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On Lockett Pairs and Lockett Conjecture for π-Soluble Fitting Classes

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Abstract. In this paper, we construct a new and wide family of Lockett pairs in the class of all finite π-soluble groups and give a new characteristic of the validity of Lockett conjecture. As application, some known results are followed.

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1. Introduction

In the theory of finite soluble groups, many well known results related to research of structures of Fitting classes and canonical subgroups are closed connected with the operators "$*$" and "$_*$" defined by Lockett [13] (see also [8, chapter X]). In fact, every nonempty Fitting class $\mathfrak{F}$ has the associated Fitting classes $\mathfrak{F}^*$ and $\mathfrak{F}_*$, where $\mathfrak{F}^*$ is the smallest Fitting class containing $\mathfrak{F}$ such that the $\mathfrak{F}^*$-radical of the direct product $G \times H$ of any two groups G and H is equal to the direct product of the $\mathfrak{F}^*$-radical of G and the $\mathfrak{F}^*$-radical of H; $\mathfrak{F}_*$ is the intersection of all Fitting classes $\mathfrak{X}$ such that $\mathfrak{X}^*$=$\mathfrak{F}^*$ (see [13] or [8, Chapter X]). If $\mathfrak{F}^* = \mathfrak{F}$, then the Fitting class is called a Lockett class. The interest for the research of $\mathfrak{F}^*$ and $\mathfrak{F}_*$ is determined mainly by the following circumstances. Firstly, the family of Fitting classes satisfying $\mathfrak{F}^* = \mathfrak{F}$ is vast. In fact, by [8, Theorem X.1.25], every Fitting class closed about homomorphic images or closed about finite subdirect products, and every Fischer class (see [8, IX.3.3]) are all Lockett classes. Secondly, Lockett [13] formulated the conjecture that for every Fitting class $\mathfrak{F}$, there exists a normal Fitting class $\mathfrak{X}$ such that $\mathfrak{F} = \mathfrak{F}^* \cap \mathfrak{X}$. Later, in this case, we say that $\mathfrak{F}$ satisfies Lockett conjecture in $\mathfrak{S}$, where $\mathfrak{S}$ is the class of all soluble groups.

About the Lockett conjecture, Bryce and Cossey [6] proved that Lockett conjecture holds for all soluble S-closed local Fitting classes and that every soluble Fitting class $\mathfrak{F}$ satisfies Lockett conjecture if and only if $\mathfrak{F}_* = \mathfrak{F}^* \cap \mathfrak{S}_*$, where $\mathfrak{S}_*$ is the smallest normal Fitting class. In the paper [6], Bryce and Cossey also gave the concept of Lockett pair $(\mathfrak{F}, \mathfrak{H})$ (see

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[6, 5.1]). They call **an** ordered pair $(\mathfrak{F},\mathfrak{H})$ of two Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ a Lockett pair (or in brevity, an $\mathfrak{L}$-pair) if $\mathfrak{F}\cap\mathfrak{H}_* = (\mathfrak{F}\cap\mathfrak{H})_*$.

It is easy to see that if $\mathfrak{F}$ is a Lockett class, $\mathfrak{F}\subseteq\mathfrak{H}$ and $(\mathfrak{F},\mathfrak{H})$ is a Lockett pair, then $\mathfrak{F}$ satisfies Lockett conjecture in $\mathfrak{H}$. In particular, if $\mathfrak{F}$ is a Lockett class, $\mathfrak{F}\subseteq\mathfrak{S}$, $\mathfrak{H}=\mathfrak{S}$ and $(\mathfrak{F},\mathfrak{H})$ is a Lockett pair, then $\mathfrak{F}$ satisfies Lockett conjecture in $\mathfrak{S}$. We say that $\mathfrak{F}$ is an $\mathfrak{L}_{\mathfrak{H}}$-class if $\mathfrak{F}$ satisfies Lockett conjecture in $\mathfrak{H}$. If $\mathfrak{F}$ satisfies Lockett conjecture in $\mathfrak{S}$, then $\mathfrak{F}$ is called an $\mathfrak{L}$-class. Recall that a Fitting class $\mathfrak{F}$ is said to be S-closed if every subgroup of $G\in\mathfrak{F}$ is in $\mathfrak{F}$.

Bryce and Cossey [6] in the universe $\mathfrak{S}$ proved the existence of Lockett pairs. They showed that if $\mathfrak{F}$ and $\mathfrak{H}$ are S-closed Fitting classes, then $(\mathfrak{F},\mathfrak{H})$ is a Lockett pair. In connection with this, the following problem arises.

Problem 1.1. *Which Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ satisfy that $(\mathfrak{F},\mathfrak{H})$ is a Lockett pair? In particular, which Fitting classes $\mathfrak{F}$ satisfies Lockett conjecture in $\mathfrak{H}$?*

Note that, up to now, the problem was resolved only in the following special cases:

(1) $\mathfrak{F}\in\{\mathfrak{X}\mathfrak{N},\mathfrak{X}\mathfrak{S}_\pi\}$, where $\mathfrak{X}$ is some nonempty soluble Fitting class, $\mathfrak{H}$ is $\mathfrak{F}\mathfrak{S}_\pi$-injector and $\mathfrak{L}_p(\mathfrak{F})$-injector closed for all primes p (see Beidleman and Hauck [1]);
(2) $\mathfrak{F}=\mathfrak{S}_\pi$ and $\mathfrak{H}\supseteq\mathfrak{S}_\pi$ (see Brison [4]);
(3) $\mathfrak{F}$ is an arbitrary soluble local Fitting class, and $\mathfrak{H}$ is ψ-injector closed, where ψ is a local function of $\mathfrak{F}$ (see Vorob'ev [15]);
(4) $\mathfrak{F}=\mathfrak{S}$, and $\mathfrak{H}=\mathfrak{E}$ is the class of all finite groups (see Berger [3]);
(5) $\mathfrak{H}$ is a Fischer class(i.e., $\mathfrak{H}$ is a Fitting class which is closed under taking subgroups of the form PN where P is a Sylow subgroup and N is a normal subgroup) or is closed under taking $\mathfrak{F}$-subgroups, whose intersection with $\mathfrak{F}$-radical of G is a normal subgroup of G, $\mathfrak{F}$ satisfies the property that $\mathfrak{X}\mathfrak{E}_p\subseteq\mathfrak{F}\subseteq\mathfrak{X}\mathfrak{E}_p\mathfrak{E}_{p'}$ for some Fitting class $\mathfrak{X}$ and all $p\in\mathrm{Char}(\mathfrak{X})$ (see Gallego [9]);
(6) $\mathfrak{F}$ is ω-local with $\mathrm{char}(\mathfrak{F})\subseteq\omega$ and $\mathfrak{H}=\mathfrak{E}$ is the class of all finite groups (see [12]).

In this paper, we will construct a new family of Lockett pairs in the class $\mathfrak{S}^\pi$ of all finite π-soluble groups. In order to achieve the purpose, in Section 3, we will give the concept of π-*HR*-closed Fitting class for some Fitting class $\mathfrak{X}$, which, in fact, is a generalized S-closed Fitting class. Base on this, in Section 4, we obtain a family of Lockett pairs and also give a new characteristic of the validity of Lockett conjecture. As application, some known results in [1,3,4,9,12,15] are obtained as corollaries of our results. Throughout this paper, all groups are finite. All unexplained notation and terminology are standard. The reader is referred to [8,10] if necessary.

2. Preliminaries

Recall that a class $\mathfrak{F}$ of groups is called a Fitting class provided the following two conditions are satisfied:

(i) if $G\in\mathfrak{F}$ and $N\trianglelefteq G$, then $N\in\mathfrak{F}$.
(ii) if $N_1,N_2\trianglelefteq G$ and $N_1,N_2\in\mathfrak{F}$, then $N_1N_2\in\mathfrak{F}$.

From the condition (ii) in the definition, we see that, for every non-empty Fitting class $\mathfrak{F}$, every group G has a largest normal $\mathfrak{F}$-subgroup which is called the $\mathfrak{F}$-radical of G and denote by $G_{\mathfrak{F}}$. The product $\mathfrak{F}\mathfrak{H}$ of two Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ is the class $(G : G/G_{\mathfrak{F}}\in\mathfrak{H})$.

It is well known that the product of any two Fitting classes is also a Fitting class and the multiplication of Fitting classes satisfies associative law (see [8, IX.1.12]).

Recall that a class $\mathfrak{H}$ of groups is said to be a saturated homomorph if the following conditions hold:

(1) $\mathfrak{H}$ is closed about homomorphic images, that is, if $G \in \mathfrak{H}$ and $N \trianglelefteq G$, then $G/N \in \mathfrak{H}$;
(2) If $G/\Phi(G) \in \mathfrak{H}$, then $G \in \mathfrak{H}$.

A nonempty Fitting class $\mathfrak{H}$ is said to be a Fischer class if $H \in \mathfrak{H}$ whenever $K \trianglelefteq G \in \mathfrak{H}$ and H/K is a nilpotent subgroup of G/K (see [8, IX.3.3]). Obviously, any S-closed Fitting class is a Fischer class.

We here cite some properties of the operators "$_*$" and "*", which are used in later proof.

Lemma 2.1. ([13] and [8, X]). *Let $\mathfrak{F}$ and $\mathfrak{H}$ be two non-empty Fitting classes. Then:*

(a) *If $\mathfrak{F} \subseteq \mathfrak{H}$, then $\mathfrak{F}^* \subseteq \mathfrak{H}^*$ and $\mathfrak{F}_* \subseteq \mathfrak{H}_*$;*
(b) $(\mathfrak{F}_*)_* = \mathfrak{F}_* = (\mathfrak{F}^*)_* \subseteq \mathfrak{F} \subseteq \mathfrak{F}^* = (\mathfrak{F}_*)^* = (\mathfrak{F}^*)^*$;
(c) $\mathfrak{F}^* \subseteq \mathfrak{F}_*\mathfrak{A}$, *where $\mathfrak{A}$ is the class of all abelian groups;*
(d) *If $\{\mathfrak{F}_i \mid i \in I\}$ is a set of Fitting classes, then $(\cap_{i\in I}\mathfrak{F}_i)^* = \cap_{i\in I}\mathfrak{F}_i^*$.*
(e) *If $\mathfrak{H}$ is a saturated homomorph, then $(\mathfrak{F}\mathfrak{H})^* = \mathfrak{F}^*\mathfrak{H}$.*
(f) *If $\mathfrak{F}$ is a homomorph (in particular, a formation) or a Fischer class, then $\mathfrak{F}$ is a Lockett class.*

Suppose that G is a group, $\mathfrak{X}$ is a class of groups and $\mathbb{P}$ is the set of all primes. Then we let $\sigma(G) = \{p \in \mathbb{P} : p||G|\}$, $\sigma(\mathfrak{X}) = \cup\{\sigma(G) : G \in \mathfrak{X}\}$ and $\mathrm{Char}(\mathfrak{X}) = \{p \in \mathbb{P} : Z_p \in \mathfrak{X}\}$.

Lemma 2.2. [8, X.1.20]. $\mathrm{Char}(\mathfrak{F}^*) = \mathrm{Char}(\mathfrak{F})$ *and* $\sigma(\mathfrak{F}^*) = \sigma(\mathfrak{F})$ *for every Fitting class $\mathfrak{F}$.*

Let $\emptyset \neq \omega \subseteq \mathbb{P}$. We denote by $\mathcal{E}_\omega$ the class of all finite ω-groups, $\mathfrak{N}$ denotes the class of all finite nilpotent groups, $\mathfrak{S}$ denotes the class of all finite soluble groups. For a class $\mathfrak{F}$ of groups, put $\mathfrak{F}_\omega = \mathfrak{F} \cap \mathcal{E}_\omega$. Following [14], a map

$$f:\ \omega \cup \{\omega'\} \longrightarrow \{\text{Fitting class}\}$$

is said to be a ω-local Hatley function (or in brevity, a ω-local H-function). Then $LR_\omega(f)$ denotes the Fitting class $(\cap_{p\in\pi_2}\mathfrak{E}_{p'}) \cap (\cap_{p\in\pi_1} f(p)\mathfrak{N}_p\mathfrak{E}_{p'}) \cap \mathfrak{E}_\omega f(\omega')$, where $\pi_1 = Supp(f) \cap \omega$, $\pi_2 = \omega\backslash\pi_1$. Here, $Supp(f) := \{a \in \omega \cup \{\omega\} : f(a) \neq \emptyset\}$ is called the support of the ω-local H-function f. A Fitting class $\mathfrak{F}$ is said to be a ω-local if there exists an ω-local H-function f such that $\mathfrak{F} = LR_\omega(f)$. If $\omega = \mathbb{P}$, then the ω-local Fitting class $\mathfrak{F}$ is said to be local.

Lemma 2.3. [17, Theorem]. *If $\mathfrak{F}$ is an ω-local Fitting class and a Lockett class as well, then $\mathfrak{F} = LR_\omega(F)$ and $F(a)$ is a Lockett class for all $a \in \omega \cup \{\omega'\}$. Moreover, $F(p)\mathfrak{N}_p = F(p) \subseteq \mathfrak{F} \subseteq F(p)\mathfrak{E}_{p'}$, for all $p \in \omega$.*

For constructing Lockett pairs, we need the concept of the normal subgroup $N(G)$ of G and its properties which given by Gallego [9]. We use $H\ sn\ G$ to denote that H is a subnormal subgroup of G. A subnormal embedding of a subnormal subgroup S of G is a monomorphism $\alpha : S \to G$ such that $S\alpha\ sn\ G$. Let $Snemb(S \to G)$ denote the set of all subnormal embedding of S in G. Then let $N(G) = \langle x^{-1}x^\alpha : x \in S\ sn\ G \text{ and } \alpha \in Snemb(S \to G)\rangle$.

Lemma 2.4. [9, Proposition (3.1)(3.2)]. *Let G be a group and $\mathfrak{F}$ a Fitting class. Then*

(i) *$N(G)$ is a characteristic subgroup of G.*
(ii) *$N(G) \subseteq G_{\mathfrak{F}_*}$ for all $G \in \mathfrak{F}$.*

Lemma 2.5. [9, Proposition (4.1)]. *Suppose that $\mathfrak{F}$, $\mathfrak{H}$ and $\mathfrak{X}$ are Fitting classes. Then the following statements are equivalent:*

(a) $\mathfrak{F}_* \cap \mathfrak{H} \subseteq \mathfrak{X}$.
(b) *$N(G) \cap G_{\mathfrak{H}} \leq G_{\mathfrak{X}}$, for all $G \in \mathfrak{F}$.*

Lemma 2.6.

(1) (Čunihin [7]) *Every π-soluble group G has a Hall π-subgroup and any two Hall π-subgroups of G are conjugate in G.*
(2) [11] *If $\mathfrak{F}$ is a Fitting class, then $\mathfrak{K}_\pi(\mathfrak{F}) = (G \in \mathfrak{S}^\pi : \mathrm{Hall}_\pi(G) \subseteq \mathfrak{F})$ is a Fitting class.*

We use $\mathrm{Hall}_\pi(G)$ to denote the set of all Hall π-subgroups of a π-soluble group G. In this connection, we have the following generalized result of [9, Proposition 4.3].

Lemma 2.7. *Let G be a group, $H \in \mathrm{Hall}_\pi(G)$, $H \in \mathfrak{N}$ and $\mathfrak{F}$ a Fitting class. Then $H \cap N(G) \subseteq N(HG_{\mathfrak{F}})$.*

Proof. Let

$$H_0 = \langle x^{-1}x^\alpha : x \in S \ sn \ G, \alpha \in Snemb(S \to G) \text{ and } x, x^\alpha \in H\rangle.$$

We first prove that $H \cap N(G) = H_0$. By the definition of $N(G)$, every generator of $N(G)$ has the form $g^{-1}g^\alpha$, where $g \in S \ sn \ G$ and $\alpha \in Snemb(S \to G)$. Let $g = xy$, where $x, y \in \langle g\rangle \leq S$ such that x is a π-element and y is a π'-element. Since $H \in \mathrm{Hall}_\pi(G)$, there exist elements a and b of G such that $x^a \in H$ and $(x^\alpha)^b \in H$. It is clear that

$$g^{-1}g^\alpha O^\pi(G)G' = x^{-1}x^\alpha O^\pi(G)G' = (x^a)^{-1}(x^\alpha)^b O^\pi(G)G'.$$

Note that S^a *sn* G, $(S^\alpha)^b$ *sn* G and there exists an isomorphism from S^a onto $(S^\alpha)^b$ such that the image of x^α is $(x^\alpha)^b$. Therefore $(x^a)^{-1}(x^\alpha)^b \in H_0$ and so $g^{-1}g^\alpha \in H_0O^\pi(G)G'$. Since N(G) is generated by such elements $g^{-1}g^\alpha$, we have $N(G) \leq H_0O^\pi(G)G'$ and hence $H \cap N(G) \leq H_0(H \cap O^\pi(G)G')$. Since $O^\pi(G)G'/G'$ is a π'-group, $H \cap O^\pi(G)G' = H \cap G'$. By [2, 21.3(2)], $H \cap G' \subseteq H_0$. Hence $H \cap N(G) \leq H_0$. On the other hand, obviously, $H_0 \leq H \cap N(G)$. Therefore $H \cap N(G) = H_0$. This shows that $H \cap N(G)$ is generated by the elements $x^{-1}x^\alpha$, where $x \in S \ sn \ G$, $\alpha \in Snemb(S \to G)$ and $x, x^\alpha \in H$. Note that the subgroup $\langle x\rangle S_{\mathfrak{F}} = \langle x\rangle(S \cap G_{\mathfrak{F}}) = S \cap \langle x\rangle G_{\mathfrak{F}}$ is subnormal in $HG_{\mathfrak{F}}$. Analogously, $(\langle x\rangle S_{\mathfrak{F}})^\alpha = \langle x^\alpha\rangle S^\alpha_{\mathfrak{F}}$ is subnormal in $HG_{\mathfrak{F}}$. Therefore $x^{-1}x^\alpha \in N(HG_{\mathfrak{F}})$ and $H \cap N(G) \leq N(HG_{\mathfrak{F}})$. ∎

3. HR-classes

In order to construct a new family of Lockett pairs, in this section, we will define the following generalized S-closed Fitting classes.

Definition 3.1. *Suppose that $\pi \subseteq \mathbb{P}$ and $\mathfrak{X}$ is a Fitting class.*

(a) *A subgroup T of G is called a π-HR-subgroup if $T = HG_{\mathfrak{X}}$ for some Hall π-subgroup H of G.*
(b) *A Fitting class $\mathfrak{F}$ is called π-HR-closed if every π-HR-subgroup of G belongs to $\mathfrak{F}$ whenever $G \in \mathfrak{F}$. If $\mathfrak{F}$ is σ-HR-closed for any $\sigma \subseteq \mathbb{P}$, then $\mathfrak{F}$ is called an HR-class.*
(c) *If $\mathfrak{X} = (1)$, we write "π-H-class" instead "π-HR-class", and write "H-class" instead "HR-class".*

The following examples show that the family of the Fitting classes defined in Definition 3.1 is wide.

Example 3.1.

(1) Suppose that $\mathfrak{F}$ is a S-closed Fitting class. Then, obviously, $\mathfrak{F}$ is an π-$HG_{\mathfrak{X}}$-class, for any nonempty Fitting class $\mathfrak{X}$.

(2) Recall that for a Fitting class $\mathfrak{F}$ and a group $G \in \mathfrak{F}$, if $H \in \mathfrak{F}$ for every $H \in \mathrm{Hall}_\pi(G)$, then $\mathfrak{F}$ is said to be π-Hall closed [4]. Obviously, a Fitting class $\mathfrak{F}$ is a π-H-class if and only if it is π-Hall closed.

(3) Let $\pi = \mathbb{P}$ and $\mathfrak{S}_*$ be the smallest normal Fitting class. By the result in [6], $\mathfrak{S}_*$ is an H-class.

(4) For any set $\pi \subseteq \mathbb{P}$ and any Fitting class $\mathfrak{H}$, the Fitting class $\mathfrak{K}_\pi(\mathfrak{H}) = (G \in \mathfrak{S}^\pi : \mathrm{Hall}_\pi(G) \subseteq \mathfrak{H})$ was defined in [11] (see Lemma 2.6(2)). Obviously, $\mathfrak{K}_\pi(\mathfrak{H})$ is π-H-closed if and only if $\mathfrak{H} \subseteq \mathfrak{K}_\pi(\mathfrak{H})$. Moreover, by the proof of [5, Proposition 4.4], we can see that for any $\tau \subseteq \pi$, the Fitting class $\mathfrak{K}_\tau(\mathfrak{H})\mathfrak{Z}$ is π-H-closed for any π-H-closed Fitting class $\mathfrak{Z}$.

4. On problem of the construction of $\mathfrak{L}$-pairs and $\mathfrak{L}$-classes

In this section, we construct a family of Lockett pairs and give a new characteristic of the validity of Lockett conjecture.

Definition 4.1. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be two Fitting classes.*

(i) *We say that $\mathfrak{F}$ and $\mathfrak{H}$ satisfy Property (α_σ) if $\sigma \subseteq \pi$ and there exists a Fitting class $\mathfrak{X}$ such that $\mathfrak{X}\mathfrak{S}_\sigma \subseteq \mathfrak{F} \subseteq \mathfrak{X}\mathfrak{S}_\sigma\mathfrak{S}^\pi_{\sigma'}$, $\mathfrak{H} \subseteq \mathfrak{K}_\sigma(\mathfrak{N})$ and $\mathfrak{H}$ is a σ-HR-class.*

(ii) *Let $\mathrm{Char}(\mathfrak{F})$ be the characteristic of $\mathfrak{F}$ and $\mathrm{Char}(\mathfrak{F}) = \cup_{i\in I}\sigma_i$, where $\sigma_i \neq \emptyset$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i, j \in I (i \neq j)$. We say that $\mathfrak{F}$ and $\mathfrak{H}$ satisfy Property (α) if $\mathfrak{F}$ and $\mathfrak{H}$ satisfy Property (α_{σ_i}) for all $i \in I$.*

Lemma 4.1. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be Fitting classes. If $\mathfrak{F}$ and $\mathfrak{H}$ satisfies Property (α_σ), then $\mathfrak{F} \cap \mathfrak{H}_* \subseteq (\mathfrak{F} \cap \mathfrak{H})_*\mathfrak{S}^\pi_{\sigma'}$.*

Proof. Let $(\mathfrak{F}\cap\mathfrak{H})_*\mathfrak{S}^\pi_{\sigma'} = \mathfrak{M}$. In order to prove $\mathfrak{F}\cap\mathfrak{H}_* \subseteq \mathfrak{M}$, by Lemma 2.5, we only need to prove that $N(G)\cap G_{\mathfrak{F}} \leq G_{\mathfrak{M}}$ for every $G \in \mathfrak{H}$. Suppose that $G \in \mathfrak{H}$ and $H \in \mathrm{Hall}_\sigma(G)$. Then $HG_{\mathfrak{X}}/G_{\mathfrak{X}} \in \mathrm{Hall}_\sigma(G/G_{\mathfrak{X}})$ and so $HG_{\mathfrak{X}}/G_{\mathfrak{X}} \in \mathfrak{S}_\sigma$. Hence, from [8, IX, 1.11], we have $H \in \mathfrak{X}\mathfrak{S}_\sigma \subseteq \mathfrak{F}$. Since $\mathfrak{H}$ is a σ-HR-class by hypothesis, $HG_{\mathfrak{X}} \in \mathfrak{H}$. Hence, $HG_{\mathfrak{X}} \in \mathfrak{F}\cap\mathfrak{H}$. Besides, since H is a Hall σ-subgroup, by Lemma 2.7, we have $H\cap N(G) \subseteq N(HG_{\mathfrak{X}})$. Hence, $H\cap N(G)\cap G_{\mathfrak{F}} \subseteq N(HG_{\mathfrak{X}})\cap G_{\mathfrak{F}}$.

We claim that $N(HG_{\mathfrak{X}})\cap G_{\mathfrak{F}} \subseteq (HG_{\mathfrak{X}}\cap G_{\mathfrak{F}})_{(\mathfrak{F}\cap\mathfrak{H})_*}$. In fact, since $HG_{\mathfrak{X}} \in \mathfrak{F}\cap\mathfrak{H}$, by Lemma 2.4 $N(HG_{\mathfrak{X}}) \subseteq (HG_{\mathfrak{X}})_{(\mathfrak{F}\cap\mathfrak{H})_*}$ and so $N(HG_{\mathfrak{X}})\cap G_{\mathfrak{F}} \subseteq (HG_{\mathfrak{X}})_{(\mathfrak{F}\cap\mathfrak{H})_*}\cap G_{\mathfrak{F}}$. But, because $HG_{\mathfrak{X}}\cap G_{\mathfrak{F}} \trianglelefteq HG_{\mathfrak{X}}$, by [8, IX.1.1((a)], we have

$$(HG_{\mathfrak{X}}\cap G_{\mathfrak{F}})_{(\mathfrak{F}\cap\mathfrak{H})_*} = (HG_{\mathfrak{X}})_{(\mathfrak{F}\cap\mathfrak{H})_*}\cap(HG_{\mathfrak{X}}\cap G_{\mathfrak{F}}) = (HG_{\mathfrak{X}})_{(\mathfrak{F}\cap\mathfrak{H})_*}\cap G_{\mathfrak{F}}.$$

Hence $N(HG_{\mathfrak{X}})\cap G_{\mathfrak{F}} \subseteq (HG_{\mathfrak{X}})_{(\mathfrak{F}\cap\mathfrak{H})_*}\cap G_{\mathfrak{F}} = (HG_{\mathfrak{X}}\cap G_{\mathfrak{F}})_{(\mathfrak{F}\cap\mathfrak{H})_*}$.

Now we prove $HG_{\mathfrak{X}}\cap G_{\mathfrak{F}} \trianglelefteq G$. In fact, since $G_{\mathfrak{F}} \in \mathfrak{X}\mathfrak{S}_\sigma\mathfrak{S}^\pi_{\sigma'}$, $G_{\mathfrak{F}}/G_{\mathfrak{X}\mathfrak{S}_\sigma} \in \mathfrak{S}^\pi_{\sigma'}$. Then, by theorem [8, IX.1.12],

$$(G_{\mathfrak{F}}/G_{\mathfrak{X}})/(G_{\mathfrak{X}\mathfrak{S}_\sigma}/G_{\mathfrak{X}}) = (G_{\mathfrak{F}}/G_{\mathfrak{X}})/(G/G_{\mathfrak{X}})_{\mathfrak{S}_\sigma} \in \mathfrak{S}^\pi_{\sigma'}$$

and so $G_{\mathfrak{X}\mathfrak{S}_\sigma}/G_{\mathfrak{X}} \in \mathrm{Hall}_\sigma(G_{\mathfrak{F}}/G_{\mathfrak{X}})$. On the other hand, since $H \in \mathrm{Hall}_\sigma(G)$, $(H \cap G_{\mathfrak{F}})G_{\mathfrak{X}}/G_{\mathfrak{X}} \in \mathrm{Hall}_\sigma(G_{\mathfrak{F}}/G_{\mathfrak{X}})$. Hence $G_{\mathfrak{X}\mathfrak{S}_\sigma} = (H \cap G_{\mathfrak{F}})G_{\mathfrak{X}} = HG_{\mathfrak{X}} \cap G_{\mathfrak{F}}$ and $HG_{\mathfrak{X}} \cap G_{\mathfrak{F}} \trianglelefteq G$. This implies that $(HG_{\mathfrak{X}} \cap G_{\mathfrak{F}})_{(\mathfrak{F}\cap\mathfrak{H})_*} \leq G_{(\mathfrak{F}\cap\mathfrak{H})_*}$. Therefore, $H \cap N(G) \cap G_{\mathfrak{F}} \subseteq G_{(\mathfrak{F}\cap\mathfrak{H})_*} \cap N(G) \cap G_{\mathfrak{F}}$. But by Lemma 2.4(i), $N(G) \trianglelefteq G$. Hence $G_{(\mathfrak{F}\cap\mathfrak{H})_*} \cap (N(G) \cap G_{\mathfrak{F}}) = (N(G) \cap G_{\mathfrak{F}})_{(\mathfrak{F}\cap\mathfrak{H})_*}$ by [8, IX, 1.1 (a)]. It follows that $H \cap (N(G) \cap G_{\mathfrak{F}}) \subseteq (N(G) \cap G_{\mathfrak{F}})_{(\mathfrak{F}\cap\mathfrak{H})_*}$. Since $H_1 := H \cap (N(G) \cap G_{\mathfrak{F}})$ is a Hall σ-subgroup of $N(G) \cap G_{\mathfrak{F}}$, $|(N(G) \cap G_{\mathfrak{F}}) : H_1|$ is a σ'-number. But since $|H_1| \mid |(N(G) \cap G_{\mathfrak{F}})_{(\mathfrak{F}\cap\mathfrak{H})_*}|$, $(N(G) \cap G_{\mathfrak{F}})/(N(G) \cap G_{\mathfrak{F}})_{(\mathfrak{F}\cap\mathfrak{H})_*} \in \mathfrak{S}^{\pi}_{\sigma'}$. Hence $N(G) \cap G_{\mathfrak{F}} \in (\mathfrak{F}\cap\mathfrak{H})_*\mathfrak{S}^{\pi}_{\sigma'} = \mathfrak{M}$. Consequently $N(G) \cap G_{\mathfrak{F}} \leq G_{\mathfrak{M}}$. This completes that proof. ∎

The following theorem describes a new and wide family of Lockett pairs. In particular, the theorem give some new Fitting classes which satisfy Lockett conjecture.

Theorem 4.1. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be two Fitting classes. If $\mathfrak{F}$ and $\mathfrak{H}$ satisfy Property (α), then $(\mathfrak{F},\mathfrak{H})$ is an $\mathfrak{L}$-pair. In particularly, if $\mathfrak{F} \subseteq \mathfrak{H}$, then $\mathfrak{F}$ is an $\mathfrak{L}_{\mathfrak{H}}$-class, that is, $\mathfrak{F}$ satisfies Lockett conjecture in $\mathfrak{H}$.*

Proof. By Lemma 4.1, we only need to prove that if $\mathfrak{F} \cap \mathfrak{H}_* \subseteq (\mathfrak{F}\cap\mathfrak{H})\mathfrak{S}^{\pi}_{\sigma_i'}$ for every $i \in I$, then $\mathfrak{F} \cap \mathfrak{H}_* = (\mathfrak{F}\cap\mathfrak{H})_*$. Firstly, by Lemma 2.1, we have $(\mathfrak{F}\cap\mathfrak{H})_* \subseteq \mathfrak{F}\cap\mathfrak{H}_*$.

Conversely, assume that it is not true and let G be a group in $\mathfrak{F}\cap\mathfrak{H}_* \setminus (\mathfrak{F}\cap\mathfrak{H})_*$ of minimal order. Then G has a unique maximal normal subgroup $M = G_{(\mathfrak{F}\cap\mathfrak{H})_*}$. Since $G \in \mathfrak{F}\cap\mathfrak{H}_*$, $G \in \mathfrak{F}\cap\mathfrak{H}$ by Lemma 2.1(a)(b). Then by using Lemma 2.1(b)(c), we obtain that $G \in (\mathfrak{F}\cap\mathfrak{H})_*\mathfrak{A}$, where $\mathfrak{A}$ is the class of all abelian groups. Hence, G/M has a unique maximal normal subgroup of order p and so $G/M \simeq Z_p$. Since $G \in \mathfrak{F}\cap\mathfrak{H}$, $p \in \mathrm{Char}(\mathfrak{F}\cap\mathfrak{H})$ by [8, Lemma IX.1.7]. It follows from Lemma 2.2 that there exists $\sigma_{i_0} \subseteq \mathrm{Char}(\mathfrak{F}\cap\mathfrak{H})$ for $i_0 \in I$ such that $p \in \sigma_{i_0} \subseteq \mathrm{Char}((\mathfrak{F}\cap\mathfrak{H})_*)$. Therefore $G/M \in \mathfrak{S}_{\sigma_{i_0}}$.

On the other hand, since σ, $\mathfrak{F}$ and $\mathfrak{H}$ satisfy conditions of Lemma 4.1, $\mathfrak{F}\cap\mathfrak{H}_* \subseteq (\mathfrak{F}\cap\mathfrak{H})_*\mathfrak{S}^{\pi}_{\sigma'_{i_0}}$. Since $G \in \mathfrak{F}\cap\mathfrak{H}_*$ and $M = G_{(\mathfrak{F}\cap\mathfrak{H})_*}$, we have $G/M \in \mathfrak{S}^{\pi}_{\sigma'_{i_0}}$. This implies that $G = M \in (\mathfrak{F}\cap\mathfrak{H})_*$. This contradiction shows that $(\mathfrak{F},\mathfrak{H})$ is an $\mathfrak{L}$-pair.

Now assume that $\mathfrak{F} \subseteq \mathfrak{H}$. In order to prove that $\mathfrak{F}$ is an $\mathfrak{L}$-class (that is, $\mathfrak{F}$ satisfies Lockett conjecture), clearly, we only need to prove that $\mathfrak{F}$ is a Lockett class (that is, $\mathfrak{F}^* = \mathfrak{F}$). By the hypothesis, $\mathfrak{F}$ satisfies Property (α). Hence $\mathfrak{X}\mathfrak{S}_{\sigma_i} \subseteq \mathfrak{F} \subseteq \mathfrak{X}\mathfrak{S}_{\sigma_i}\mathfrak{S}^{\pi}_{\sigma_i'}$ for all $i \in I$. Then by Lemma 2.1(a), we see that $\mathfrak{F}^* \subseteq (\mathfrak{X}\mathfrak{S}_{\sigma_i}\mathfrak{S}^{\pi}_{\sigma_i'})^*$. But by [15, Corollary], $\mathfrak{X}\mathfrak{S}_{\sigma_i}\mathfrak{S}^{\pi}_{\sigma_i'}$ is local and so it is a Lockett class by [15, Lemma 5]. Therefore, $(\mathfrak{X}\mathfrak{S}_{\sigma_i}\mathfrak{S}^{\pi}_{\sigma_i'})^* = \mathfrak{X}\mathfrak{S}_{\sigma_i}\mathfrak{S}^{\pi}_{\sigma_i'}$ and thereby $\mathfrak{F}^* \subseteq \mathfrak{F}\mathfrak{S}^{\pi}_{\sigma_i'}$ for all $i \in I$. By Lemma 2.2, $\mathrm{Char}(\mathfrak{F}) = \mathrm{Char}(\mathfrak{F}^*)$. Now analogous proof from $\mathfrak{F}_* \cap \mathfrak{H} \subseteq (\mathfrak{F}\cap\mathfrak{H})_*\mathfrak{S}^{\pi}_{\sigma_i'}$ to $\mathfrak{F}_* \cap \mathfrak{H} \subseteq (\mathfrak{F}\cap\mathfrak{H})_*$, we obtain that $\mathfrak{F}^* = \mathfrak{F}$. This completes the proof. ∎

5. Applications

By using Theorem 4.1, we immediately obtain the following known results about description of $\mathfrak{L}$-pair and $\mathfrak{L}$-class.

Corollary 5.1. (Bryce, Cossey [6]). *If $\mathfrak{F}$ and $\mathfrak{H}$ are soluble S-closed Fitting classes, then $(\mathfrak{F},\mathfrak{H})$ is an $\mathfrak{L}$-pair. In particularly, $(\mathfrak{F},\mathfrak{S})$ is an $\mathfrak{L}$-pair, that is, every S-closed Fitting class is an $\mathfrak{L}$-class.*

Proof. By [17, Theorem], $\mathfrak{F}$ and $\mathfrak{H}$ are local Fitting classes. Put $\sigma_i = \{p\}$ in Theorem 4.1, for all $p \in \mathrm{Char}(\mathfrak{F})$ and $i \in I$. Since $\mathfrak{H}$ is S-closed, $\mathfrak{H}$ is an HR-class for any Fitting class $\mathfrak{X}$. Besides, by Lemma 2.3, for $\omega = \mathbb{P}$ and every $p \in \mathrm{Char}(\mathfrak{F})$, we have $F(p)\mathfrak{N}_p \subseteq \mathfrak{F} \subseteq F(p)\mathfrak{N}_p\mathfrak{S}_{p'}$. Thus, by Theorem 4.1, $(\mathfrak{F},\mathfrak{H})$ is an $\mathfrak{L}$-pair. Besides, by Lemma 2.1(f), $\mathfrak{F} = \mathfrak{F}^*$. If $\mathfrak{H} = \mathfrak{S}$, then $\mathfrak{F}$ is an $\mathfrak{L}$-class, that is, $\mathfrak{F}$ satisfies Lockett conjecture. ∎

Corollary 5.2. [see 12, Theorem B]. *If $\mathfrak{F} = LR_\omega(F)$ is ω-local Fitting class with* $\mathrm{Char}(\mathfrak{F}) \subseteq \omega$*, then $(\mathfrak{F},\mathfrak{S}^\pi)$ is an $\mathfrak{L}$-pair and $\mathfrak{F}$ is an $\mathfrak{L}$-class.*

Proof. By Lemma 2.3, $F(p)\mathfrak{N}_p \subseteq \mathfrak{F} \subseteq F(p)\mathfrak{N}_p\mathfrak{S}^\pi_{p'}$ for all $p \in \mathrm{Char}(\mathfrak{F})$. Hence, if put $\sigma_i = \{p\}$ for all $p \in \mathrm{Char}(\mathfrak{F})$ and $i \in I$ and let $\mathfrak{H} = \mathfrak{S}^\pi$, the $\mathfrak{F}$ and $\mathfrak{H}$ satisfy the hypothesis of Theorem 4.1. Thus, by Theorem 4.1, $(\mathfrak{F},\mathfrak{S}^\pi)$ is an $\mathfrak{L}$-pair and so, clearly, $\mathfrak{F}$ is an $\mathfrak{L}$-class since $\mathfrak{F} \subseteq \mathfrak{H}$. ∎

Put $\omega = \pi$, then by Corollary 5.2, we have

Corollary 5.3. *If $\mathfrak{F}$ is a local Fitting class, then $(\mathfrak{F},\mathfrak{S}^\pi)$ is an $\mathfrak{L}$-pair.*

Put $\omega = \pi = \mathbb{P}$, then by Corollary 5.2, we obtain

Corollary 5.4. [15]. *Lockett conjecture holds for every soluble local Fitting class $\mathfrak{F}$, that is, if $\mathfrak{F} \subseteq \mathfrak{S}$, then pair $(\mathfrak{F},\mathfrak{S})$ is an $\mathfrak{L}$-pair.*

Corollary 5.5. [1]. *Let $\mathfrak{F} \in \{\mathfrak{X}\mathfrak{N}, \mathfrak{X}\mathfrak{S}_\pi\mathfrak{S}_{\pi'}\}$, where $\mathfrak{X}$ is some nonempty soluble Fitting class. Then $\mathfrak{F}$ satisfies Lockett conjecture.*

Proof. By [15, Corollary], $\mathfrak{X}\mathfrak{N}$ and $\mathfrak{X}\mathfrak{S}_\pi\mathfrak{S}_{\pi'}$ are all local Fitting classes. Hence by Corollary 5.3, the statement holds. ∎

Corollary 5.6. *Let $\sigma \subseteq \pi$ and $\mathfrak{F}$, $\mathfrak{H}$ be Fitting classes such that $\mathfrak{F} = \mathfrak{S}_\sigma \subseteq \mathfrak{H}$. Then $(\mathfrak{F},\mathfrak{H})$ is an $\mathfrak{L}$-pair and $\mathfrak{S}_\sigma$ is an $\mathfrak{L}_\mathfrak{H}$-class.*

Proof. Obviously, $\mathfrak{F}$ is local. By Lemma 2.3, for $\omega = \mathbb{P}$, $\mathfrak{F}$ satisfies the related conditions of Theorem 4.1 for $\mathfrak{F}$ if put $\sigma_i = \{p\}$ for all $p \in \mathrm{Char}(\mathfrak{F})$ and $i \in I$. Besides, by [15, Lemma 5], $\mathfrak{F}$ is a Lockett class, that is, $\mathfrak{F}^* = \mathfrak{F}$.

Now we prove that $(\mathfrak{F},\mathfrak{H})$ is an $\mathfrak{L}$-pair, that is, $\mathfrak{F} \cap \mathfrak{H}_* = (\mathfrak{F} \cap \mathfrak{H})_*$. If $\sigma = \phi$, then $\mathfrak{F} = (1)$ and so it is trivial. If $\sigma = \{p\}$, then by [8, X.1.23], $(\mathfrak{N}_p)_* = \mathfrak{N}_p$ and so $\mathfrak{N}_p \cap \mathfrak{H}_* = (\mathfrak{N}_p \cap \mathfrak{H})_* = (\mathfrak{N}_p)_*$. Put $|\sigma| \geq 2$. Since $\mathfrak{S}_\sigma \subseteq \mathfrak{H}$, then $\mathfrak{H}$ is a p-H-class and $\mathfrak{H} \subseteq \mathfrak{K}_p(\mathfrak{N})$ for every $p \in \mathrm{Char}(\mathfrak{F}) = \sigma$. Thus by Theorem 4.1, $(\mathfrak{S}_\sigma,\mathfrak{H})$ is an $\mathfrak{L}$-pair. ∎

For $\pi = \mathbb{P}$ we have

Corollary 5.7. [4]. *Let $\mathfrak{F}$, $\mathfrak{H}$ be soluble Fitting classes and $\sigma \subseteq \mathbb{P}$. If $\mathfrak{F} = \mathfrak{S}_\sigma \subseteq \mathfrak{H}$, then $(\mathfrak{F},\mathfrak{H})$ is an $\mathfrak{L}$-pair and $\mathfrak{F}$ is an $\mathfrak{L}_\mathfrak{H}$-class.*

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