Multiple Results for Critical Quasilinear Elliptic Systems Involving Concave-Convex Nonlinearities and Sign-Changing Weight Functions

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Abstract. This paper is devoted to study the multiplicity of nontrivial nonnegative or positive solutions to the following systems

$$\begin{cases} -\triangle_p u = \lambda a_1(x)|u|^{q-2}u + b(x)F_u(u,v), & \text{in } \Omega, \\ -\triangle_p v = \lambda a_2(x)|v|^{q-2}v + b(x)F_v(u,v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$; $1 < q < p < N$, $p^* = (Np)/(N-p)$; $\triangle_p w = \text{div}(|\nabla w|^{p-2}\nabla w)$ denotes the p-Laplacian operator; $\lambda > 0$ is a positive parameter; $a_i \in L^\Theta(\Omega)(i = 1,\ 2)$ with $\Theta = p^*/(p^*-q)$ and $b \in L^\infty(\Omega)$ are allowed to change sign; $F \in C^1((R^+)^2, R^+)$ is positively homogeneous of degree p^*, that is, $F(tz) = t^{p^*}F(z)$ holds for all $z \in (R^+)^2$ and $t > 0$, here, $R^+ = [0, +\infty)$. The multiple results of weak solutions for the above critical quasilinear elliptic systems are obtained by using the Ekeland's variational principle and the mountain pass theorem.

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1. Introduction and main results

Let $\Omega \subset R^N$ be a bounded domain with smooth boundary $\partial\Omega$. We are concerned with the following problems

$$(1.1) \quad \begin{cases} -\triangle_p u = \lambda a_1(x)|u|^{q-2}u + b(x)F_u(u,v), & \text{in } \Omega, \\ -\triangle_p v = \lambda a_2(x)|v|^{q-2}v + b(x)F_v(u,v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $1 < q < p < N$, $p^* = (Np)/(N-p)$; $\triangle_p w = \text{div}(|\nabla w|^{p-2}\nabla w)$ denotes the p-Laplacian operator; $\lambda > 0$ is a positive parameter; $a_i \in L^\Theta(\Omega)(i = 1,\ 2)$ with $\Theta = p^*/(p^*-q)$ and

$b \in L^\infty(\Omega)$ are allowed to change sign; $F \in C^1((R^+)^2, R^+)$ is positively homogeneous of degree p^*, that is, $F(tz) = t^{p^*}F(z)$ holds for all $z \in (R^+)^2$ and $t > 0$, here, $R^+ = [0, +\infty)$.

In recent years, more and more attention have been paid to the existence and multiplicity of nonnegative or positive solutions for the elliptic problems involving concave-convex nonlinearities and critical Sobolev exponent. Results relating to the semilinear problems can be found in [2, 6, 8, 9, 16, 19, 21, 22, 24, 27, 28], and the references therein. For the quasilinear problems, the corresponding results can be found in [1, 7, 10–12, 14, 15, 17, 18, 26, 29], and the references therein. By the results of the above papers we know that the number of nontrivial solutions for problem (1.1) is affected by the concave-convex nonlinearities, and the sign of solutions for problem (1.1) depends largely on the sign of the weight functions. In general, the nonnegative solutions of problem (1.1) are obtained when the weight functions satisfy some conditions. In order to apply the maximum principle to guarantee the positivity of solutions for problem (1.1), they usually require that the weight functions are nonnegative.

Set $u = v$, $a_1(x) = a_2(x)$. Then problem (1.1) reduces to the quasilinear scalar elliptic equations with concave-convex nonlinearities

$$(1.2) \qquad \begin{cases} -\triangle_p u = \lambda a_1(x)|u|^{q-2}u + b(x)|u|^{p^*-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Problem (1.2) was originally considered in [2] when $p = 2$, $a_1(x) = b(x) \equiv 1$. They proved that problem (1.2) has at least two positive solutions for $\lambda > 0$ small enough. Subsequently, for the case $p = 2$, the multiple results for solutions of problem (1.2) were extended to the variable $a_1(x)$ and the variable $b(x)$ (see [8, 9, 21, 28]). For the case $p > 1$, [14] and [15] have considered the case $a_1(x) = b(x) \equiv 1$ of problem (1.2). By using the variational approach, they obtained that problem (1.2) has at least two positive solutions for $\lambda > 0$ small enough. For more general cases, the multiple results for solutions of problem (1.2) can be found in [10] and [18].

For the semilinear elliptic systems, Wu in [29] has considered the following semilinear elliptic systems

$$(1.3) \qquad \begin{cases} -\triangle u = \lambda a_1(x)|u|^{q-2}u + \frac{\alpha}{\alpha+\beta}b(x)|u|^{\alpha-2}u|v|^\beta, & \text{in } \Omega, \\ -\triangle v = \lambda a_2(x)|v|^{q-2}v + \frac{\beta}{\alpha+\beta}b(x)|u|^\alpha|v|^{\beta-2}v, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $1 < q < 2$; $\alpha > 1$, $\beta > 1$ satisfy $2 < \alpha + \beta < 2^* (2^* = (2N)/(N-2)$ if $N \geq 3$, $2^* = \infty$ if $N = 2$); $\lambda > 0$ is a positive parameter; a_1, $a_2 \in L^{(\alpha+\beta)/(\alpha+\beta-q)}(\Omega)$ are allowed to change sign; $b(x) \in C(\overline{\Omega})$ with $\|b\|_\infty = 1$ and $b \geq 0$. By the variational approach involving the Nehari manifold, he proved that problem (1.3) has at least two nontrivial nonnegative solutions for $\lambda > 0$ small enough. Subsequently, Hsu and Lin in [19] considered the case $a_1(x) = b(x) \equiv 1$, $a_2(x) \equiv \mu/\lambda$ ($\mu > 0$) and $\alpha + \beta = 2^*$ of problem (1.3). By using the similar methods of [29], they obtained that problem (1.3) has at least two positive solutions for $\lambda > 0$ small enough.

In a recent paper, Hsu in [17] has considered the case $a_1(x) = b(x) \equiv 1$, $a_2(x) \equiv \mu/\lambda$ ($\mu > 0$) and $F(u,v) = 2/(\alpha+\beta)|u|^\alpha|v|^\beta$, $\alpha > 1$, $\beta > 1$ satisfy $\alpha + \beta = p^*$ of problem (1.1). With the help of the Nehari manifold, he proved that problem (1.1) has at least two positive solutions for $\lambda > 0$ small enough. The variational approach involving the Nehari manifold require that the nonlinearity F is second order derivative about u and v. However,

we only assume that $F \in C^1((R^+)^2, R^+)$, the Nehari manifold technique is invalid. Hence, in order to obtain the multiple results for solutions of problem (1.1), we need to look for other methods. In addition, as far as we know, very few multiple results of problem (1.1) have been obtained in the literate for a more general term F when the weight functions are allowed to change sign. So our goal will be to extend the corresponding results in [17] to a more general term F and the sign-changing weight functions. In the present paper, under the conditions of the weight functions are allowed to change sign, we consider problem (1.1) and obtain the existence and multiplicity of nontrivial nonnegative or positive solutions for problem (1.1) by applying the Ekeland's variation principle and the mountain pass theorem. The main results of this paper extend the corresponding results in [17] and our proof is completely different with him. In fact, our proof for the principal result is much simpler than that of [17, Theorem 1.2]. Moreover, even in the elliptic equation case, our results also extend the corresponding results in [10] and [15].

Before stating our results, we give some notations and assumptions. Let $z = (u, v)$, $|z|^p = |u|^p + |v|^p$, $E = W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$, $\|w\|_s = (\int_\Omega |w|^s dx)^{1/s} (1 < s < \infty)$, $\|w\|_\infty = \sup_{x \in \Omega}\{|w(x)|\}$, $\|w\| = (\int_\Omega |\nabla w|^p dx)^{1/p}$, $\|z\|_E^p = \|u\|^p + \|v\|^p$, $B(x_0, \delta) = \{x \in \Omega : |x - x_0| < \delta\}$.

- (a_1) $a_i(x) \in L^\Theta(\Omega)$ and $a_i^+ = \max\{a_i(x), 0\} \neq 0$, where $i = 1, 2$;
- (a_2) There exist positive constants β_0, δ_0 and $x_0 \in \Omega$ such that $B(x_0, 2\delta_0) \subset \Omega$ and $a_i(x) \geq \beta_0 (i = 1, 2)$ in $B(x_0, 2\delta_0)$.
- (b_1) $b(x) \in L^\infty(\Omega)$ and $b^+ = \max\{b(x), 0\} \neq 0$;
- (b_2) $b(x_0) = \|b\|_\infty$ and $b(x) > 0$ for all $x \in B(x_0, 2\delta_0)$;
- (b_3) There exists $k > (N - p)/(p - 1)$ such that

$$b(x) = b(x_0) + o(|x - x_0|^k) \text{ as } x \to x_0.$$

In addition, we denote positive constants by C, C_1, C_2, $\cdots$. The main results of this paper are as follows.

Theorem 1.1. *Let $1 < q < p < N$, (a_1) and (b_1) hold. Assume that $F \in C^1((R^+)^2, R^+)$ is positively homogeneous of degree p^* and $F_u(0, v) = F_v(u, 0) = 0$ for all u, $v \in R^+$, then there exists $\Lambda > 0$ such that problem (1.1) for all $\lambda \in (0, \Lambda)$ has at least one solution $z_\lambda = (u_\lambda, v_\lambda)$ satisfies that $u_\lambda \geq 0$, $v_\lambda \geq 0$ in Ω and $u_\lambda \neq 0$, $v_\lambda \neq 0$. Moreover, if $a_i(x) \geq 0 (i = 1, 2)$ and $b(x) \geq 0$, then z_λ is a positive solution, that is, $u_\lambda > 0$, $v_\lambda > 0$ in Ω.*

Theorem 1.2. *Let $1 < p < N$, $(N(p - 1))/(N - p) < q < p < N$, (a_1), (a_2), (b_1), (b_2) and (b_3) hold. Assume that $F \in C^1((R^+)^2, R^+)$ is positively homogeneous of degree p^* and $F_u(u, 0) = F_u(0, v) = F_v(u, 0) = F_v(0, v) = 0$ for all u, $v \in R^+$, then there exists $\lambda^* > 0$ such that problem (1.1) for all $\lambda \in (0, \lambda^*)$ has at least two nontrivial nonnegative solutions $z_\lambda = (u_\lambda, v_\lambda)$ and $\bar{z}_\lambda = (\bar{u}_\lambda, \bar{v}_\lambda)$ satisfy that $u_\lambda \neq 0$, $v_\lambda \neq 0$, and $\bar{u}_\lambda \neq 0$, $\bar{v}_\lambda \neq 0$ in Ω. Moreover, if $a_i(x) \geq 0 (i = 1, 2)$ and $b(x) \geq 0$, then z_λ and $\bar{z}_\lambda$ are two positive solutions of problem (1.1), that is, $u_\lambda > 0$, $v_\lambda > 0$, and $\bar{u}_\lambda > 0$, $\bar{v}_\lambda > 0$ in Ω.*

From elliptic systems reduce to elliptic equations, our Theorem 1.2 can be described as

Corollary 1.1. *Let $1 < p < N$ and $(N(p - 1))/(N - p) < q < p$. Assume that (a_1), (a_2), (b_1), (b_2) and (b_3) hold. Then there exists $\lambda^* > 0$ such that problem (1.2) for all $\lambda \in (0, \lambda^*)$ has at least two nontrivial nonnegative solutions u_λ and $\bar{u}_\lambda$ satisfy that $u_\lambda \neq 0$ and $\bar{u}_\lambda \neq 0$ in Ω. Moreover, if $a_1(x) \geq 0$ and $b(x) \geq 0$, then $u_\lambda > 0$ and $\bar{u}_\lambda > 0$ in Ω.*

Remark 1.1. [17, Theorem 1.1 and 1.2] are the special cases of our Theorem 1.1 and 1.2 corresponding to $a_1(x) = b(x) \equiv 1$, $a_2(x) \equiv \mu/\lambda$ $(\mu > 0)$ and $F(u,v) = 2/(\alpha+\beta)|u|^\alpha|v|^\beta$, $\alpha > 1$, $\beta > 1$ satisfy $\alpha + \beta = p^*$. There are other functions F satisfying the conditions of our Theorem 1.1. For example,

$$F(u,v) = |u|^{\alpha_1}|v|^{\beta_1} + |u|^{\alpha_2}|v|^{\beta_2}$$

where $\alpha_i > 1$, $\beta_i > 1$ $(i = 1,2)$, $\alpha_1 \neq \alpha_2$ and $\alpha_1 + \beta_1 = \alpha_2 + \beta_2 = p^*$. Obviously, F satisfies the conditions of our theorems.

Remark 1.2. In the constant coefficients case, it implies from $p \geq 3$ that $(N(p-1))/(N-p) \leq p^* - 2/(p-1)$. Hence our Corollary 1.1 extend [15, Theorem 2].

Remark 1.3. In the case $p \geq 3$, our Corollary 1.1 extend [10, Theorem 6.3]. In fact, in the case $p \geq 3$, the condition $p > q > p^* - 2/(p-1)$ deduce that $N > p(1 + p(p-1)/2)$(see Remark 1 in [15]), which implies that our condition (b_3) is weaker than the condition (b) of [10, Theorem 6.3].

2. Palais-Smale condition

In this section, we show that the corresponding functional of problem (1.1) satisfies the $(PS)_c$ condition. Let $u^\pm = \max\{\pm u, 0\}$, the corresponding functional of problem (1.1) is

$$I(u,v) = \frac{1}{p}\int_\Omega (|\nabla u|^p + |\nabla v|^p)dx - \frac{\lambda}{q}\int_\Omega [a_1(x)(u^+)^q + a_2(x)(v^+)^q]dx - \int_\Omega b(x)F(u^+, v^+)dx$$

for $(u,v) \in E$. Under the hypotheses of our theorems, it is obvious that I is a C^1 functional. It is well known that any critical point of I in E is a weak solution of problem (1.1). Hence, in order to obtain the nontrivial solutions of problem (1.1), we only need to look for the nontrivial critical points of I in E. In addition, since F is positively homogeneous of degree p^*, we have the so-called Euler identity

$$(2.1) \qquad\qquad\qquad z \cdot \nabla F(z) = p^* F(z),$$

and

$$(2.2) \qquad\qquad F(z) \leq M|z|^{p^*} \quad \text{for all} \quad z \in (R^+)^2,$$

where $M = \max\limits_{\{z \in (R^+)^2 : |z|=1\}} F(z) > 0$. Now we first give some preliminaries.

Definition 2.1. *Let $c \in R$, E^* denote the dual space of the Banach space E.*

    (i) *A sequence $\{z_n\} \subset E$ is called a $(PS)_c$ sequence of I if $I(u_n) \to c$ and $I'(z_n) \to 0$ in E^* as $n \to \infty$;*

    (ii) *We call that I satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence $\{z_n\} \subset E$ of I has a convergent subsequence.*

Lemma 2.1. *Assume that $F \in C^1((R^+)^2, R^+)$ is positively homogeneous of degree p^* such that $F_u(0,v) = F_v(u,0) = 0$ for all u, $v \in R^+$, (a_1) and (b_1) hold. Let $\{z_n\} = \{(u_n, v_n)\} \subset E$ be a $(PS)_c$ sequence of I, then $\{z_n\}$ is bounded.*

Proof. By the Sobolev imbedding theorem, there exists $C > 0$ such that

$$(2.3) \qquad\qquad \|w\|_s \leq C\|w\|, \quad \text{for all} \quad w \in W_0^{1,p}(\Omega) \quad \text{and} \quad 1 \leq s \leq p^*.$$

Let $A = \max\{\|a_1\|_\Theta, \|a_2\|_\Theta\}$. According to the hypothesis (a_1), for each $\varepsilon > 0$, by the Hölder inequality and the Young inequality, we imply from (2.3) that

$$\left| \frac{\lambda}{q} \int_\Omega [a_1(x)(u^+)^q + a_2(x)(v^+)^q] dx \right|$$

(2.4)
$$\leq \frac{\lambda}{q} \left(\|a_1\|_\Theta \|u\|_{p^*}^q + \|a_2\|_\Theta \|v\|_{p^*}^q \right) \leq \frac{\lambda}{q} AC^q (\|u\|^q + \|v\|^q)$$

$$\leq \varepsilon(\|u\|^p + \|v\|^p) + \frac{2}{q}(p-q)\varepsilon^{-\frac{q}{p-q}} \left(\frac{\lambda A}{p} C^q \right)^{\frac{p}{p-q}} = \varepsilon \|z\|_E^p + C(\varepsilon)\lambda^{\frac{p}{p-q}},$$

where $C(\varepsilon) = 2/q(p-q)\varepsilon^{-q/(p-q)}(A/pC^q)^{p/(p-q)}$. Let $\{z_n\}$ be a $(PS)_c$ sequence of I. Using the hypotheses that F is positively homogeneous of degree p^* such that $F_u(0,v) = F_v(u,0) = 0$ for all $u, v \in R^+$, we derive from (2.4) that

$$p^* I(z_n) - \langle I'(z_n), z_n \rangle = \left(\frac{p^*}{p} - 1 \right) \|z_n\|_E^p + \frac{\lambda}{q}(q-p^*) \int_\Omega [a_1(x)(u_n^+)^q + a_2(x)(v_n^+)^q] dx$$

$$\geq \left(\frac{p^*}{p} - 1 - (p^* - q)\varepsilon \right) \|z_n\|_E^p - (p^* - q)C(\varepsilon)\lambda^{\frac{p}{p-q}}.$$

It follows that

$$\left(\frac{p^*}{p} - 1 - (p^* - q)\varepsilon \right) \|z_n\|_E^p \leq p^* c + (p^* - q)C(\varepsilon)\lambda^{\frac{p}{p-q}} + o(\|z_n\|_E).$$

Let $\varepsilon < (p^* - p)/(p(p^* - q))$, we obtain $\{z_n\}$ is bounded in E. ∎

Let
$$S = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p dx}{(\int_\Omega |u|^{p^*} dx)^{p/p^*}}$$

denote the best Sobolev constant for the imbedding $W_0^{1,p}(\Omega)$ in $L^{p^*}(\Omega)$. S is achieved on $\Omega = R^N$ by the function $W(x) = K/((1 + |x|^{p/(p-1)})^{(N-p)/p^2})$, where

$$K = \left[N((N-p)/(p-1))^{p-1} \right]^{(N-p)/p^2}$$

(see [12] or [23]). Define

$$S_F := \inf_{(u,v) \in E} \left\{ \frac{\int_\Omega (|\nabla u|^p + |\nabla v|^p) dx}{(\int_\Omega F(u^+, v^+) dx)^{p/p^*}} : \int_\Omega F(u^+, v^+) dx > 0 \right\}.$$

We have the following lemmas.

Lemma 2.2. *Assume that $F \in C^1((R^+)^2, R^+)$ is positively homogeneous of degree p^* such that $F_u(0,v) = F_v(u,0) = 0$ for all $u, v \in R^+$, (a_1) and (b_1) hold. Let $\{z_n\}$ be a $(PS)_c$ sequence of I with $z_n \rightharpoonup z$ in E as $n \to \infty$, then there exists a positive constant B depending on p, q, N, S, $\|a_1\|_\Theta$ and $\|a_2\|_\Theta$ such that*

$$I'(z) = 0 \quad and \quad I(z) \geq -B\lambda^{\frac{p}{p-q}}.$$

Proof. Let $\{z_n\} = \{(u_n, v_n)\}$ be a $(PS)_c$ sequence of I with $z_n \rightharpoonup z = (u,v)$ in E. Then we have

$$I'(z_n) \to 0, \quad \text{strongly in } E^* \text{ as } n \to \infty.$$

Since $\{z_n\}$ is bounded, we can obtain a subsequence still denoted by $\{z_n\}$ such that

$$
\begin{cases}
z_n = (u_n, v_n) \rightarrow (u, v) = z, & \text{in } L^s(\Omega) \times L^s(\Omega), \quad 1 < s < p^*, \\
z_n = (u_n, v_n) \rightarrow (u, v) = z, & \text{a.e. in } \Omega, \\
\nabla u_n \rightarrow \nabla u, \quad \nabla v_n \rightarrow \nabla v, & \text{a.e. in } \Omega.
\end{cases}
$$

Consequently, passing to the limit in $\langle I'(z_n), (\varphi, \psi) \rangle$ as $n \rightarrow \infty$, and using the hypotheses of our Lemma 2.2, we have

$$
\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx - \lambda \int_\Omega a_1(x)(u^+)^{q-1} \varphi \, dx - \int_\Omega b(x) F_u(u^+, v^+) \varphi \, dx = 0
$$

and

$$
\int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla \psi \, dx - \lambda \int_\Omega a_2(x)(v^+)^{q-1} \psi \, dx - \int_\Omega b(x) F_v(u^+, v^+) \psi \, dx = 0
$$

for all $(\varphi, \psi) \in E$, that is, $I'(z) = 0$.

In particular, we have $\langle I'(z), z \rangle = 0$, which implies from (2.1) that

$$
\|z\|_E^p = \lambda \int_\Omega [a_1(x)(u^+)^q + a_2(x)(v^+)^q] \, dx + p^* \int_\Omega b(x) F(u^+, v^+) \, dx.
$$

It follows that

$$
I(z) = \left(\frac{1}{p} - \frac{1}{p^*} \right) \|z\|_E^p - \left(\frac{1}{q} - \frac{1}{p^*} \right) \lambda \int_\Omega [a_1(x)(u^+)^q + a_2(x)(v^+)^q] \, dx.
$$

Using the Hölder inequality, the Young inequality and the Sobolev imbedding theorem, one has

$$
\begin{aligned}
I(z) &= \left(\frac{1}{p} - \frac{1}{p^*} \right) \|z\|_E^p - \left(\frac{1}{q} - \frac{1}{p^*} \right) \lambda \int_\Omega [a_1(x)(u^+)^q + a_2(x)(v^+)^q] \, dx \\
&\geq \frac{1}{N} \|z\|_E^p - \frac{p^* - q}{p^* q} \lambda \left(\|a_1\|_\Theta \|u\|_{p^*}^q + \|a_2\|_\Theta \|v\|_{p^*}^q \right) \\
&\geq \frac{1}{N} \|z\|_E^p - \frac{p^* - q}{p^* q} \lambda S^{-\frac{q}{p}} \left(\|a_1\|_\Theta \|u\|^q + \|a_2\|_\Theta \|v\|^q \right) \\
&\geq \frac{1}{N} \|z\|_E^p - \left(\frac{1}{N} \|z\|_E^p + B\lambda^{\frac{p}{p-q}} \right) = -B\lambda^{\frac{p}{p-q}},
\end{aligned}
$$

where B is a positive constant depending on p, q, N, S, $\|a_1\|_\Theta$ and $\|a_2\|_\Theta$. ∎

Now we introduce the following version of the Brezis-Lieb Lemma (see [4] or [5]).

Lemma 2.3. *Assume that $G \in C^1(R^2)$ with $G(0,0) = 0$ and $|\partial G(z)/\partial u|$, $|\partial G(z)/\partial v| \leq C_1 |z|^{s-1}$, some $1 \leq s < \infty$. Let z_n be a bounded sequence in $L^s(\Omega) \times L^s(\Omega)$, and such that $z_n \rightharpoonup z$ a.e. Ω. Then, as $n \rightarrow \infty$,*

$$
\int_\Omega G(z_n) \, dx = \int_\Omega G(z_n - z) \, dx + \int_\Omega G(z) \, dx + o(1).
$$

Lemma 2.4. *Assume that $F \in C^1((R^+)^2, R^+)$ is positively homogeneous of degree p^* such that $F_u(0, v) = F_v(u, 0) = 0$ for all u, $v \in R^+$, (a_1) and (b_1) hold. then I satisfies the $(PS)_c$ condition with c satisfying*

$$
c < \frac{p}{N-p} \|b\|_\infty^{-\frac{N-p}{p}} \left(\frac{S_F}{p^*} \right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}},
$$

where B is the positive constant give in Lemma 2.2.

Proof. Let $\{z_n = (u_n, v_n)\} \subset E$ be a $(PS)_c$ sequence of I with $c < p/(N-p)\|b\|_\infty^{-(N-p)/p}$ $(S_F/p^*)^{N/p} - B\lambda^{p/(p-q)}$. By Lemma 2.1, we know that $\{z_n\}$ is bounded. Up to a subsequence, we may assume that

$$\begin{cases} z_n = (u_n, v_n) \rightharpoonup (u, v) = z, & \text{in } E, \\ z_n = (u_n, v_n) \to (u, v) = z, & \text{a.e. on } \Omega, \\ z_n = (u_n, v_n) \to (u, v) = z, & \text{in } L^s(\Omega) \times L^s(\Omega), \quad 1 \le s < p^*. \end{cases}$$

From Lemma 2.2, we have that $I'(z) = 0$ and

$$\lambda \int_\Omega [a_1(x)(u_n^+)^q + a_2(x)(v_n^+)^q]dx = \lambda \int_\Omega [a_1(x)(u^+)^q + a_2(x)(v^+)^q]dx + o(1).$$

Let $\widetilde{z}_n = (\widetilde{u}_n, \widetilde{v}_n)$, where $\widetilde{u}_n = u_n - u$, $\widetilde{v}_n = v_n - v$. From Lemma 2.3, one has

$$\|\widetilde{z}_n\|_E^p = \|z_n\|_E^p - \|z\|_E^p + o(1).$$

Since $F \in C^1((R^+)^2, R^+)$ is positively homogeneous of degree p^* such that $F_u(0, v) = F_v(u, 0) = 0$ for all u, $v \in R^+$, it follows from Lemma 2.3 that

$$\int_\Omega F((\widetilde{z}_n)^+)dx = \int_\Omega F(z_n^+)dx - \int_\Omega F(z^+)dx + o(1),$$

which implies from (b_1) that

$$\int_\Omega b(x)F((\widetilde{z}_n)^+)dx = \int_\Omega b(x)F(z_n^+)dx - \int_\Omega b(x)F(z^+)dx + o(1).$$

Since $I(z_n) = c + o(1)$ and $I'(z_n) = o(1)$, we obtain

$$(2.5) \qquad \frac{1}{p}\|\widetilde{z}_n\|_E^p - \int_\Omega b(x)F((\widetilde{z}_n)^+)dx = c - I(z) + o(1).$$

and

$$(2.6) \qquad \|\widetilde{z}_n\|_E^p - p^* \int_\Omega b(x)F((\widetilde{z}_n)^+)dx = o(1).$$

From (2.6), we may assume that

$$\|\widetilde{z}_n\|_E^p \to p^*l, \quad \int_\Omega b(x)F((\widetilde{z}_n)^+)dx \to l.$$

Assume that $l > 0$, by the definition of S_F, we have

$$\|\widetilde{z}_n\|_E^p \ge S_F \left(\int_\Omega F((\widetilde{z}_n)^+)dx \right)^{\frac{p}{p^*}} \ge S_F \left(\int_\Omega \frac{b(x)}{\|b\|_\infty} F((\widetilde{z}_n)^+)dx \right)^{\frac{p}{p^*}}$$

$$= S_F \|b\|_\infty^{-\frac{N-p}{N}} \left(\int_\Omega b(x)F((\widetilde{z}_n)^+)dx \right)^{\frac{p}{p^*}}.$$

As $n \to \infty$, we deduce that

$$l \ge \|b\|_\infty^{-\frac{N-p}{p}} \left(\frac{S_F}{p^*} \right)^{\frac{N}{p}}.$$

It follows from (2.5) and Lemma 2.2 that

$$c = \left(\frac{p^*}{p} - 1 \right) l + I(z) \ge \frac{p}{N-p} \|b\|_\infty^{-\frac{N-p}{p}} \left(\frac{S_F}{p^*} \right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}},$$

which contradicts the fact $c < p/(N-p)\|b\|_\infty^{-(N-p)/p} (S_F/p^*)^{N/p} - B\lambda^{p/(p-q)}$. Therefore, we have $l = 0$, which implies that

$$z_n \to z \quad \text{in} \quad E.$$

Hence I satisfies the $(PS)_c$ condition with

$$c < p/(N-p)\|b\|_\infty^{-(N-p)/p} (S_F/p^*)^{N/p} - B\lambda^{p/(p-q)}. \qquad \blacksquare$$

3. The existence of solutions for problem (1.1)

In this section, we show that there exists $\Lambda > 0$ such that problem (1.1) has a nontrivial nonnegative or positive solution $z_\lambda = (u_\lambda, v_\lambda)$ with $I(z_\lambda) < 0$ for any $\lambda \in (0, \Lambda)$. More precisely, we prove our Theorem 1.1 by the Ekeland's variational principal.

Lemma 3.1. *Let* r, $s > 1$, $g \in L^s(\Omega)$ *and* $g^+ = max\{g(x), 0\} \neq 0$. *Then there exists* $w_0 \in C_0^\infty(\Omega)$ *such that* $\int_\Omega g(x)(w_0^+)^r dx > 0$.

Proof. Set $s_1 = (rs)/(s-1)$. Define

$$\chi_g = \begin{cases} 1, & \text{in } \{x| \, x \in \Omega, g(x) > 0\}, \\ 0, & \text{in } \{x| \, x \in \Omega, g(x) \leq 0\}. \end{cases}$$

Since $C_0^\infty(\Omega)$ is dense in $L^{s_1}(\Omega)$, there exist $\{w_n\} \subset C_0^\infty(\Omega)$ such that

$$w_n \to \chi_g \quad \text{in} \quad L^{s_1}(\Omega),$$

which implies that

(3.1) $w_n^+ \to \chi_g^+ = \chi_g \quad \text{in } L^{s_1}(\Omega).$

It follows from (3.1) and $g^+ \neq 0$ that

(3.2) $\int_\Omega g(x)(w_n^+)^r dx \to \int_\Omega g(x)\chi_g^r dx = \int_\Omega g^+ dx > 0, \quad \text{as } n \to \infty.$

Indeed, in view of $r > 1$, there is a positive integer m such that $0 < r - m \leq 1$. In addition, it is obvious to obtain the following inequality

$$(a+b)^{r-m} \leq a^{r-m} + b^{r-m}, \quad \text{for any } a \geq 0, \, b \geq 0.$$

By the above inequality and the Hölder inequality, we have

$$\left| \int_\Omega g(x)[(w_n^+)^r - \chi_g^r]dx \right|$$

$$\leq \sum_{i=1}^{m-1} \left| \int_\Omega g(x)(w_n^+)^{r-i}\chi_g^{i-1}(w_n^+ - \chi_g)dx \right| + \left| \int_\Omega g(x)\chi_g^m[(w_n^+)^{r-m} - \chi_g^{r-m}]dx \right|$$

$$\leq \sum_{i=1}^{m-1} \|g\|_s\|w_n^+\|_{s_1}^{r-i}\|\chi_g\|_{s_1}^{i-1}\|w_n^+ - \chi_g\|_{s_1} + \int_\Omega |g(x)|\chi_g^m|w_n^+ - \chi_g|^{r-m}dx$$

$$\leq \sum_{i=1}^{m-1} \|g\|_s\|w_n^+\|_{s_1}^{r-i}\|\chi_g\|_{s_1}^{i-1}\|w_n^+ - \chi_g\|_{s_1} + \|g\|_s\|\chi_g\|_{s_1}^m\|w_n^+ - \chi_g\|_{s_1}^{r-m},$$

which implies that (3.2) holds. It follows from (3.2) that there exists $w_0 \in C_0^\infty(\Omega)$ such that $\int_\Omega g(x)(w_0^+)^r dx > 0$. $\blacksquare$

Proof of Theorem 1.1. According to the hypothesis (a_1), we obtain that (2.4) holds. From (2.2)–(2.4), we deduce that

$$I(z) \geq \left(\frac{1}{p} - \varepsilon\right) \|z\|_E^p - C(\varepsilon)\lambda^{\frac{p}{p-q}} - 2^{\frac{p^*}{p}} M\|b\|_\infty (\|u\|_{p^*}^{p^*} + \|v\|_{p^*}^{p^*})$$

$$\geq \left(\frac{1}{p} - \varepsilon\right) \|z\|_E^p - 2^{\frac{p^*}{p}} C^{p^*} M\|b\|_\infty \|z\|_E^{p^*} - C(\varepsilon)\lambda^{\frac{p}{p-q}}$$

Let $\varepsilon < 1/p$, we can find $\rho > 0$ and $\Lambda_1 > 0$ such that

(3.3) $\qquad I(z) > 0 \quad \text{if} \quad \|z\|_E = \rho \quad \text{and} \quad I(z) > -C_2 \quad \text{if} \quad \|z\|_E \leq \rho,$

for any $\lambda \in (0, \Lambda_1)$, where $C_2 = C(\varepsilon)\lambda^{p/(p-q)}$.

From Lemma 3.1, we obtain that there exist $\varphi_0, \ \psi_0 \in C_0^\infty(\Omega) \subset W_0^{1,p}(\Omega)$ such that

(3.4) $$\int_\Omega a_1(x)(\varphi_0^+)^q dx > 0, \quad \int_\Omega a_2(x)(\psi_0^+)^q dx > 0.$$

Therefore, one has

$$I(k\varphi_0, k\psi_0) = \frac{1}{p}k^p \int_\Omega (|\nabla\varphi_0|^p + |\nabla\psi_0|^p)dx - \frac{\lambda}{q}k^q \int_\Omega [a_1(x)(\varphi_0^+)^q + a_2(x)(\psi_0^+)^q]dx$$

$$- k^{p^*} \int_\Omega b(x)F(\varphi_0^+, \psi_0^+)dx$$

$$\leq \frac{1}{p}k^p \int_\Omega (|\nabla\varphi_0|^p + |\nabla\psi_0|^p)dx - \frac{\lambda}{q}k^q \int_\Omega [a_1(x)(\varphi_0^+)^q + a_2(x)(\psi_0^+)^q]dx + C_3 k^{p^*},$$

where $C_3 = \|b\|_\infty \int_\Omega F(\varphi_0^+, \psi_0^+)dx + 1 > 0$. Fix $\lambda \in (0, \Lambda_1)$, noticing that $p > q > 1$, it implies from (3.4) that there exists $k = k(\lambda) > 0$ small enough such that

$$I(k\varphi_0, k\psi_0) < 0.$$

Thus we deduce that

$$c_\lambda = \inf_{z \in B_\rho(0)} I(z) < 0 < \inf_{z \in \partial B_\rho(0)} I(z).$$

By applying the Ekeland's variational principle in $\overline{B_\rho(0)}$ (see [13]), we obtain that there exists a $(PS)_{c_\lambda}$ sequence $\{z_n\} = \{(u_n, v_n)\} \subset \overline{B_\rho(0)}$ with c_λ.

According to (2.2) and the Minkowski inequality, we have

$$\left(\int_\Omega F(u^+, v^+)dx\right)^{\frac{p}{p^*}} \leq M^{\frac{p}{p^*}} \left(\int_\Omega |z|^{p^*}dx\right)^{\frac{p}{p^*}} \leq M^{\frac{p}{p^*}} \left[\left(\int_\Omega u^{p^*}dx\right)^{\frac{p}{p^*}} + \left(\int_\Omega v^{p^*}dx\right)^{\frac{p}{p^*}}\right]$$

$$\leq M^{\frac{p}{p^*}} \frac{1}{S} \int_\Omega (|\nabla u|^p + |\nabla v|^p)dx.$$

It implies that

(3.5) $$S_F \geq SM^{-\frac{p}{p^*}} > 0.$$

Hence we can choose $0 < \Lambda < \Lambda_1$ such that

$$0 < \frac{p}{N-p}\|b\|_\infty^{-\frac{N-p}{p}} \left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}}, \quad \forall \lambda \in (0, \Lambda).$$

It follows from $c_\lambda < 0$ and Lemma 2.4 that I satisfies the $(PS)_{c_\lambda}$ condition. Therefore, one has a subsequence still denoted by $\{z_n\}$ and $z_\lambda = (u_\lambda, v_\lambda) \in E$ such that $z_n \to z_\lambda$ in E and

$$I(z_\lambda) = c_\lambda, \quad I'(z_\lambda) = 0,$$

which implies that z_λ is a solution of problem (1.1). Using the hypothesis that $F_u(0, v^+) = F_v(u^+, 0) = 0$ for any $(u, v) \in E$, after a direct calculation, we derive that

$$\|u_\lambda^-\|^p = \langle I'_u(u_\lambda, v_\lambda), -u_\lambda^-\rangle = 0, \quad \text{and} \quad \|v_\lambda^-\|^p = \langle I'_v(u_\lambda, v_\lambda), -v_\lambda^-\rangle = 0,$$

which implies that $u_\lambda^- = 0$ and $v_\lambda^- = 0$. Hence we have $u_\lambda \geq 0$ and $v_\lambda \geq 0$.

Now we show that $u_\lambda \neq 0$ and $v_\lambda \neq 0$. Since $I(z_\lambda) = c_\lambda < 0 = I(0,0)$, we have $u_\lambda \neq 0$ or $v_\lambda \neq 0$. Without loss of generality, we may assume that $u_\lambda \neq 0$ and $v_\lambda = 0$. Then, for any $\lambda \in (0, \Lambda)$, we have

$$
\begin{aligned}
I(tu_\lambda, t\psi_0^+) = {} & \frac{1}{p}t^p \int_\Omega |\nabla u_\lambda|^p dx - \frac{\lambda}{q}t^q \int_\Omega a_1(x)u_\lambda^q dx + \frac{1}{p}t^p \int_\Omega |\nabla \psi_0^+|^p dx \\
& - \frac{\lambda}{q}t^q \int_\Omega a_2(x)(\psi_0^+)^q dx - t^{p^*} \int_\Omega b(x)F(u_\lambda, \psi_0^+)dx.
\end{aligned}
$$
(3.6)

Since $(u_\lambda, 0)$ is a critical point of I with $I(u_\lambda, 0) = c_\lambda$, it follows from (2.1) that

$$\int_\Omega |\nabla u_\lambda|^p dx - \lambda \int_\Omega a_1(x)u_\lambda^q dx - p^* \int_\Omega b(x)F(u_\lambda, 0)dx = \langle I'_u(u_\lambda, 0), u_\lambda \rangle = 0,$$

which implies that

$$c_\lambda = I(u_\lambda, 0) = \frac{p^* - p}{pp^*}\int_\Omega |\nabla u_\lambda|^p dx - \frac{p^* - q}{qp^*}\lambda \int_\Omega a_1(x)u_\lambda^q dx.$$

Let $t_1 = \min\left\{ ((p^* - p)/(p^* - q))^{1/(p-q)}, ((p^* - q)/p^*)^{1/q} \right\}$, for any $t \in (0, t_1)$, we have

$$
\begin{aligned}
& \frac{1}{p}t^p \int_\Omega |\nabla u_\lambda|^p dx - \frac{\lambda}{q}t^q \int_\Omega a_1(x)u_\lambda^q dx \\
(3.7) \quad = {} & \frac{p^*}{p^* - q}t^q \left(\frac{p^* - q}{pp^*}t^{p-q}\int_\Omega |\nabla u_\lambda|^p dx - \frac{p^* - q}{qp^*}\lambda \int_\Omega a_1(x)u_\lambda^q dx \right) \\
< {} & \frac{p^*}{p^* - q}t^q \left(\frac{p^* - p}{pp^*}\int_\Omega |\nabla u_\lambda|^p dx - \frac{p^* - q}{qp^*}\lambda \int_\Omega a_1(x)u_\lambda^q dx \right) = \frac{p^*}{p^* - q}c_\lambda t^q < c_\lambda.
\end{aligned}
$$

According to $F \in C^1((R^+)^2, R^+)$ and (3.4), there exists $t_2 > 0$ such that for any $t \in (0, t_2)$

$$
\begin{aligned}
& \frac{1}{p}t^p \int_\Omega |\nabla \psi_0^+|^p dx - \frac{\lambda}{q}t^q \int_\Omega a_2(x)(\psi_0^+)^q dx - t^{p^*}\int_\Omega b(x)F(u_\lambda, \psi_0^+)dx \\
(3.8) \quad \leq {} & \frac{1}{p}t^p \int_\Omega |\nabla \psi_0^+|^p dx - \frac{\lambda}{q}t^q \int_\Omega a_2(x)(\psi_0^+)^q dx + C_4 t^{p^*} < 0,
\end{aligned}
$$

where $C_4 = \|b\|_\infty \int_\Omega F(u_\lambda, \psi_0^+)dx + 1 > 0$. Let $t_3 = \min\{t_1, t_2\}$, from (3.6)–(3.8), for any $t \in (0, t_3)$, we obtain that

$$I(tu_\lambda, t\psi_0^+) < c_\lambda.$$

Noticing that $(u_\lambda, 0) \in B_\rho(0)$, thus we can choose t so small that $(u_\lambda, t\psi_0^+) \in B_\rho(0)$. Hence we obtain

$$\inf_{z \in B_\rho(0)} I(z) < c_\lambda,$$

which is a contradiction with $c_\lambda = \inf\limits_{z \in B_\rho(0)} I(z)$. Therefore, we have $u_\lambda \neq 0$ and $v_\lambda = 0$ are not established. Similarly, we obtain $u_\lambda = 0$ and $v_\lambda \neq 0$ are impossible. Hence we have $u_\lambda \neq 0$ and $v_\lambda \neq 0$.

Moreover, if $a_i(x) \geq 0 (i = 1, 2)$ and $b(x) \geq 0$, it follows that $-\triangle_p u_\lambda \geq 0$. By the maximum principle (see [25]), we obtain $u_\lambda > 0$ in Ω. Similarly, we have $v_\lambda > 0$ in Ω. The proof of Theorem 1.1 is completed. ∎

4. The multiplicity of solutions for problem (1.1)

In this section, we shall use the mountain pass theorem to obtain the second nontrivial nonnegative or positive solution of problem (1.1).

Lemma 4.1. *Let* $1 < p < N$, $(N(p-1))/(N-p) < q < p$, (a_1), (a_2), (b_1), (b_2) *and* (b_3) *hold. Assume that* $F \in C^1((R^+)^2, R^+)$ *is positively homogeneous of degree* p^*. *Then there exist a nonnegative function* $z \in E$ *and* $\Lambda^* > 0$ *such that*

$$\sup_{t \geq 0} I(tz) < \frac{p}{N-p} \|b\|_\infty^{-\frac{N-p}{p}} \left(\frac{S_F}{p^*} \right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}} \quad for\ all \quad 0 < \lambda < \Lambda^*,$$

where B *is the positive constant give in Lemma 2.2.*

Proof. For convenience, we consider the functional $J : E \to R$ defined by

$$J(u, v) = \frac{1}{p} \int_\Omega (|\nabla u|^p + |\nabla v|^p) dx - \int_\Omega b(x) F(u^+, v^+) dx \quad for\ all \quad (u, v) \in E.$$

According to (b_2) and (b_3), we can choose such a cut-off function $\phi(x) \in C_0^\infty(\Omega)$ that $\phi(x) = 1$ for $x \in B(x_0, \delta_0)$, $\phi(x) = 0$ for $x \in R^N \setminus B(x_0, 2\delta_0)$, $0 \leq \phi(x) \leq 1$ and $|\nabla\phi| \leq C_5$, where $C_5 > 0$ is a positive constant. Define

$$u_\eta(x) = \frac{\eta^{\frac{N-p}{p(p-1)}} \phi(x)}{\left(\eta^{\frac{p}{p-1}} + |x - x_0|^{\frac{p}{p-1}} \right)^{\frac{N-p}{p}}}.$$

After a detailed calculation, we have the following estimate(as $\eta \to 0$)

$$(4.1) \qquad \frac{\int_\Omega |\nabla u_\eta|^p dx}{(\int_\Omega |u_\eta|^{p^*} dx)^{p/p^*}} = S + O\left(\eta^{\frac{N-p}{p-1}} \right).$$

Now we show that the above estimate is valid. Indeed, we have

$$\nabla u_\eta(x) = \eta^{\frac{N-p}{p(p-1)}} \left(\frac{\nabla\phi(x)}{\left(\eta^{\frac{p}{p-1}} + |x - x_0|^{\frac{p}{p-1}} \right)^{\frac{N-p}{p}}} - \frac{N-p}{p-1} \frac{\phi(x)|x - x_0|^{\frac{2-p}{p-1}} x}{\left(\eta^{\frac{p}{p-1}} + |x - x_0|^{\frac{p}{p-1}} \right)^{\frac{N}{p}}} \right).$$

Since $\phi(x) \equiv 1$ in $B(x_0, \delta_0)$ and $|\nabla\phi| \leq C_5$, let $x = x_0 + \eta y$, one has

$$\int_\Omega |\nabla u_\eta|^p dx = \eta^{\frac{N-p}{p-1}} \int_\Omega \frac{|x - x_0|^{p/(p-1)}}{\left(\eta^{p/(p-1)} + |x - x_0|^{p/(p-1)} \right)^N} dx + O\left(\eta^{\frac{N-p}{p-1}} \right)$$

$$= \eta^{\frac{N-p}{p-1}} \int_{R^N} \frac{|x - x_0|^{p/(p-1)}}{\left(\eta^{p/(p-1)} + |x - x_0|^{p/(p-1)} \right)^N} dx + O\left(\eta^{\frac{N-p}{p-1}} \right)$$

$$= \int_{R^N} \frac{|y|^{p/(p-1)}}{\left(1 + |y|^{p/(p-1)}\right)^N} dy + O\left(\eta^{\frac{N-p}{p-1}}\right) = |\nabla U|_{L^p(R^N)}^p + O\left(\eta^{\frac{N-p}{p-1}}\right),$$

and

$$\int_\Omega |u_\eta|^{p^*} dx = \eta^{\frac{N}{p-1}} \int_\Omega \frac{\phi^{p^*} dx}{\left(\eta^{p/(p-1)} + |x - x_0|^{p/(p-1)}\right)^N}$$

$$= \eta^{\frac{N}{p-1}} \int_{B(x_0,\delta_0)} \frac{dx}{\left(\eta^{p/(p-1)} + |x - x_0|^{p/(p-1)}\right)^N} + O\left(\eta^{\frac{N}{p-1}}\right)$$

$$= \eta^{\frac{N}{p-1}} \int_{R^N} \frac{dx}{\left(\eta^{p/(p-1)} + |x - x_0|^{p/(p-1)}\right)^N} + O\left(\eta^{\frac{N}{p-1}}\right)$$

$$= \int_{R^N} \frac{dy}{\left(1 + |y|^{p/(p-1)}\right)^N} + O\left(\eta^{\frac{N}{p-1}}\right) = |U|_{L^{p^*}(R^N)}^{p^*} + O\left(\eta^{\frac{N}{p-1}}\right),$$

where $U(x) = \left(1 + |x|^{p/(p-1)}\right)^{-(N-p)/p} \in W^{1,p}(R^N)$ satisfies

$$\frac{|\nabla U|_{L^p(R^N)}^p}{|U|_{L^{p^*}(R^N)}^p} = S = \inf_{u \in W^{1,p}(R^N)\setminus\{0\}} \frac{|\nabla u|_{L^p(R^N)}^p}{|u|_{L^{p^*}(R^N)}^p}.$$

A direct calculation, we deduce that (4.1) holds.

Since F is positively homogeneous of degree p^*, we have (2.1) and (2.2) hold. It follows from $F \in C^1((R^+)^2, R^+)$ and (2.2) that there exists $(e_1, e_2) \in \{z \in (R^+)^2 : |z| = 1\}$ such that $F(e_1, e_2) = M$. Define

$$h(t) = J(te_1 u_\eta, te_2 u_\eta) = \frac{1}{p} t^p \int_\Omega |\nabla u_\eta|^p dx - M t^{p^*} \int_\Omega b(x)|u_\eta|^{p^*} dx \quad \text{for all} \quad t \geq 0.$$

Assume that $h(t)$ attains its maximum at a point t_η. According to

$$0 = h'(t_\eta) = t_\eta^{p-1}\left(\int_\Omega |\nabla u_\eta|^p dx - p^* M t_\eta^{p^*-p} \int_\Omega b(x)|u_\eta|^{p^*} dx\right),$$

one has

$$t_\eta^{p^*-p} = \frac{\int_\Omega |\nabla u_\eta|^p dx}{p^* M \int_\Omega b(x)|u_\eta|^{p^*} dx}.$$

Using the definition of u_η and (b_2), we obtain $t_\eta < \infty$. We also have

(4.2) $$\sup_{t \geq 0} J(te_1 u_\eta, te_2 u_\eta) = J(t_\eta e_1 u_\eta, t_\eta e_2 u_\eta) = \Phi(\eta) + \Psi(\eta),$$

where

$$\Phi(\eta) = \frac{1}{p} t_\eta^p \int_\Omega |\nabla u_\eta|^p dx - M\|b\|_\infty t_\eta^{p^*} \int_\Omega |u_\eta|^{p^*} dx,$$

$$\Psi(\eta) = M t_\eta^{p^*} \int_\Omega (\|b\|_\infty - b(x))|u_\eta|^{p^*} dx.$$

Using the fact

$$\max_{t \geq 0} \left(\frac{\alpha}{p} t^p - \frac{\beta}{p^*} t^{p^*}\right) = \frac{1}{N} \left(\frac{\alpha}{\beta^{p/p^*}}\right)^{\frac{N}{p}} \quad \text{for any} \quad \alpha, \beta > 0,$$

we deduce from (3.5) and (4.1) that

$$\Phi(\eta) \le \frac{1}{N}(M\|b\|_\infty)^{-\frac{N-p}{p}} \left[\frac{\int_\Omega |\nabla u_\eta|^p dx}{\left(\int_\Omega |u_\eta|^{p^*} dx\right)^{p/p^*}} \right]^{\frac{N}{p}}$$

(4.3)
$$= \frac{1}{N}(M\|b\|_\infty)^{-\frac{N-p}{p}} S^{\frac{N}{p}} + O(\eta^{\frac{N-p}{p-1}})$$

$$\le \frac{p}{N-p}\|b\|_\infty^{-\frac{N-p}{p}} \left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} + O(\eta^{\frac{N-p}{p-1}}).$$

It follows from (b_3) that there exists $\rho_0 \in (0,\delta_0)$ such that

$$0 \le b(x_0) - b(x) \le |x - x_0|^k \quad \text{for all} \quad x \in B(x_0,\rho_0).$$

From $k > \frac{N-p}{p-1}$, noticing that $t_\eta < \infty$, we have

$$\Psi(\eta) = M t_\eta^{p^*} \eta^{\frac{N}{p-1}} \int_\Omega \frac{(\|b\|_\infty - b(x))\phi^{p^*}}{\left(\eta^{p/(p-1)} + |x-x_0|^{p/(p-1)}\right)^N} dx$$

$$\le M t_\eta^{p^*} \eta^{\frac{N}{p-1}} \int_{R^N \setminus B(x_0,\rho_0)} \frac{\|b\|_\infty dx}{\left(\eta^{p/(p-1)} + |x-x_0|^{p/(p-1)}\right)^N}$$

$$+ M t_\eta^{p^*} \eta^{\frac{N}{p-1}} \int_{B(x_0,\rho_0)} \frac{|x-x_0|^k dx}{\left(\eta^{p/(p-1)} + |x-x_0|^{p/(p-1)}\right)^N}$$

(4.4)
$$\le M t_\eta^{p^*} \|b\|_\infty \eta^{\frac{N}{p-1}} \int_{R^N \setminus B(x_0,\rho_0)} |x-x_0|^{-\frac{pN}{p-1}} dx$$

$$+ \frac{M}{N} t_\eta^{p^*} \eta^{\frac{N-p}{p-1}} \int_{B(x_0,\rho_0)} |x-x_0|^{k-\frac{p(N-1)}{p-1}} dx$$

$$= MN\omega_N t_\eta^{p^*} \|b\|_\infty \eta^{\frac{N}{p-1}} \int_{\rho_0}^{+\infty} r^{-\frac{N}{p-1}-1} dr + M\omega_N t_\eta^{p^*} \eta^{\frac{N-p}{p-1}} \int_0^{\rho_0} r^{k-1-\frac{N-p}{p-1}} dr$$

$$= M(p-1)\omega_N t_\eta^{p^*} \|b\|_\infty \rho_0^{-\frac{N}{p-1}} \eta^{\frac{N}{p-1}} + \frac{M(p-1)\omega_N}{k(p-1)-N+p} t_\eta^{p^*} \rho_0^{k-\frac{N-p}{p-1}} \eta^{\frac{N-p}{p-1}}$$

$$= O\left(\eta^{\frac{N-p}{p-1}}\right),$$

where $\omega_N = (2\pi^{N/2})/(N\Gamma(N/2))$ denotes the volume of the unit ball $B(0,1)$ in R^N. From $(4.2) - (4.4)$, we have

(4.5)
$$\sup_{t \ge 0} J(te_1 u_\eta, te_2 u_\eta) \le \frac{p}{N-p}\|b\|_\infty^{-\frac{N-p}{p}} \left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} + O(\eta^{\frac{N-p}{p-1}}).$$

By the inequality (3.5), we can choose δ_1 such that

$$\frac{p}{N-p}\|b\|_\infty^{-\frac{N-p}{p}} \left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}} > 0, \quad \forall \lambda \in (0,\delta_1).$$

Using the definitions of I and u_η, from (a_2) and (b_2), we have

$$I(te_1 u_\eta, te_2 u_\eta) \le \frac{1}{p} t^p \int_\Omega |\nabla u_\eta|^p dx = \frac{1}{p} t^p \left[|\nabla U|_{L^p(R^N)}^p + O\left(\eta^{\frac{N-p}{p-1}}\right) \right]$$

for all $t \geq 0$ and $\lambda > 0$. It follows from (3.5) and (4.1) that there exist $T \in (0,1)$ and $\eta_1 > 0$ such that for all $0 < \lambda < \delta_1$ and $0 < \eta < \eta_1$

$$(4.6) \qquad \sup_{0 \leq t \leq T} I(te_1 u_\eta, te_2 u_\eta) \leq \frac{p}{N-p} \|b\|_\infty^{-\frac{N-p}{p}} \left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}}.$$

Moreover, using the definitions of I and u_η, it follows from (a_2) and (4.5) that

$$\sup_{t \geq T} I(te_1 u_\eta, te_2 u_\eta) = \sup_{t \geq T} \left(J(te_1 u_\eta, te_2 u_\eta) - \frac{\lambda}{q} t^q \int_\Omega [a_1(x) + a_2(x)] |u_\eta|^q dx \right)$$

$$\leq \frac{p}{N-p} \|b\|_\infty^{-\frac{N-p}{p}} \left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} + O(\eta^{\frac{N-p}{p-1}}) - \frac{2\lambda\beta_0 T^q}{q} \int_{B(x_0,\delta_0)} |u_\eta|^q dx.$$

By the Lemma $A5$ of [15], it implies from $N(p-1)/N-p < q < p < N$ that

$$\int_{B(x_0,\delta_0)} |u_\eta|^q dx \geq C_6 \eta^{\frac{N(p-q)+pq}{p}}.$$

By the above two inequalities, we have

$$(4.7) \quad \sup_{t \geq T} I(te_1 u_\eta, te_2 u_\eta) \leq \frac{p}{N-p} \|b\|_\infty^{-\frac{N-p}{p}} \left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} + O(\eta^{\frac{N-p}{p-1}}) - \frac{2C_6\lambda\beta_0 T^q}{q} \eta^{\frac{N(p-q)+pq}{p}}.$$

By the hypothesis $N(p-1)/N-p < q < p < N$, we obtain $((N-p)(p-q))/(p(N-Np+Nq-pq)) > 0$. For some positive constants C_7 and C_8, let $\eta = \lambda^{(p(p-1))/((p-q)(N-p))}$ and $\lambda < (C_8/(B+C_7))^{((N-p)(p-q))/(p(N-Np+Nq-pq))}$, we have

$$C_7 \eta^{\frac{N-p}{p-1}} - C_8\lambda \eta^{\frac{N(p-q)+pq}{p}} = C_7\lambda^{\frac{p}{p-q}} - C_8\lambda^{\frac{(Np-Nq+pq-p)p}{(N-p)(p-q)}} < -B\lambda^{\frac{p}{p-q}},$$

which implies that there exists $\delta_2 > 0$ such that for all $\eta = \lambda^{(p(p-1))/((p-q)(N-p))}$ and $\lambda < \delta_2$

$$(4.8) \qquad O(\eta^{\frac{N-p}{p-1}}) - \frac{2C_6\lambda\beta_0 T^q}{q} \eta^{\frac{N(p-q)+pq}{p}} < -B\lambda^{\frac{p}{p-q}}.$$

From (4.7) and (4.8), for all $\eta = \lambda^{(p(p-1))/((p-q)(N-p))}$ and $\lambda < \delta_2$, one has

$$(4.9) \qquad \sup_{t \geq T} I(te_1 u_\eta, te_2 u_\eta) \leq \frac{p}{N-p} \|b\|_\infty^{-\frac{N-p}{p}} \left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}}.$$

Set $\Lambda^* = \min\left\{\delta_1, \delta_2, \eta_1^{((N-p)(p-q))/(p(p-1))}\right\}$. Combining (4.6) with (4.9), for all $\eta = \lambda^{(p(p-1))/((p-q)(N-p))}$ and $\lambda \in (0, \Lambda^*)$, we obtain

$$\sup_{t \geq 0} I(te_1 u_\eta, te_2 u_\eta) \leq \frac{p}{N-p} \|b\|_\infty^{-\frac{N-p}{p}} \left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}}. \qquad \blacksquare$$

Proof of Theorem 1.2. Choose $\lambda^* \leq \Lambda$, from the proof of Theorem 1.1, we have already obtained that problem (1.1) has a nontrivial nonnegative solution z_λ with $I(z_\lambda) < 0$ for any $\lambda \in (0, \lambda^*)$. Now we only need to prove that problem (1.1) has a nontrivial nonnegative solution $\bar{z}_\lambda$ with $I(\bar{z}_\lambda) > 0$ for any $\lambda \in (0, \lambda^*)$. According to (a_1), we can obtain (2.4) and (3.3) hold. It follows from (b_1) and Lemma 3.1 that there exists $\phi_0 \in C_0^\infty(\Omega)$ such that

$$\int_\Omega b(x)(\phi_0^+)^{p^*} dx > 0.$$

Since F is positively homogeneous of degree p^*, we have (2.2) holds. It follows from $F \in C^1((R^+)^2, R^+)$ and (2.2) that there exists $(e_1, e_2) \in \{z \in (R^+)^2 : |z| = 1\}$ such that $F(e_1, e_2) = M$. Let $z_0 = (e_1\phi_0, e_2\phi_0)$, from (2.4), we have

$$I(tz_0) \le \left(\frac{1}{p} + \varepsilon\right) t^p \|z_0\|_E^p - Mt^{p^*} \int_\Omega b(x)(\phi_0^+)^{p^*} dx + C(\varepsilon)\lambda^{\frac{p}{p-q}},$$

which implies that

$$I(tz_0) \to -\infty \quad \text{as} \quad t \to +\infty.$$

Hence, there exists a positive number t_0 such that $\|t_0 z_0\|_E > \rho$ and $I(t_0 z_0) < 0$ for any $\lambda \in (0, \lambda^*)$. Therefore, the functional I has the mountain pass geometry. Define

$$\Gamma = \{\gamma \in C([0,1], E) | \ \gamma(0) = 0, \ \gamma(1) = t_0 z_0\}, \quad \widetilde{c} = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)).$$

From Lemma 4.1, we have

$$\widetilde{c} < \frac{p}{N-p} \|b\|_\infty^{-\frac{N-p}{p}} \left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}}.$$

Applying Lemma 2.4, we know that I satisfies the $(PS)_{\widetilde{c}}$ condition. By the mountain pass theorem (see [3]), we obtain that problem (1.1) has the second solution $\bar{z}_\lambda = (\bar{u}_\lambda, \bar{v}_\lambda)$ with $I(\bar{z}_\lambda) > 0$. Noticing that $F_u(0, (\bar{v}_\lambda)^+) = F_v((\bar{u}_\lambda)^+, 0) = 0$, after a direct calculation, we derive that

$$\|(\bar{u}_\lambda)^-\|^p = \langle I_u'(\bar{u}_\lambda, \bar{v}_\lambda), -(\bar{u}_\lambda)^-\rangle = 0, \quad \text{and} \quad \|(\bar{v}_\lambda)^-\|^p = \langle I_v'(\bar{u}_\lambda, \bar{v}_\lambda), -(\bar{v}_\lambda)^-\rangle = 0,$$

which implies that $(\bar{u}_\lambda)^- = 0$ and $(\bar{v}_\lambda)^- = 0$. Hence we have $\bar{u}_\lambda \ge 0$ and $\bar{v}_\lambda \ge 0$.

Next, we show that $\bar{u}_\lambda \neq 0$ and $\bar{v}_\lambda \neq 0$. Since $I(\bar{z}_\lambda) > 0 = I(0,0)$, we have $\bar{u}_\lambda \neq 0$ or $\bar{v}_\lambda \neq 0$. Without loss of generality, we may assume that $\bar{u}_\lambda \neq 0$ and $\bar{v}_\lambda = 0$. Using the hypothesis that $F_u(u, 0) = F_v(0, v) = 0$ for all $u, v \in R^+$, it is easy to obtain $\bar{u}_\lambda$ satisfies that

$$(4.10) \qquad \begin{cases} -\triangle_p \bar{u}_\lambda = \lambda a_1(x)(\bar{u}_\lambda)^{q-1}, & \text{in } \Omega, \\ \bar{u}_\lambda = 0, & \text{on } \partial\Omega. \end{cases}$$

Acting on (4.10) with $\bar{u}_\lambda \in W_0^{1,p}(\Omega)$ yields

$$\int_\Omega |\nabla \bar{u}_\lambda|^p dx = \lambda \int_\Omega a_1(x)(\bar{u}_\lambda)^q dx.$$

Therefore, we have

$$I(\bar{u}_\lambda, 0) = \frac{1}{p} \int_\Omega |\nabla \bar{u}_\lambda|^p dx - \frac{\lambda}{q} \int_\Omega a(x)(\bar{u}_\lambda)^q dx = \left(\frac{1}{p} - \frac{1}{q}\right) \int_\Omega |\nabla \bar{u}_\lambda|^p dx < 0,$$

which is a contradiction with $I(\bar{u}_\lambda, 0) = I(\bar{z}_\lambda) > 0$. Therefore, we have $\bar{u}_\lambda \neq 0$ and $\bar{v}_\lambda = 0$ are not established. Similarly, we obtain $\bar{u}_\lambda = 0$ and $\bar{v}_\lambda \neq 0$ are impossible. Hence we have $\bar{u}_\lambda \neq 0$ and $\bar{v}_\lambda \neq 0$.

Moreover, if $a_i(x) \ge 0 (i = 1, 2)$ and $b(x) \ge 0$, a same argument with the proof of Theorem 1.1, we obtain that $u_\lambda > 0$, $v_\lambda > 0$, and $\bar{u}_\lambda > 0$, $\bar{v}_\lambda > 0$ in Ω. The proof of Theorem 1.2 is completed. $\blacksquare$

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