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On Maximally Irregular Graphs

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Abstract. Let G be a connected graph with maximum degree $\Delta(G)$. The *irregularity index* $t(G)$ of G is defined as the number of distinct terms in the degree sequence of G. We say that G is *maximally irregular* if $t(G) = \Delta(G)$. The purpose of this note, apart from pointing out that every highly irregular graph is maximally irregular, is to establish upper bounds on the size of maximally irregular graphs and maximally irregular triangle-free graphs.

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1. Introduction

Let $G = (V,E)$ be a finite connected graph with vertex set V and edge set E. The *degree* $\deg_G(v)$ of a vertex v of G is the number of vertices adjacent to v. We denote by $\Delta(G)$ the maximum value of the degrees of vertices of G. A graph that is not regular is called *irregular*. The *irregularity index* $t(G)$ of G, introduced by Mukwembi [6], is defined as the number of distinct terms in the degree sequence of G. Clearly, for any connected graph G,

$$t(G) \leq \Delta(G). \tag{1.1}$$

This inequality inspires us to propose a new natural class of graphs which we, for lack of a better term, call “maximally irregular graphs”. More formally, we say that a connected graph G is *maximally irregular* if $t(G) = \Delta(G)$. In this note we study maximally irregular graphs. A well-studied [1, 2, 3, 4] class of graphs is that of highly irregular graphs introduced in [2]. The notion of highly irregular graphs was inspired by Chartrand, Erdős and Oellermann’s question concerning how to define irregularity in graphs. Formally, a graph G is *highly irregular* if it is connected and each of its vertices is adjacent only to vertices with distinct degrees. The existence and enumeration of highly irregular graphs was investigated in [2] where it was uncovered that this class of highly irregular graphs is sufficiently numerous and diverse so as to be an appropriate answer to the question concerning how to define irregularity. We will begin by noting that every highly irregular graph is maximally irregular implying that the class of maximally irregular graphs is at least as vast as that of

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highly irregular graphs. We will, by construction, show that the class of maximally irregular graphs is in fact much larger than the class of highly irregular graphs. Further, we will establish asymptotically tight upper bounds on the size of maximally irregular graphs and maximally irregular triangle-free graphs in terms of order.

For a vertex u of G we will denote the open neigbourhood of u, i.e., the set of all vertices adjacent to u, by $N(u)$, and the closed neigbourhood of u, i.e., the set $N(u) \cup \{u\}$, by $N[u]$. For a subset $S \subseteq V(G)$, we will denote by $G[S]$ the subgraph of G induced by S.

2. Results

Theorem 2.1. *Every highly irregular graph is maximally irregular.*

Proof. Assume that G is a highly irregular graph and let v be a vertex of degree Δ. Since G is highly irregular, all neighbours of v have distinct degrees. Thus, $t(G) \geq \Delta(G)$. This, in conjunction with (1.1), yields $t(G) = \Delta(G)$, and so G is maximally irregular as desired. ∎

In general, the converse of Theorem 2.1 does not hold. For instance, the path graph, P_n, $n \geq 3$, is maximally irregular but not highly irregular. We now present a construction which proves that the class of maximally irregular graphs is much larger than that of highly irregular graphs.

Theorem 2.2. *Every highly irregular graph G of order $n \geq 2$ and maximum degree Δ is an induced subgraph of a maximally irregular graph H of order $n + \Delta + 1$. Moreover, the graph H is not highly irregular.*

Proof. Assume that G is a highly irregular graph of order $n \geq 2$. If $\Delta(G) = 1$, then $G = K_2$ and the graph obtained by taking G and attaching one end vertex of G to each of the two vertices in a disjoint copy of K_2 has the desired properties. Now assume that $\Delta(G) \geq 2$. Let v be a vertex of G with maximum degree Δ. Let H be the graph obtained by joining a single vertex u, say, of $K_{\Delta+1}$ to v. Note that $|V(H)| = n + \Delta(G) + 1$, and since G is maximally irregular because of Theorem 2.1, $t(H) = t(G) + 1 = \Delta(G) + 1 = \Delta(H)$, showing that H is maximally irregular. Since $\Delta(G) \geq 2$, u is adjacent to two vertices in $K_{\Delta+1}$ with the same degree. Therefore, H has the desired properties. ∎

We now turn to the size of maximally irregular graphs. In [2], the maximum size of a highly irregular graph of order n was proved to be $(n(n+2))/8$ with equality possible for n even. Majcher and Michael [4] improved this bound for odd n. They proved that if n is odd and G is a highly irregular graph of order n, then the size m of G is at most $((n-1)(n+1))/8 + \lfloor (n+1)/10 \rfloor$. For maximally irregular graphs none of these bounds apply, for example consider the graph in Figure 1.

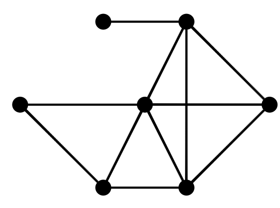

Figure 1. A maximally irregular graph.

In the next theorem, we establish an asymptotically tight upper bound on the size of maximally irregular graphs in terms of order. The following simple observation is useful.

Fact 1. Let G be a maximally irregular graph. Then there exists a set $S = \{v_1, v_2, \ldots, v_t\} \subseteq V(G)$ such that

$$\deg_G(v_i) = i,$$

for all $i = 1, 2, \ldots, t = \Delta$.

Theorem 2.3. *Let G be a maximally irregular graph of order n. Then the size m of G satisfies*

$$m \leq \frac{(n+2)(n-1)}{4}.$$

Moreover, the coefficient of n^2 is best possible.

Proof. Let $t = \Delta$ be the irregularity index of G and for $i = 1, 2, \ldots, t$, let $A_i := \{x \in V(G) \mid \deg(x) = i\}$ and $|A_i| = n_i$. By Fact 1,

$$n_i \geq 1 \text{ for all } i = 1, 2, \ldots, t. \tag{2.1}$$

Furthermore,

$$n_1 + n_2 + \cdots + n_t = n. \tag{2.2}$$

Then, subject to (2.1) and (2.2), we have

$$2m = \sum_{v \in A_1} \deg(v) + \sum_{v \in A_2} \deg(v) + \cdots + \sum_{v \in A_t} \deg(v) = \sum_{i=1}^{t} i n_i$$
$$\leq 1 + 2 + \ldots + (t-1) + [n - (t-1)]t = nt - \frac{t^2}{2} + \frac{t}{2}.$$

It follows that

$$m \leq \frac{nt}{2} - \frac{t^2}{4} + \frac{t}{4} = f(t),$$

say. A simple differentiation shows that, subject to $1 \leq t \leq n-1$, the function f is maximized for $t = n-1$. Therefore,

$$m \leq f(t) \leq f(n-1) = \frac{(n+2)(n-1)}{4},$$

as desired.To see that the coefficient of n^2 in the bound is best possible, consider the graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ in which $v_i v_j \in E(G)$ if and only if $i + j > n$. Then G is maximally irregular and $|E(G)| = (n(n-1))/4 + 1/2\lfloor n/2 \rfloor$. ∎

In 1907 Mantel [5], and subsequently Turán [7] in 1941, showed that the size m of a general triangle-free graph of order n is at most

$$m \leq \left\lfloor \frac{n^2}{4} \right\rfloor,$$

and equality holding iff G is the Turán graph $T_2(n)$, i.e., the complete bipartite graph whose classes are as nearly equal as possible. For $n \geq 4$, the extremal graph is not maximally irregular. In the remainder of this paper, we will prove an upper bound on the size of triangle-free maximally irregular graphs. First we need a lemma which establishes an upper bound on the irregularity index of maximally irregular graphs with no triangles.

Lemma 2.1. *Let G be a maximally irregular graph of order n with no triangles. Then the irregularity index of G satisfies*

$$t(G) \leq \frac{2}{3}(n+1).$$

Proof. Let $t = t(G)$ and, suppose to the contrary that

$$t > \frac{2}{3}(n+1). \tag{2.3}$$

By Fact 1, let $S = \{v_1, v_2, \ldots, v_t\} \subseteq V(G)$ be such that $\deg_G(v_i) = i$, for all $i = 1, 2, \ldots, t$. Consider the open neighbourhood $N(v_t)$ of v_t.

Claim 1. For $i \in \{\lfloor t/2 \rfloor, \lfloor t/2 \rfloor + 1, \ldots, t-1\}$, we have that $v_i \notin N(v_t)$.

Proof of Claim 1: If the claim were false, then $v_i v_t$ is an edge in G for some i, $\lfloor t/2 \rfloor \leq i \leq t-1$. Since G has no triangles, $N(v_i) \cap N(v_t) = \emptyset$; hence

$$n \geq |N(v_i)| + |N(v_t)| = i + t \geq t + \left\lfloor \frac{t}{2} \right\rfloor \geq \frac{3}{2}t - \frac{1}{2},$$

and so $t \leq 2/3(n + 1/2)$, contradicting (2.3). Thus the claim is proven.

We deduce from Claim 1 that $\{v_{\lfloor t/2 \rfloor}, v_{\lfloor t/2 \rfloor + 1}, \ldots, v_{t-1}\} \cap N[v_t] = \emptyset$. Therefore,

$$n \geq |\{v_{\lfloor \frac{t}{2} \rfloor}, v_{\lfloor \frac{t}{2} \rfloor + 1}, \ldots, v_{t-1}\}| + |N[v_t]| = 2t + 1 - \left\lfloor \frac{t}{2} \right\rfloor \geq 2t + 1 - \frac{t+1}{2},$$

from which it follows that $t \leq 2/3(n - 1/2)$, contradicting (2.3). This proves the lemma. ∎

Proposition 2.1. *Let G be a maximally irregular graph of order n. If G is triangle-free, then the size m of G satisfies*

$$m \leq \frac{1}{18}(n+1)(4n+1).$$

Proof. As in Theorem 2.3, we have that $m \leq nt/2 - t^2/4 + t/4$. Since G is triangle-free, by Lemma 2.1, $t \leq 2/3(n+1)$. Subject to this constraint, the function $nt/2 - t^2/4 + t/4$ is maximized for $t = 2/3(n+1)$ to give $m \leq 1/18(n+1)(4n+1)$, and the proposition is proven. ∎

The bound in Proposition 2.1 seems not best possible. It is conceivable that the correct bound is:

Conjecture 2.1. *Let G be a maximally irregular graph of order n. If G is triangle-free, then the size m of G satisfies*

$$m \leq \frac{n(n+1)}{6}.$$

Moreover, the inequality is tight.

It is not hard to construct maximally irregular graphs G with no triangles, of order n, for which $|E(G)| = (n(n+1))/6$. For instance, for n a multiple of 3, let $K_{n/3,n/3}$ be the complete bipartite graph with partite sets V_1 and V_2. Let $V_1 = \{v_1, v_2, \ldots, v_{n/3}\}$ and let $W = \{w_1, w_2, \ldots, w_{n/3}\}$ be a set of vertices disjoint from $V_1 \cup V_2$. Form a graph G by taking $K_{n/3,n/3}$ and joining each vertex w_i in W to a vertex v_j in V_1 if and only if $i + j > n/3$. Then G is maximally irregular, triangle-free and

$$|E(G)| = \frac{n}{3} \cdot \frac{n}{3} + 1 + 2 + \cdots + \frac{n}{3} = \frac{n(n+1)}{6}.$$

Conjecture 2.1 seems difficult at present. We conclude this note by giving some support to the conjecture.

Proposition 2.2. *Let G be a maximally irregular graph of order n with no triangles. If $t(G) \leq (1/3)n$, then the size m of G satisfies*

$$m \leq \frac{n(n+1)}{6}.$$

Proof. As in Theorem 2.3, we have that $m \leq nt/2 - t^2/4 + t/4$. The proposition follows from maximizing $nt/2 - t^2/4 + t/4$ subject to $t \leq (1/3)n$. ∎

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