds,$$

$$\sum_{i=1}^{n}\delta y\int_{t-\tau_i}^{t}f_i'(w(s))u(s)ds\leqslant\frac{1}{2}\sum_{i=1}^{n}(\delta L_i\tau_i)y^2+\frac{\delta}{2}\sum_{i=1}^{n}L_i\int_{t-\tau_i}^{t}u^2(s)ds,$$

$$\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\sum_{i=1}^{n}z\int_{t-\tau_i}^{t}f_i'(w(s))u(s)ds\leqslant\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{2(\alpha_1\alpha_4-\alpha_5)}\sum_{i=1}^{n}(L_i\tau_i)z^2$$
$$+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{2(\alpha_1\alpha_4-\alpha_5)}\sum_{i=1}^{n}L_i\int_{t-\tau_i}^{t}u^2(s)ds.$$

Then we have

$$\begin{aligned}\frac{d}{dt}V(x_t,y_t,z_t,w_t,u_t)\leqslant&-\left[\frac{\varepsilon\alpha_4}{2}-\frac{\delta}{2}\sum_{i=1}^{n}(L_i\tau_i)\right]y^2-\left[\frac{\varepsilon\alpha_2}{2}-\frac{\alpha_4L(\alpha_1\alpha_2-\alpha_3)}{2(\alpha_1\alpha_4-\alpha_5)}\sum_{i=1}^{n}(L_i\tau_i)\right]z^2\\&-\left[\frac{\varepsilon}{4}-\frac{\alpha_1}{2}\sum_{i=1}^{n}(L_i\tau_i)\right]w^2-\left[\frac{\varepsilon_0}{2}-\frac{1}{2}\sum_{i=1}^{n}(L_i+2\lambda_i)\tau_i\right]u^2\\&-\left[\sum_{i=1}^{n}\lambda_i-\left(\frac{1}{2}+\frac{\alpha_1}{2}+\frac{\delta}{2}+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{2(\alpha_1\alpha_4-\alpha_5)}\right)\sum_{i=1}^{n}L_i\right]\int_{t-\tau_i}^{t}u^2(s)ds\\&-\sum_{k=4}^{8}V_k,\end{aligned}$$

where

$$V_4=\frac{1}{4}\left[\sum_{i=1}^{n}\varphi_i(u(t-\tau_i))-\alpha_1\right]u^2+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\left[\sum_{i=1}^{n}\varphi_i(u(t-\tau_i))-\alpha_1\right]uz+\frac{\varepsilon\alpha_2}{4}z^2,$$

$$V_5 = \frac{1}{4}\left[\sum_{i=1}^{n}\varphi_i(u(t-\tau_i))-\alpha_1\right]u^2+\alpha_1\left[\sum_{i=1}^{n}\varphi_i(u(t-\tau_i))-\alpha_1\right]uw+\frac{\varepsilon}{4}w^2,$$

$$V_6 = \frac{1}{4}\left[\sum_{i=1}^{n}\varphi_i(u(t-\tau_i))-\alpha_1\right]u^2+\delta\left[\sum_{i=1}^{n}\varphi_i(u(t-\tau_i))-\alpha_1\right]uy+\frac{\varepsilon\alpha_4}{4}y^2,$$

$$V_7 = \frac{\varepsilon}{4}w^2+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\left[\sum_{i=1}^{n}\frac{f_i(w)}{w}-\alpha_2\right]wz+\frac{\varepsilon\alpha_2}{4}z^2,$$

$$V_8 = \frac{\varepsilon}{4}w^2+\delta\left[\sum_{i=1}^{n}\frac{f_i(w)}{w}-\alpha_2\right]wy+\frac{\varepsilon\alpha_4}{4}y^2.$$

It is clear that the expressions given by $V_4, V_5, V_6, V_7$ and $V_8$ represent certain specific quadratic forms, respectively. Making use of the basic information on the positive semi-definite of a quadratic form, one can easily conclude that $V_4 \geqslant 0, V_5 \geqslant 0, V_6 \geqslant 0, V_7 \geqslant 0$ and $V_8 \geqslant 0$ provided that

$$\sum_{i=1}^{n}\varphi_i(u(t-\tau_i))-\alpha_1 \leqslant \frac{\varepsilon\alpha_2(\alpha_1\alpha_4-\alpha_5)^2}{4\alpha_4^2(\alpha_1\alpha_2-\alpha_3)^2},$$

$$\sum_{i=1}^{n}\varphi_i(u(t-\tau_i))-\alpha_1 \leqslant \frac{\varepsilon}{4\alpha_1^2},$$

$$\sum_{i=1}^{n}\varphi_i(u(t-\tau_i))-\alpha_1 \leqslant \frac{\varepsilon\alpha_4}{4\delta^2},$$

$$\left(\sum_{i=1}^{n}\frac{f_i(w)}{w}-\alpha_2\right)^2 \leqslant \frac{\varepsilon^2\alpha_2(\alpha_1\alpha_4-\alpha_5)^2}{4\alpha_4^2(\alpha_1\alpha_2-\alpha_3)^2}$$

$$\left(\sum_{i=1}^{n}\frac{f_i(w)}{w}-\alpha_2\right)^2 \leqslant \frac{\varepsilon^2\alpha_4}{4\delta^2}.$$

Thus, in view of the above discussion and the estimate (3.5), it follows that

$$\begin{aligned}\frac{d}{dt}V(x_t,y_t,z_t,w_t,u_t) \leqslant &-\left[\frac{\varepsilon\alpha_4}{2}-\frac{\delta}{2}\sum_{i=1}^{n}(L_i\tau_i)\right]y^2-\left\{\frac{\varepsilon\alpha_2}{2}-\left[\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{2(\alpha_1\alpha_4-\alpha_5)}\right]\sum_{i=1}^{n}(L_i\tau_i)\right\}z^2\\ &-\left[\frac{\varepsilon}{4}-\frac{\alpha_1}{2}\sum_{i=1}^{n}(L_i\tau_i)\right]w^2-\left\{\frac{\varepsilon_0}{2}-\frac{1}{2}\left[\sum_{i=1}^{n}(L_i+2\lambda_i)\tau_i\right]\right\}u^2\\ &-\left[\sum_{i=1}^{n}\lambda_i-\left(\frac{1}{2}+\frac{\alpha_1}{2}+\frac{\delta}{2}+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{2(\alpha_1\alpha_4-\alpha_5)}\right)\sum_{i=1}^{n}L_i\right]\int_{t-\tau_i}^{t}u^2(s)ds.\end{aligned}$$

Let $\sum_{i=1}^{n}L_i = L$, $\sum_{i=1}^{n}\lambda_i=\lambda$ and $\tau=\max\{\tau_1,\tau_2,...,\tau_n\}$. Hence, we have

$$\begin{aligned}\frac{d}{dt}V(x_t,y_t,z_t,w_t,u_t) \leqslant &-\left(\frac{\varepsilon\alpha_4}{2}-\frac{\delta L}{2}\tau\right)y^2-\left\{\frac{\varepsilon\alpha_2}{2}-\left[\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)L}{2(\alpha_1\alpha_4-\alpha_5)}\right]\tau\right\}z^2\\ &-\left(\frac{\varepsilon}{4}-\frac{\alpha_1 L}{2}\tau\right)w^2-\left[\frac{\varepsilon_0}{2}-\frac{(L+2\lambda)}{2}\tau\right]u^2\end{aligned}$$

$$-\left\{\lambda-\left[\frac{1}{2}+\frac{\alpha_1}{2}+\frac{\delta}{2}+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{2(\alpha_1\alpha_4-\alpha_5)}\right]L\right\}\int\limits_{t-\tau_i}^{t}u^2(s)ds.$$

Let

$$\lambda=\left[\frac{1}{2}+\frac{\alpha_1}{2}+\frac{\delta}{2}+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{2(\alpha_1\alpha_4-\alpha_5)}\right]L.$$

Then we get

$$\frac{d}{dt}V(x_t,y_t,z_t,w_t,u_t)\leqslant-\left(\frac{\varepsilon\alpha_4}{2}-\frac{\delta L}{2}\tau\right)y^2-\left\{\frac{\varepsilon\alpha_2}{2}-\left[\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)L}{2(\alpha_1\alpha_4-\alpha_5)}\right]\tau\right\}z^2$$
$$-\left(\frac{\varepsilon}{4}-\frac{\alpha_1 L}{2}\tau\right)w^2-\left[\frac{\varepsilon_0}{2}-\frac{(L+2\lambda)}{2}\tau\right]u^2.$$

The above estimate implies

$$\frac{d}{dt}V(x_t,y_t,z_t,w_t,u_t)\leqslant-D_8y^2-D_9z^2-D_{10}w^2-D_{11}u^2\leqslant0$$

for some positive constants $D_i,(i=8,\ 9,\ 10,\ 11)$ provided that

$$\tau<\min\left\{\frac{\varepsilon\alpha_4}{\delta L},\frac{\varepsilon\alpha_2(\alpha_1\alpha_4-\alpha_5)}{\alpha_4(\alpha_1\alpha_2-\alpha_3)L},\frac{\varepsilon}{2\alpha_1 L},\frac{\varepsilon_0}{L+2\lambda}\right\}.$$

In conclusion, we have shown that $\frac{d}{dt}V(x_t,y_t,z_t,w_t,u_t)\leqslant0$. Further, it follows that $\frac{d}{dt}V(x_t,y_t,z_t,w_t,u_t)=0$ if and only if $y=z=w=u=0$. In view of $\frac{d}{dt}V(x_t,y_t,z_t,w_t,u_t)\equiv0$ and system (1.2) together, we can easily obtain $x=y=z=w=u=0$ by the assumption $f_i(w)\neq0,(w\neq0)$. Thus, all conditions of Lemma 2.1 (see [14]) hold. Therefore, by noting the above discussion, we arrive at the conclusion that the zero solution of (1.1) is asymptotically stable. This completes the proof of Theorem 3.1.

Let $p(t,x(t),x(t-\tau_1),\dots,x(t-\tau_n),\dots,u(t),u(t-\tau_1),\dots,u(t-\tau_n))\neq0$.

Our second main result is the following theorem.

**Theorem 3.2.** *We assume that all the assumptions of Theorem 3.1 and*

$$|p(t,x(t),x(t-\tau_1),\dots,x(t-\tau_n),\dots,u(t),u(t-\tau_1),\dots,u(t-\tau_n))|\leqslant|q(t)|,$$

$$\int\limits_0^t|q(s)|ds\leqslant P_0<\infty$$

*hold.*

*Then, there exists a finite positive constant M such that the solution $x(t)$ of (1.1) defined by the initial function*

$$x(t)=\phi(t),\quad x'(t)=\phi'(t),\quad x''(t)=\phi''(t),x'''(t)=\phi'''(t),x^{(4)}(t)=\phi^{(4)}(t)$$

*satisfies*

$$|x(t)|\leqslant\sqrt{M},\quad\left|x'(t)\right|\leqslant\sqrt{M},\quad\left|x''(t)\right|\leqslant\sqrt{M},\left|x'''(t)\right|\leqslant\sqrt{M},\left|x^{(4)}(t)\right|\leqslant\sqrt{M}$$

*for all $t\geqslant t_0\geqslant0$, where $\phi\in C^4([t_0-r,t_0],\ \Re)$, provided that*

$$\tau<\min\left\{\frac{\varepsilon\alpha_4}{\delta L},\frac{\varepsilon\alpha_2(\alpha_1\alpha_4-\alpha_5)}{\alpha_4(\alpha_1\alpha_2-\alpha_3)L},\frac{\varepsilon}{2\alpha_1 L},\frac{\varepsilon_0}{L+2\lambda}\right\}.$$

*Proof.* Subject to the assumptions of Theorem 3.2, the result of Theorem 3.1 can be revised as follows

$$\begin{aligned}\frac{d}{dt}V(x_t,y_t,z_t,w_t,u_t) &\leqslant -D_8y^2 - D_9z^2 - D_{10}w^2 - D_{11}u^2\\ &\quad + \left|u+\alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\right| \ |q(t)|\\ &\leqslant D_{12}(|y|+|z|+|w|+|u|)\ |q(t)|,\end{aligned}$$

where $D_{12}=\max\{1,\ \alpha_1,\ \alpha_4(\alpha_1\alpha_2-\alpha_3)(\alpha_1\alpha_4-\alpha_5)^{-1},\ \delta\}$.

The estimate $|a|<1+a^2$ implies

$$\frac{d}{dt}V(x_t,y_t,z_t,w_t,u_t)\leqslant D_{12}[4+(y^2+z^2+w^2+u^2)]\ |q(t)|.$$

Using the estimate

$$x^2+y^2+z^2+w^2+u^2\leqslant D_7^{-1}V(x_t,y_t,z_t,w_t,u_t)$$

we obtain

$$\frac{d}{dt}V(x_t,y_t,z_t,w_t,u_t)\leqslant 4D_{12}\,|q(t)|+\frac{D_{12}}{D_7}V(x_t,y_t,z_t,w_t,u_t)\,|q(t)|.$$

Integrating this estimate from 0 to $t$ and using the assumption $\int_0^t |q(s)|ds\leqslant P_0<\infty$, we get

$$V(x_t,y_t,z_t,w_t,u_t)\leqslant V(x_0,y_0,z_0,w_0,u_0)+4D_{12}P_0+\frac{D_{12}}{D_7}\int_0^t V(x_s,y_s,z_s,w_s,u_s)\,|q(s)|\,ds.$$

Hence, using Gronwall inequality, one can easily obtain for some positive constant $K_1$ that

$$V(x_t,y_t,z_t,w_t,u_t)\leqslant [V(x_0,y_0,z_0,w_0,u_0)+4D_{12}P_0]\exp(D_{12}D_7^{-1}P_0)=K_1<\infty$$

so that

$$x^2+y^2+z^2+w^2+u^2\leqslant D_7^{-1}V(x_t,y_t,z_t,w_t,u_t)\leqslant D_7^{-1}K_1=M.$$

This completes the proof of Theorem 3.2. ∎

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