BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **36***(3) (2013), 783–787*

Total Colorings of Planar Graphs with Small Maximum Degree

[1]BING WANG, [2]JIAN-LIANG WU AND [3]SI-FENG TIAN
[1]Department of Mathematics, Zaozhuang University, Shandong, 277160, P. R. of China
[1,2]School of Mathematics, Shandong University, Jinan, 250100, P. R. of China
[3]Zaozhuang No.3 Middle School, Shandong , 277100, P. R. of China
[1]wangbing456@uzz.edu.cn, [2]jlwu@sdu.edu.cn, [3]tiansifeng1969@163.com

Abstract. Let G be a planar graph of maximum degree Δ and girth g, and there is an integer $t(> g)$ such that G has no cycles of length from $g+1$ to t. Then the total chromatic number of G is $\Delta+1$ if $(\Delta, g, t) \in \{(5,4,6), (4,4,17)\}$; or $\Delta = 3$ and $(g,t) \in \{(5,13), (6,11), (7,11), (8,10), (9,10)\}$, where each vertex is incident with at most one g-cycle.

2010 Mathematics Subject Classification: 05C15

Keywords and phrases: Total coloring, planar graph, cycle, girth.

1. Introduction

All graphs considered in this paper are simple, finite and undirected, and we follow [2] for the terminologies and notations not defined here. Let G be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ (or simply V, E, Δ and δ) to denote the vertex set, the edge set, the maximum degree and the minimum degree of G, respectively. For a vertex $v \in V$, let $N(v)$ denote the set of vertices adjacent to v, and let $d(v) = |N(v)|$ denote the degree of v. A *k-vertex*, a *k^+-vertex* or a *k^--vertex* is a vertex of degree k, at least k or at most k respectively. A k-cycle is a cycle of length k, and a 3-cycle is usually called a triangle.

A *total-k-coloring* of a graph G is a coloring of $V \cup E$ using k colors such that no two adjacent or incident elements receive the same color. The *total chromatic number* $\chi''(G)$ of G is the smallest integer k such that G has a total-k-coloring. Clearly, $\chi''(G) \geq \Delta+1$. Behzad [1] posed independently the following famous conjecture, which is known as the *Total Coloring Conjecture* (TCC).

Conjecture 1.1. *For any graph G,* $\Delta+1 \leq \chi''(G) \leq \Delta+2$.

This conjecture was confirmed for a general graph with $\Delta \leq 5$. But for planar graph, the only open case is $\Delta = 6$ (see [11, 15]). Interestingly, planar graphs with high maximum degree allow a stronger assertion,that is, every planar graph with high maximum degree Δ is $(\Delta+1)$-totally-colorable. This result was first established in [3] for $\Delta \geq 14$, which was extended to $\Delta \geq 9$ (see [12]). For $4 \leq \Delta \leq 8$, it is not known whether that the assertion still

Communicated by Xueliang Li .
Received: May 16, 2011; *Revised:* November 9, 2011.

holds true. But there are many related results by adding girth restrictions, see [7–10, 13, 14, 16]. We present our new results in this paper.

Theorem 1.1. *Let G be a planar graph of maximum degree* Δ *and girth g, and there is an integer* $t(>g)$ *such that G has no cycles of length from* $g+1$ *to t. Then the total chromatic number of G is* $\Delta+1$ *if (a)* $(\Delta,g,t)=(5,4,6)$ *or (b)*$(\Delta,g,t)=(4,4,17)$.

Borodin *et al.* [6] obtained that if a planar graph G of maximum degree three contains no cycles of length from 3 to 9, then $\chi''(G)=\Delta+1$. In the following, we make further efforts on the total-colorability of planar graph on the condition that G contains some k-cycle,where $k\in(5,\cdots,9)$. We get the following result.

Theorem 1.2. *Let G be a planar graph of maximum degree* 3 *and girth g, each vertex is incident with at most one g-cycle and there is an integer* $t(>g)$ *such that G has no cycles of length from* $g+1$ *to t. Then* $\chi''(G)=\Delta+1$ *if one of the following conditions holds.*
(a) $g=5$ *and* $t\geq 13$, (b) $g=6$ *and* $t\geq 11$, (c) $g=7$ *and* $t\geq 11$,
(d) $g=8$ *and* $t\geq 10$, (e) $g=9$ *and* $t\geq 10$.

We will introduce some more notations and definitions here for convenience. Let $G=(V,E,F)$ be a planar graph, where F is the face set of G. The degree of a face f, denoted by $d(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A k-face or a k^+-face is a face of degree k or at least k, respectively. Let $n_k(v)$ be the number of k-vertices adjacent to v and $n_k(f)$ be the number of k-vertices incident with f.

2. Proof of Theorem 1.1

Let G be a minimal counterexample to Theorem 1.1 in terms of the number of vertices and edges. Then every proper subgraph of G is $(\Delta+1)$-totally-colorable. Firstly, we investigate some structural properties of G,which will be used to derive the desired contradiction completing our proof.

Lemma 2.1. *G is* 2*-connected and hence, it has no vertices of degree* 1 *and the boundary* $b(f)$ *of each face f in G is exactly a cycle (i.e.* $b(f)$ *cannot pass through a vertex v more than once).*

Lemma 2.2. [5] *G contains no edge uv with* $\min\{d(u),d(v)\}\leq\lfloor\Delta/2\rfloor$ *and* $d(u)+d(v)\leq\Delta+1$.

Lemma 2.3. [3] *The subgraph of G induced by all edges joining* 2*-vertices to* Δ *-vertices is a forest.*

Lemma 2.4. [6] *If* $\Delta\geq 5$*, then no* 3*-vertex is adjacent two* 3*-vertices.*

Let G_2 be the subgraph induced by all edges incident with 2-vertices of G. Then G_2 is a forest by Lemma 2.3. We root it at a 5-vertex. In this case, every 2-vertex has exactly one parent and exactly one child, which are 5-vertices.

Since G is a planar graph, by Euler's formula, we have

$$\sum_{v\in V}(d(v)-4)+\sum_{f\in F}(d(f)-4)=-8<0.$$

Now we define the initial charge function $ch(x)$ of $x\in V\cup F$ to be $ch(v)=d(v)-4$ if $v\in V$ and $ch(f)=d(f)-4$ if $f\in F$. It follows that $\sum_{x\in V\cup F}ch(x)<0$. Note that any discharging procedure preserves the total charge of G. If we can define suitable discharging rules to change the initial charge function ch to the final charge function ch' on $V \cup F$, such that $ch'(x) \geq 0$ for all $x \in V \cup F$, then we get an obvious contradiction. Now we design appropriate discharging rules and redistribute weights accordingly.

For a vertex v, we define $f_k(v)$ or $f_k^+(v)$ to be the number of k-faces or k^+-faces incident with v, respectively. To prove (a), our discharging rules are defined as follows.

R11. Each 2-vertex receives 2 from its child.
R12. Each 3-vertex v receives $1/(f_7^+(v))$ from each of its incident 7^+-faces.
R13. Each 5^+-vertex receives $1/3$ from each of its incident 7^+-faces.

Next,we will check $ch'(x) \geq 0$ for all $x \in V \cup F$. Let $f \in F(G)$. If $d(f) = 4$, then $ch'(f) = ch(f) = 0$. Suppose $d(f) = 7$. Then $n_3(f) \leq 4$ by Lemma 2.4. Moreover, every 7^+-face sends at most $1/2$ to its incident 3-vertices by R12 and $1/3$ to its incident 5-vertices by R13. So we have $ch'(f) \geq ch(f) - 4 \times 1/2 - 3 \times 1/3 = 0$. Suppose $d(f) \geq 8$. Then $n_3(v) \leq \lfloor (2d(f))/3 \rfloor$ by Lemma 2.2 and Lemma 2.4. Thus, $ch'(f) \geq ch(f) - (\lfloor (2d(f))/3 \rfloor \times 1/2) - (d(f) - \lfloor (2d(f))/3 \rfloor) \times 1/3 \geq (5d(f) - 38)/9 \geq 0$ by R12 and R13.

Let $v \in V(G)$. If $d(v) = 2$, then $ch'(v) = ch(v) + 2 = 0$. If $d(v) = 3$, then $f_7^+(v) \geq 2$ and it follows from R12 that $ch'(v) = ch(v) + f_7^+(v) \times 1/(f_7^+(v)) = 0$. If $d(v) = 4$, then $ch'(v) = ch(v) = 0$. If $d(v) = 5$, then $f_7^+(v) \geq 3$. Moreover, it may be the child of at most one 2-vertex. Thus $ch'(v) \geq ch(v) + 1/3 \times 3 - 2 = 0$ by R13. Suppose $d(v) \geq 6$. Then v is incident with at most $\lfloor (d(v))/2 \rfloor$ 4-faces and it may be the the parent of at most one 2-vertex. So $ch'(v) \geq ch(v) + (d(v) - \lfloor (d(v))/2 \rfloor) \times 1/3 - 2 = (7d(v) - 36)/6 > 0$.

Note that (a) implies that (b) is true if $\Delta \geq 5$. So it suffice to prove (b) by assuming $\Delta = 4$. Since G is a planar graph, by Euler's formula, we have

$$\sum_{v \in V} (2d(v) - 6) + \sum_{f \in F} (d(f) - 6) = -12 < 0.$$

Now we define the initial charge function $ch(x)$ of $x \in V \cup F$ to be $ch(v) = 2d(v) - 6$ if $v \in V$ and $ch(f) = d(f) - 6$ if $f \in F$. It follows that $\sum_{x \in V \cup F} ch(x) < 0$.

To prove (b), we construct the new charge $ch'(x)$ on G as follows.

R21. Each $d(f)(d(f) \geq 18)$-face gives $1 - 6/(d(f))$ to its incident vertices.
R22. Each 2-vertex gets $3/2$ from its child and $1/2$ from its parent.
R23. Let f be a 4-face. If f is incident with a 2-vertex, then it gets $2/3$ from each of its incident 3^+-vertices. If f is incident with no 2-vertices, then it gets $1/2$ from each of its incident vertices.

The rest of this paper is devoted to checking $ch'(x) \geq 0$ for all $x \in V \cup F$. Let $f \in F(G)$. If $d(f) = 4$, then $ch'(f) = ch(f) + \max\{2/3 \times 3, 1/2 \times 4\} = 0$. If $d(f) \geq 18$, then $ch'(f) = ch(f) - r \times (1 - 6/r) = 0$ by R21.

Let $v \in V(G)$. If $d(v) = 2$, then $ch'(v) = ch(v) + 3/2 + 1/2 = 0$ by R22. If $d(v) = 3$, then $f_{18}^+(v) \geq 2$ and $f_4(v) \leq 1$, and it follows from R21 and R23 that $ch'(v) = ch(v) + 2 \times 2/3 - 2/3 > 0$. Suppose that $d(v) = 4$. Then $ch(v) = 2 \times 4 - 6 = 2$. If $n_2(v) \geq 1$, then v sends at most $(n_2(v) + 2)/2$ to all its adjacent 2-vertices by R22. If $3 \leq n_2(v) \leq 4$, then $f_4(v) \leq 1$ by Lemma 2.3, and it follows that $ch'(v) \geq ch(v) - (n_2(v) + 2)/2 + 2/3 \times 3 - 2/3 = (14 - n_2(v) \times 3)/6 > 0$ by R21 and R23. If $1 \leq n_2(v) \leq 2$, then $f_4(v) \leq 2$, and it follows that $ch'(v) \geq ch(v) - (n_2(v) + 2)/2 + 2/3 \times 2 - 2/3 \times 2 = (2 - n_2(v))/2 \geq 0$. If $n_2(v) = 0$, we have $f_4(v) \leq 2$. Moreover, each 4-face incident with v contains no 2-vertices. By R23, we have $ch'(v) \geq ch(v) + 2/3 \times 2 - 1/2 \times 2 > 0$. Now we complete the proof of Theorem 1.1.

3. Proof of Theorem 1.2

A $3(k)$-*vertex* is a 3-vertex adjacent to exactly k 2-vertices. Let G be a minimal counterexample to Theorem 1.2 in terms of the number of vertices and edges. By minimality of G, it has the following result.

Lemma 3.1. [6]

(a) *no* 2-*vertex is adjacent to two* 2-*vertices;*
(b) *no* 2-*vertex is adjacent to a* 2-*vertex and a* $3(2)$-*vertex;*
(c) *no* 3-*vertex is adjacent to three* 2-*vertices.*

Let G_{23} be the bipartite subgraph of G comprising V and all edges of G that join a 2-vertex to a 3-vertex. Then G_{23} has no isolated 2-vertices by Lemma 3.1(a), and the maximum degree is at most 2 by Lemma 3.1(c), and any component of G_{23} is a path with more than one edges must end in two 3-vertices by Lemma 3.1(b). It follows that $n_3 \geq n_2$. So we can find a matching M in G saturating all 2-vertices. If $uv \in M$ and $d(u) = 2$,v is called the 2-master of u. Each 2-vertex has one 2-master and each vertex of degree Δ can be the 2-master of at most one 2-vertex.

Since G is a planar graph, by Euler's formula, we have

$$\sum_{v\in V}(d(v)-6)+\sum_{f\in F}(2d(f)-6) = -12 < 0.$$

Now we define the initial charge function $ch(x)$ of $x \in V \cup F$ to be $ch(v) = d(v) - 6$ if $v \in V$ and $ch(f) = 2d(f) - 6$ if $f \in F$. It follows that $\sum_{x\in V\cup F} ch(x) < 0$. Note that any discharging procedure preserves the total charge of G. If we can define suitable discharging rules to change the initial charge function ch to the final charge function ch' on $V \cup F$, such that $ch'(x) \geq 0$ for all $x \in V \cup F$, then we get an obvious contradiction. Now we design appropriate discharging rules and redistribute weights accordingly.

R31. Each $d(f)(d(f) \geq 5)$-face gives $2 - 6/(d(f))$ to its incident vertices.

R32. Each 2-vertex receives $3 - 12/(t+1) - 6/g$ from its 2-master.

Let $ch'(x)$ be the new charge obtained by the above rules for all $x \in V \cup F$. If $f \in F(G)$, then $ch'(f) = ch(f) - d(f) \times (2d(f) - 6)/(d(f)) = 0$ by R31. Let $v \in V(G)$. Suppose $d(v) = 3$. Then v can be the 2-master of at most one 2-vertex, and v sends at most $3 - 12/(t+1) - 6/g$ to 2-vertex by R32. In addition, If v is incident with a g-face, then the other faces incident with v are two $(t+1)^+$-faces, for G has no cycles of length from $g+1$ to t. Thus, v receives $(2 - 6/g)$ from its incident g-face and $(2 - 6/(t+1))$ from each of its incident $(t+1)^+$-face by R31. So $ch'(v) \geq ch(v) + 2(2 - 6/(t+1)) + (2 - 6/g) - (3 - 12/(t+1) - 6/g) = 0$ for all g and t. Otherwise, v is incident with three $(t+1)^+$-faces, then $ch'(v) \geq ch(v) + 3(2 - 6/(t+1)) - (3 - 12/(t+1) - 6/g) = 6/g - 6/(t+1) > 0$, for $t+1 > g$. Suppose $d(v) = 2$. Then v receives at most $3 - 12/(t+1) - 6/g$ from its 2-master by R31. If v is incident with a g-face, since G has no cycles of length from $g+1$ to t, then the other face incident with v is a $(t+1)^+$-face, and it follows that $ch'(v) \geq ch(v) + (2 - 6/(t+1)) + (2 - 6/g) + (3 - 12/(t+1) - 6/g) = 0$ for all g and t. Otherwise, v is incident with two $(t+1)^+$-faces, then $ch'(v) \geq ch(v) + 2(2 - 6/(t+1)) + (3 - 12/(t+1) - 6/g) = 3 - 24/(t+1) - 6/g > 0$.

From the above, we can see that $ch'(f) = ch(f) - d(f) \times (2d(f) - 6)/(d(f)) = 0$ for all $f \in F(G)$. Suppose $d(v) = 3$. So $ch'(v) \geq ch(v) + 2(2 - 6/(t+1)) + (2 - 6/g) - (3 - 12/(t+1) - 6/g) = 0$ for all g and t. When v is incident with three $(t+1)^+$-faces, then

$ch'(v) \geq ch(v) + 3(2 - 6/(t+1)) - (3 - 12/(t+1) - 6/g) = 6/g - 6/(t+1) > 0$, for $t+1 > g$. Suppose $d(v) = 2$. If v is incident with a g-face and a $(t+1)^+$-face, then $ch'(v) \geq ch(v) + (2 - 6/(t+1)) + (2 - 6/g) + (3 - 12/(t+1) - 6/g) = 0$ for all g and t.When v is incident with two $(t+1)^+$-faces, then $ch'(v) \geq ch(v) + 2(2 - 6/(t+1)) + (3 - 12/(t+1) - 6/g) = 3 - 24/(t+1) - 6/g$. So when $g = 5$, then $t \geq 13$; when $g = 6$, then $t \geq 11$; when $g = 7$, then $t \geq 11$; when $g = 8$, then $t \geq 10$; when $g = 9$, then $t \geq 10$, and it follows that $ch'(v) \geq 0$.

Our proof of Theorem 1.2 is now complete.

Acknowledgement. This work is supported by research grants NSFC (10971121, 11271106) of China and the Natural Science Foundation of Shandong Province (ZR2012AL08).

References

[1] M. Behzad, *Graphs and Their Chromatic Numbers*, ProQuest LLC, Ann Arbor, MI, 1965.

[2] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, American Elsevier Publishing Co., Inc., New York, 1976.

[3] O. V. Borodin, On the total coloring of planar graphs, *J. Reine Angew. Math.* **394** (1989), 180–185.

[4] O. V. Borodin, A. V. Kostochka and D. R. Woodall, List edge and list total colourings of multigraphs, *J. Combin. Theory Ser. B* **71** (1997), no. 2, 184–204.

[5] O. V. Borodin, A. V. Kostochka and D. R. Woodall, Total colorings of planar graphs with large maximum degree, *J. Graph Theory* **26** (1997), no. 1, 53–59.

[6] O. V. Borodin, A. V. Kostochka and D. R. Woodall, Total colourings of planar graphs with large girth, *European J. Combin.* **19** (1998), no. 1, 19–24.

[7] H. Y. Chen, X. Tan and J. L. Wu, The linear arboricity of planar graphs without 5-cycles with chords, *Bull. Malays. Math. Sci. Soc. (2)*, accepted.

[8] A. J. Dong, X. Zhang and G. J. Li, Equitable coloring and equitable choosability of planar graphs without 5- and 7-cycles, *Bull. Malays. Math. Sci. Soc. (2)* **35** (2012), no. 4, 897-910.

[9] D. Du, L. Shen and Y. Wang, Planar graphs with maximum degree 8 and without adjacent triangles are 9-totally-colorable, *Discrete Appl. Math.* **157** (2009), no. 13, 2778–2784.

[10] J. Hou, Y. Zhu, G. Liu, J. Wu and M. Lan, Total colorings of planar graphs without small cycles, *Graphs Combin.* **24** (2008), no. 2, 91–100.

[11] A. V. Kostochka, The total chromatic number of any multigraph with maximum degree five is at most seven, *Discrete Math.* **162** (1996), no. 1-3, 199–214.

[12] Ł. Kowalik, J.-S. Sereni and R. Škrekovski, Total-coloring of plane graphs with maximum degree nine, *SIAM J. Discrete Math.* **22** (2008), no. 4, 1462–1479.

[13] J. Li and G. Liu, On f-edge cover coloring of nearly bipartite graphs, *Bull. Malays. Math. Sci. Soc. (2)* **34** (2011), no. 2, 247–253.

[14] Q. Ma, J.-L. Wu and X. Yu, Planar graphs without 5-cycles or without 6-cycles, *Discrete Math.* **309** (2009), no. 10, 2998–3005.

[15] D. P. Sanders and Y. Zhao, On total 9-coloring planar graphs of maximum degree seven, *J. Graph Theory* **31** (1999), no. 1, 67–73.

[16] P. Wang and J.-L. Wu, A note on total colorings of planar graphs without 4-cycles, *Discuss. Math. Graph Theory* **24** (2004), no. 1, 125–135.