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Some Properties of Planar p-Harmonic and log-p-Harmonic Mappings

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Abstract. A $2p$-times continuously differentiable complex-valued function $F = u + iv$ in a domain $\Omega \subseteq \mathbb{C}$ is *p-harmonic* if F satisfies the p-harmonic equation $\Delta^p F = 0$. We say that F is log-*p-harmonic* if $\log F$ is p-harmonic. In this paper, we investigate several basic properties of p-harmonic and log-p-harmonic mappings. In particular, we discuss the problem of when the composite mappings of p-harmonic mappings with a fixed analytic function are q-harmonic, where $q \in \{1, \dots, p\}$. Also, we obtain necessary and sufficient conditions for a function to be p-harmonic (resp. log-p-harmonic). We study the local univalence of p-harmonic and log-p-harmonic mappings, and in particular, we obtain two sufficient conditions for a function to be a locally univalent p-harmonic or a locally univalent log-p-harmonic. The starlikeness of log-p-harmonic mappings is considered.

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1. Introduction and main results

One of the most fundamental articles on univalent harmonic (sense preserving) mappings is due to Clunie and Sheil-Small [11] (see the work of Mocanu [17] for many basic results about univalent C^1-mappings and the monograph of Duren [12] about univalent harmonic mappings).

Biharmonic mappings arise in a lot of physical situations, particularly in fluid dynamics and elasticity problems, and have many important applications in engineering and biology, see [13–15]. However, the investigation of biharmonic mappings in the context of geometric function theory is a recent one (see [2, 4, 5, 7, 8] and the references therein). Many physical problems are modeled by log-biharmonic mappings, particularly those arising in fluid flow theory and elasticity. The log-biharmonic mappings are closely associated with the biharmonic mappings, which appear in Stokes flow problems. There is an enormous number of problems involving Stokes flow which arise in engineering and biological transport phenomena (for details see [13–15]). It will also be interesting investigate in directions of applications. Recently, the properties of log-harmonic and log-biharmonic mappings have

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been investigated by several authors, see [1,3,16]. In the following subsections, we include the definitions of these classes of mappings and state the main results together with their implications and some interesting examples about these classes of mappings.

1.1. p-harmonic mappings

A $2p$-times continuously differentiable complex-valued function $F = u + iv$ in a domain $\Omega \subseteq \mathbb{C}$ is *p-harmonic* if F satisfies the p-harmonic equation $\Delta^p F = 0$, where $p\ (\geq 1)$ is an integer, Δ represents the Laplacian operator

$$\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \ \text{ and } \ \Delta^p F := \underbrace{\Delta \cdots \Delta}_{p} F = \Delta^{p-1}(\Delta F).$$

Obviously, when $p = 1$ (resp. $p = 2$), F is harmonic (resp. biharmonic) in Ω. Also, it is clear that every harmonic mapping is p-harmonic for each $p \geq 2$.

If $\Omega \subset \mathbb{C}$ is a simply connected domain, then it is easy to see that (see [9]) every p-harmonic mapping F can be written as

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z),$$

where each G_{p-k+1} is harmonic, i.e., $\Delta G_{p-k+1} = 0$ for $k \in \{1, \ldots, p\}$. We refer to [9] for many interesting results on p-harmonic mappings of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

1.2. Composition mappings

Although a harmonic mapping of an analytic function is known to be harmonic, a p-harmonic mapping $(p > 1)$ precomposition with an analytic function may not be p-harmonic. This can be easily seen by taking

$$F(z) = |z|^{2(p-1)} \ \text{ and } \ f(z) = 2z^2 + z + 1.$$

We see that F is p-harmonic, whereas $F \circ f$ is not. Therefore, it is natural to ask

Problem 1.1. *For an analytic function f, under what condition, is the composite mapping $F \circ f$ still p-harmonic (resp. q-harmonic, where $q \in \{1, \ldots, p-1\}$) for any p-harmonic mapping F?*

We now begin to state two elementary results concerning this problem.

Theorem 1.1. Let f be an analytic function in $\mathbb{D}$. Then for any p-harmonic mapping F with $p > 1$, $F \circ f$ is p-harmonic if and only if $f(z) = az + b$, where a and b are constants.

Theorem 1.2. Let f be an analytic function in $\mathbb{D}$ and q an integer in $\{1, \ldots, p-1\}$, where $p > 1$. Then for any p-harmonic function F, $F \circ f$ is q-harmonic if and only if f is a constant.

We present the proofs of Theorems 1.1 and 1.2 in Section 2.

1.3. log-p-harmonic mappings

A *logharmonic mapping* defined on $\mathbb{D}$ is a solution of the nonlinear elliptic partial differential equation

$$\overline{f_{\bar{z}}} = (\mu \overline{f}/f) f_z, \quad f(0) = 0, \tag{1.1}$$

where the *second dilatation* μ is analytic in $\mathbb{D}$ such that $|\mu(z)| < 1$ on $\mathbb{D}$. In general the solution of the equation (1.1) is not univalent. For instance, the functions

$$f_1(z) = |z|^4 z^4 \ \text{ and } \ f_2(z) = |z|^2 z$$

are the solutions of (1.1) with $\mu = 1/3$ and $\mu = 1/2$, respectively. We observe that the function f_1 is not univalent whereas f_2 is univalent in $\mathbb{D}$. It follows that the Jacobian

$$J_f := |f_z|^2 - |f_{\bar{z}}|^2 = |f_z|^2 (1 - |\mu|^2)$$

is positive and hence, non-constant logharmonic mappings are sense-preserving and open on $\mathbb{D}$. If in addition f is univalent then we say that f is univalent logharmonic on $\mathbb{D}$ (vanishing at the origin). Such mapping f admits the representation

$$f(z) = z|z|^{2\beta} h(z) \overline{g(z)}, \tag{1.2}$$

where $\operatorname{Re}\beta > -1/2$ and, h and g are non-vanishing analytic functions in $\mathbb{D}$ with $g(0) = 1$ and $h(0) \neq 0$ (cf. [6]). We see that the exponent β in (1.2) depends only on $\mu(0)$ and can be expressed by

$$\beta = \frac{\overline{\mu(0)}(1 + \mu(0))}{1 - |\mu(0)|^2}.$$

If f is univalent logharmonic on $\mathbb{D}$ such that $\mu(0) = 0$, then $\beta = 0$ and so f in this case has the form $f(z) = zh(z)\overline{g(z)}$.

In case $0 \notin f(\mathbb{D})$, then every univalent logharmonic f on $\mathbb{D}$ takes the form $f(z) = h(z)\overline{g(z)}$.

We say that F is log-*p-harmonic* if $\log F$ is p-harmonic. Throughout "log" denotes the principal branch of the logarithm. It can be easily shown that every log-p-harmonic function F in a simply connected domain Ω has the form

$$F(z) = \prod_{k=1}^{p} \left(G_{p-k+1}(z)\right)^{|z|^{2(k-1)}},$$

where all G_{p-k+1} are nonvanishing logharmonic mappings in Ω for $k \in \{1, \ldots, p\}$. When $p = 1$ (resp. $p = 2$), log-p-harmonic F is called log-*harmonic* (resp. log-*biharmonic*).

Throughout this paper we will discuss p-harmonic and log-p-harmonic mappings defined on the unit disk $\mathbb{D}$. In order to state our results about the local univalence of p-harmonic and log-p-harmonic mappings, we need some preparations.

A complex-valued function $f\colon \Omega \to \mathbb{C}$ is said to belong to the class $C^1(\Omega)$ (resp. $C^2(\Omega)$) if $\operatorname{Re} f$ and $\operatorname{Im} f$ have continuous first order (resp. second order) partial derivatives in Ω. For $f \in C^1(\Omega)$, consider the complex linear differential operators D and $\mathscr{D}$ defined on $C^1(\Omega)$ by

$$D = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}, \ \text{ and } \ \mathscr{D} = z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}},$$

respectively. A number of interesting algebraic and analytic properties of these operators are discussed by Mocanu [17]. For instance, it is easy to see that the operator D preserves *p-harmonicity*. It is well-known that f is locally univalent if $J_f(z) \neq 0$ for $z \in \mathbb{D}$. If $J_f(z) > 0$ for $z \in \Omega$, then f is locally diffeomorphism preserving the orientation. In [17], Mocanu obtained extra conditions by using geometric concepts such as starlikeness, convexity and close-to-convexity so that the map $f : \Omega \to f(\Omega)$ is globally diffeomorphism.

1.4. Log-harmonic starlike mappings

A continuous function $f: \mathbb{D} \to \mathbb{C}$, $f(0) = 0$, is called starlike in $\mathbb{D}$ if it is univalent and the range $f(\mathbb{D})$ is a starlike (with respect to the origin) domain.

Definition 1.1. *We say that a univalent function $f \in C^1(\mathbb{D})$ with $f(0) = 0$ is said to be fully starlike if the curve $f(re^{it})$ is starlike for each $r \in (0,1)$. In other words,*

$$\frac{\partial}{\partial t}\left(\arg f(re^{it})\right) = \mathrm{Re}\left(\frac{Df(z)}{f(z)}\right) > 0$$

for $z = re^{it} \in \mathbb{D}\backslash\{0\}$ (see also [10] in order to distinguish starlikeness in the analytic and the harmonic cases).

Throughout the paper, we treat fully starlike functions as starlike functions although this is not the case in strict sense. At this place, it is also important to observe that Dg for C^1-functions behaves much like zg' for analytic functions, for example in the sense that for g univalent and analytic, g is starlike if and only if $\mathrm{Re}\,(zg'(z)/g(z)) > 0$. A similar characterization has also been obtained by Mocanu [17] for convex (C^2) functions. Lately, interesting distortion theorems and coefficients estimates for convex and close-to-convex harmonic mappings were given by Clunie and Sheil-Small [11]. We now state our next main result.

Theorem 1.3. Let $F(z) = |z|^{2(p-1)}G(z) + K(z)$, where G and K belong to the class $C^1(\mathbb{D})$ such that G is univalent starlike (not necessarily harmonic) and K is a sense-preserving. If for $z \in \mathbb{D}\backslash\{0\}$,

$$(p-1)|G|^2\mathrm{Re}\left(\frac{zK_z - \bar{z}K_{\bar{z}}}{G}\right) + |z|^2\mathrm{Re}\left(G_z\overline{K_z} - G_{\bar{z}}\overline{K_{\bar{z}}}\right) > 0 \tag{1.3}$$

or equivalently

$$(p-2)|G|^2\mathrm{Re}\left(\frac{zK_z - \bar{z}K_{\bar{z}}}{G}\right) + \mathrm{Re}\left(K_z\overline{(|z|^2G)_z} - K_{\bar{z}}\overline{(|z|^2G)_{\bar{z}}}\right) > 0 \tag{1.4}$$

then $J_F > 0$ and F is locally univalent.

As every biharmonic mapping F has the form $F(z) = |z|^2G(z) + K(z)$ for some harmonic mappings G and K, Theorem 1.3 for $p = 2$ contains a refined version of [4, Theorem 3.1] although the proof for the general case follows from the same lines of those in [4].

Corollary 1.1. *Let $F(z) = |z|^{2(p-1)}G(z)$, where G is starlike (not necessarily harmonic) in $\mathbb{D}$. Then F is starlike and univalent in $\mathbb{D}$.*

The proofs of Theorem 1.3 and Corollary 1.1 will be given in Section 3.

If G in this corollary is analytic and univalent in $\mathbb{D}$, then $G_{\bar{z}} = 0$ and therefore the corresponding Jacobian J_F takes the form

$$J_F = 2(p-1)|z|^{2(2p-3)}|G|^2 \mathrm{Re}\left(\frac{zG'}{G}\right) + |z|^{4(p-1)}|G'|^2,$$

or equivalently

$$J_F = |z|^{2(2p-3)}|G|^2 \left[\left|\frac{zG'}{G} + p - 1\right|^2 - (p-1)^2\right].$$

This observation gives the following

Corollary 1.2. *Suppose that G is analytic and univalent in $\mathbb{D}$ and $F(z) = |z|^{2(p-1)}G(z)$. Then we have*

(i) $J_F = 0$ *if and only if* $\left|\frac{zG'}{G} + p - 1\right| = p - 1$ *or* $z = 0$*;*

(ii) $J_F > 0$ *if and only if* $\left|\frac{zG'}{G} + p - 1\right| > p - 1$ *and* $z \neq 0$*;*

(iii) $J_F < 0$ *if and only if* $\left|\frac{zG'}{G} + p - 1\right| < p - 1$ *and* $z \neq 0$.

Theorem 1.4. Let $f(z) = k(z)g(z)^{|z|^{2(p-1)}}$, where both k and g are nonvanishing C^1-functions in $\mathbb{D}$ such that $g(0) = 1$ and $\log g$ is starlike (not necessarily harmonic) and univalent in $\mathbb{D}$, and that k is sense-preserving. If (1.3) holds for $z \in \mathbb{D}\backslash\{0\}$, with $G = \log g$ and $K = \log k$, then $J_f > 0$ and f is locally univalent.

Proof. Set $F = \log f$. As $G = \log g$ and $K = \log k$, the function F takes the form given in Theorem 1.3 and the proof follows easily from the hypotheses. ∎

As remarked for earlier theorem, the case $p = 2$ of Theorem 1.4 is again a refined version of corresponding result from [3]. Also, Theorem 1.4 includes a result for certain log-p-harmonic mappings as a special case. Thus, Theorems 1.3 and 1.4 are natural generalizations (indeed under weaker hypotheses) of the corresponding results obtained in [3, 4] for biharmonic and log-biharmonic mappings.

1.5. Examples

Example 1.1. Set $f(z) = ze^{\lambda\bar{z}}$, where $|\lambda| \leq 1$. Then $f_{z\bar{z}} = \lambda e^{\lambda\bar{z}}$ and therefore, f is not harmonic unless $\lambda = 0$. On the other hand, for $|\lambda| \leq 1$, we see that

$$\frac{Df(z)}{f(z)} = 1 - \lambda\bar{z} \text{ and } J_f(z) = |e^{\lambda\bar{z}}|^2(1 - |\lambda\bar{z}|^2),$$

showing that f is starlike and sense preserving in $\mathbb{D}$. We observe that f is a solution of (1.1) with the second complex dilatation $\mu(z) = \overline{\lambda}z$ and is therefore, log-harmonic in $\mathbb{D}$.

Example 1.2. Set $f(z) = z - \lambda|z|^2$, where $0 < |\lambda| < 1/2$. Then $f_{z\bar{z}} = -\lambda$ and therefore, f is not harmonic. It is easy to see that f is log-harmonic in $\mathbb{D}$, as a solution of (1.1) with the second complex dilatation as

$$\mu(z) = \frac{\overline{\lambda}z}{1 + \overline{\lambda}z}$$

which is analytic in $\mathbb{D}$ and $|\mu(z)| < 1$ in $\mathbb{D}$ (as $0 < |\lambda| < 1/2$). Indeed, a simple calculation shows that for $0 < |\lambda| < 1/2$,

$$\operatorname{Re}\frac{Df(z)}{f(z)} = \operatorname{Re}\left(\frac{1}{1-\lambda\bar{z}}\right) \geq \frac{1}{1+|\lambda|} > 0$$

and

$$J_f(z) = |1-\lambda\bar{z}|^2 - |\lambda z|^2 = 1 - 2\operatorname{Re}(\lambda\bar{z}) \geq 1 - 2|\lambda| > 0$$

showing that f is starlike and sense preserving in $\mathbb{D}$.

Example 1.3. Consider the functions

$$f(z) = ze^{\bar{z}},\ g(z) = \frac{z(1-\bar{z})}{1-z}, \text{ and } h(z) = \frac{ze^{(1/2)\bar{z}}}{1-z}.$$

In Example 1.1, we have shown that f is starlike in $\mathbb{D}$. For the function g, we see that

$$J_g(z) = \frac{1-|z|^2}{|1-z|^2} > 0 \text{ and } \frac{Dg(z)}{g(z)} = \frac{1}{1-z} + \frac{\bar{z}}{1-\bar{z}}$$

so that

$$\operatorname{Re}\left(\frac{Dg(z)}{g(z)}\right) = \operatorname{Re}\left(\frac{1}{1-z} + \frac{z}{1-z}\right) > 0.$$

Thus, g is starlike (univalent) in $\mathbb{D}$. Similarly, it is a simple exercise to show that h is starlike (univalent) in $\mathbb{D}$. According to Corollary 1.1,

$$F_1(z) = |z|^{2(p-1)}ze^{\bar{z}},\ F_2(z) = |z|^{2(p-1)}\frac{z(1-\bar{z})}{1-z} \text{ and } F_3(z) = |z|^{2(p-1)}\frac{ze^{(1/2)\bar{z}}}{1-z}$$

are all starlike (univalent) in $\mathbb{D}$. Also, it is easy to see that g is (univalent) log-harmonic in $\mathbb{D}$ with $\mu(z) = -z$ whereas h is (univalent) log-harmonic in $\mathbb{D}$ with $\mu(z) = z(1-z)/2$. The mapping properties of $F_j(z)$ $(j = 1,2,3)$ for the values of $p = 1,3$ are shown in Figures 1–6. The figures show the images of concentric circles and equally spaced rays of the unit disk $\mathbb{D}$.

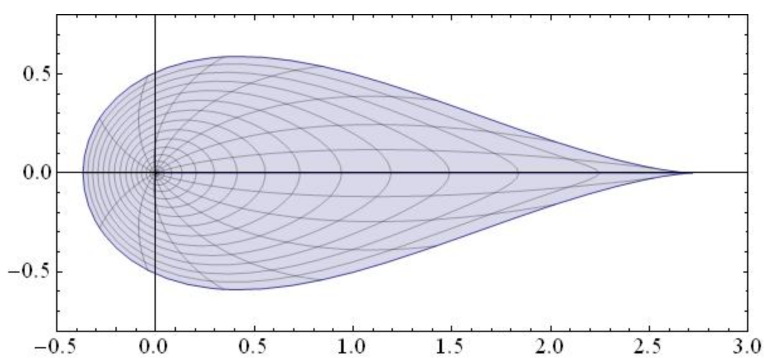

Figure 1. Graph of $f(z) = |z|^{2(p-1)}ze^{\bar{z}}$ for $p = 1$

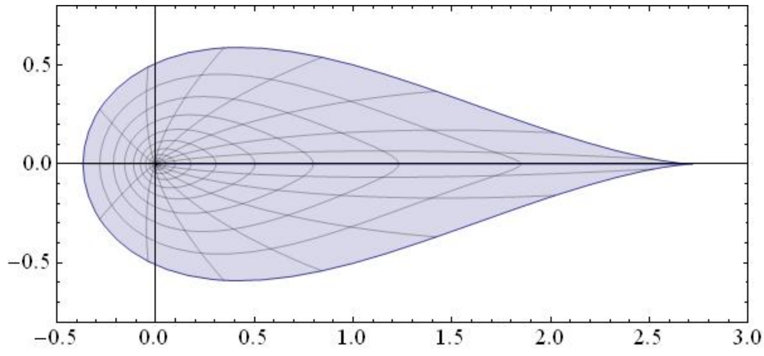

Figure 2. Graph of $f(z) = |z|^{2(p-1)} z e^{\bar{z}}$ for $p = 3$

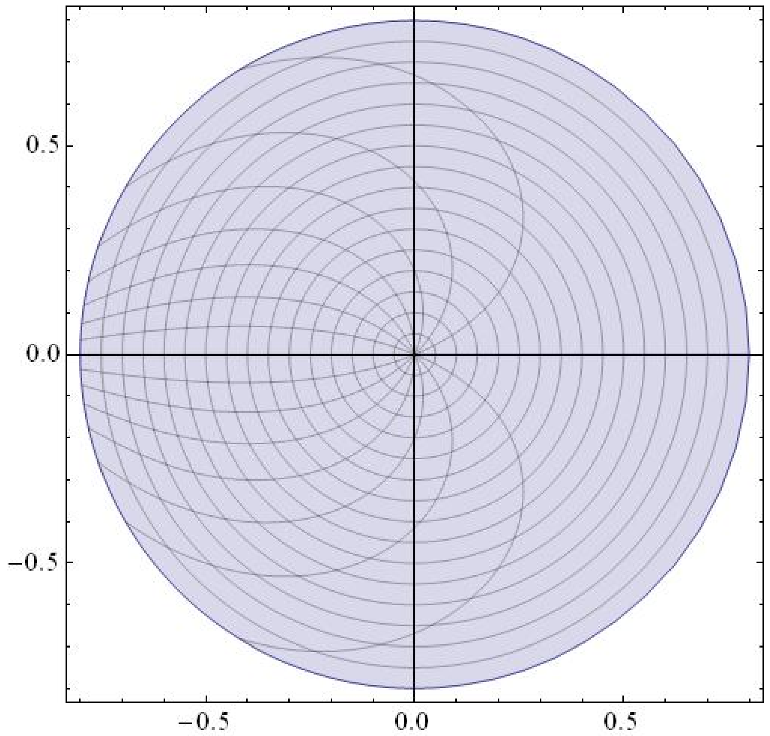

Figure 3. Graph of $f(z) = |z|^{2(p-1)} \dfrac{z(1-\bar{z})}{1-z}$ for $p = 1$

Example 1.4. Let

$$f(z) = -\frac{\log(1-z)}{1-z} = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{1}{k} \right) z^n.$$

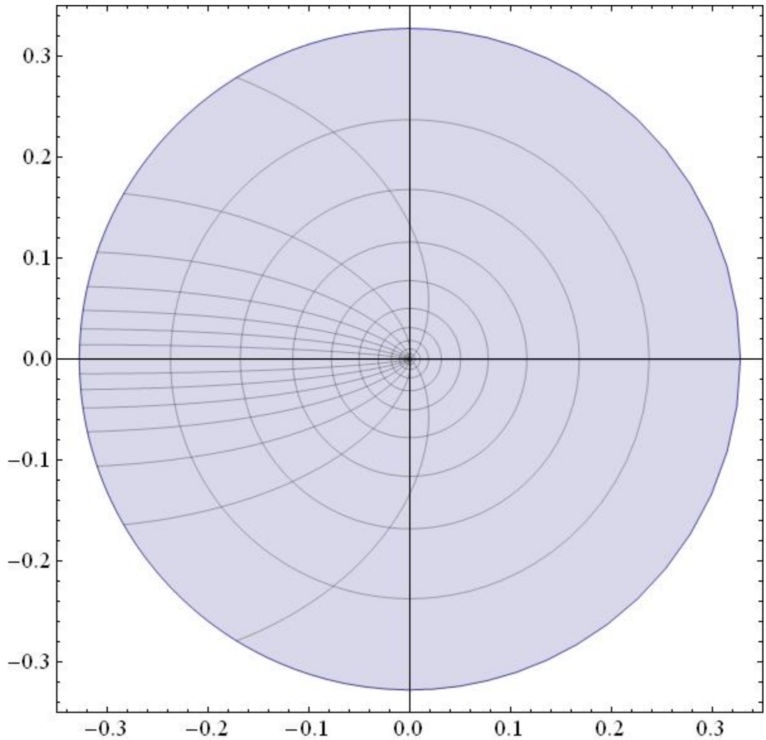

Figure 4. Graph of $f(z) = |z|^{2(p-1)}\dfrac{z(1-\bar{z})}{1-z}$ for $p = 3$

Then we see that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) = \operatorname{Re}\left(\frac{z}{1-z} - \frac{z}{(1-z)\log(1-z)}\right) > -\frac{1}{2} + \frac{1}{2\log 2} > 0, \quad z \in \mathbb{D},$$

and thus f is analytic and starlike (univalent) in $\mathbb{D}$. According to Corollary 1.1,

$$F(z) = |z|^{2(p-1)} f(z) = -|z|^{2(p-1)}\frac{\log(1-z)}{1-z}$$

is starlike (univalent) in $\mathbb{D}$. The mapping properties of $F(z)$ for the values of $p = 1,2,3$ are shown in Figures 7–9.

1.6. log-p-harmonic convex mappings

A continuous function $f\colon \mathbb{D} \to \mathbb{C}$ is called convex in $\mathbb{D}$ if it is univalent and the range $f(\mathbb{D})$ is a convex domain.

Next, we consider the starlikeness of log-p-harmonic mappings which is a generalization of the corresponding result in [3] where starlikeness of log-biharmonic mappings was discussed.

Definition 1.2. *We say that a univalent* log-p-*harmonic mapping* F *with* $F(0) = 0$ *and* $\frac{\partial F(re^{it})}{\partial t} \neq 0$ *whenever* $0 < r < 1$, *is said to be fully convex if the curve* $F(re^{it})$ *is convex for*

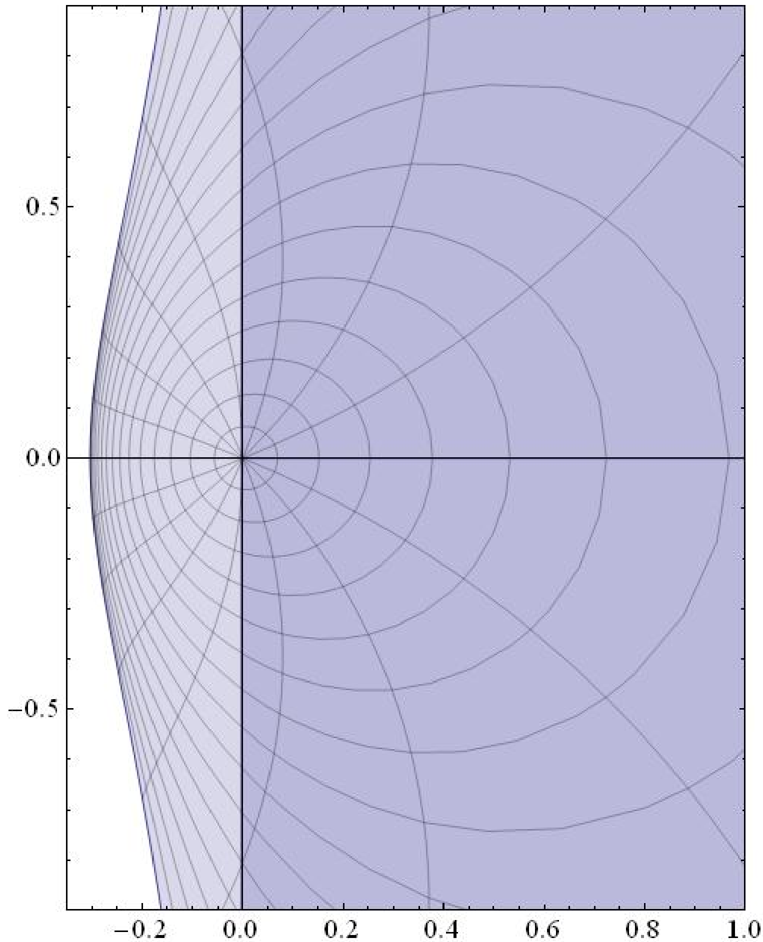

Figure 5. Graph of $f(z) = |z|^{2(p-1)}\left(\frac{z}{1-z}\right)e^{\frac{1}{2}\bar{z}}$ for $p = 1$

each $r \in (0,1)$. *In other words,*

$$\frac{\partial}{\partial t}\left(\arg \frac{\partial}{\partial t} F(re^{it})\right) = Re\,\frac{zF_z(z) + \bar{z}F_{\bar{z}}(z) - 2|z|^2 F_{z\bar{z}}(z) + z^2 F_{zz}(z) + \bar{z}^2 F_{\bar{z}\bar{z}}(z)}{zF_z(z) - \bar{z}F_{\bar{z}}(z)} > 0$$

for $z = re^{it} \in \mathbb{D}\backslash\{0\}$ *(see also [10] in order to distinguish convexity in the analytic and the harmonic cases).*

Thus, a mapping F is log-p-harmonic convex if and only if the mapping DF is log-p-harmonic starlike.

Theorem 1.5. Let F be a log-p-harmonic mapping of $\mathbb{D}$. Suppose F has the form $F(z) = G(z)^{|z|^{2(p-1)}}$, where G is a nonvanishing logharmonic mapping and $G(0) = 1$. If $\log G(z)$ is a starlike mapping, then $\log F(z)$ is starlike and univalent.

We remark that Theorem 1.5 is a generalization of [3, Lemma 2].

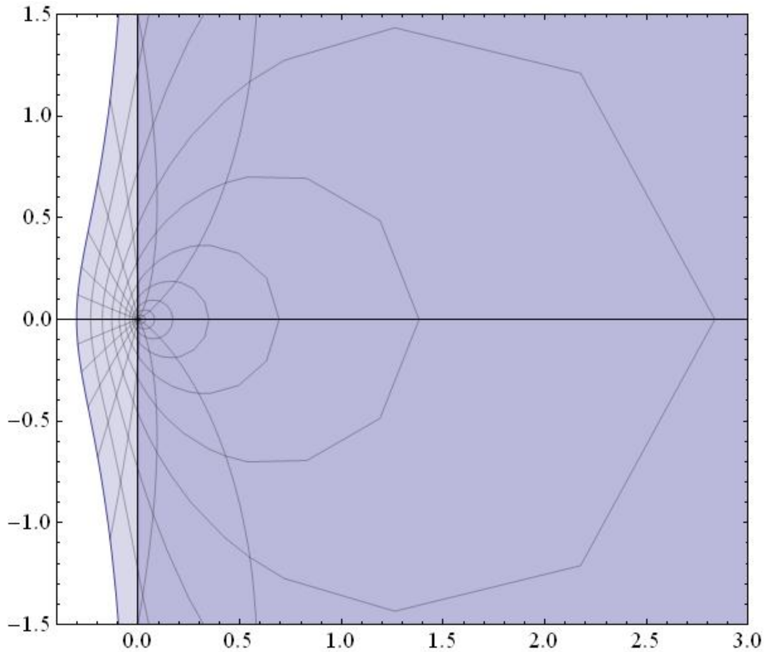

Figure 6. Graph of $f(z)=|z|^{2(p-1)}\left(\frac{z}{1-z}\right)e^{\frac{1}{2}\bar{z}}$ for $p=3$

2. Composition of p-harmonic mappings with analytic functions

2.1. Proof of Theorem 1.1

It suffices to prove the necessity since the proof of the sufficiency is obvious. Let $H=F\circ f$, where f is analytic and F is p-harmonic. In particular, we set $F(z)=|z|^{2(p-1)}$. Then we see that H is p-harmonic, $H=f^{p-1}\overline{f}^{p-1}$ and

$$\Delta^p H(z)=4^p\frac{\partial^p f(z)^{p-1}}{\partial z^p}\frac{\partial^p \overline{f}(z)^{p-1}}{\partial \overline{z}^p}=0,$$

which yields

$$\frac{\partial^p f(z)^{p-1}}{\partial z^p}=0.$$

Thus, $f(z)^{p-1}$ must be of the form

$$f(z)^{p-1}=a_0+a_1z+\cdots+a_{p-1}z^{p-1},$$

which implies that f must be linear. ∎

2.2. Proof of Theorem 1.2

Again, it suffices to prove the necessity since the proof of the sufficiency is obvious. As in the proof of Theorem 1.1, assume that f is analytic and let $F(z)=|z|^{2(p-1)}$. Then $H=$

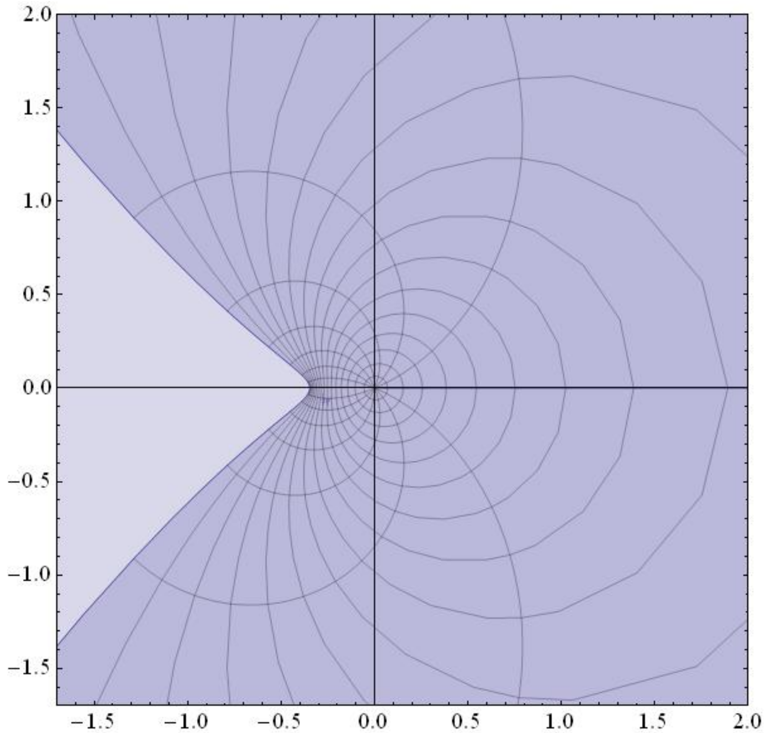

Figure 7. Graph of $f(z) = -|z|^{2(p-1)} \dfrac{\log(1-z)}{1-z}$ for $p = 1$

$F \circ f = f^{p-1}\overline{f}^{p-1}$ is q-harmonic and

$$\Delta^q H(z) = 4^q \frac{\partial^q f(z)^{p-1}}{\partial z^q} \frac{\partial^q \overline{f}(z)^{p-1}}{\partial \overline{z}^q} = 0,$$

which yields

$$\frac{\partial^q f(z)^{p-1}}{\partial z^q} = 0.$$

It follows that

$$f(z)^{p-1} = a_0 + a_1 z + \cdots + a_{q-1} z^{q-1},$$

which implies that f must be a constant, since $q \leq p-1$. ∎

3. Local univalence

3.1. Proof of Theorem 1.3

It suffices to prove the theorem for the case $p > 1$. Consider $F(z) = |z|^{2(p-1)} G(z) + K(z)$. It follows that

$$F_z(z) = (p-1) z^{p-2} (\overline{z})^{p-1} G(z) + |z|^{2(p-1)} G_z(z) + K_z(z)$$

which may be rewritten as

$$F_z = |z|^{2(p-2)} \overline{z} [(p-1)G + zG_z] + K_z.$$

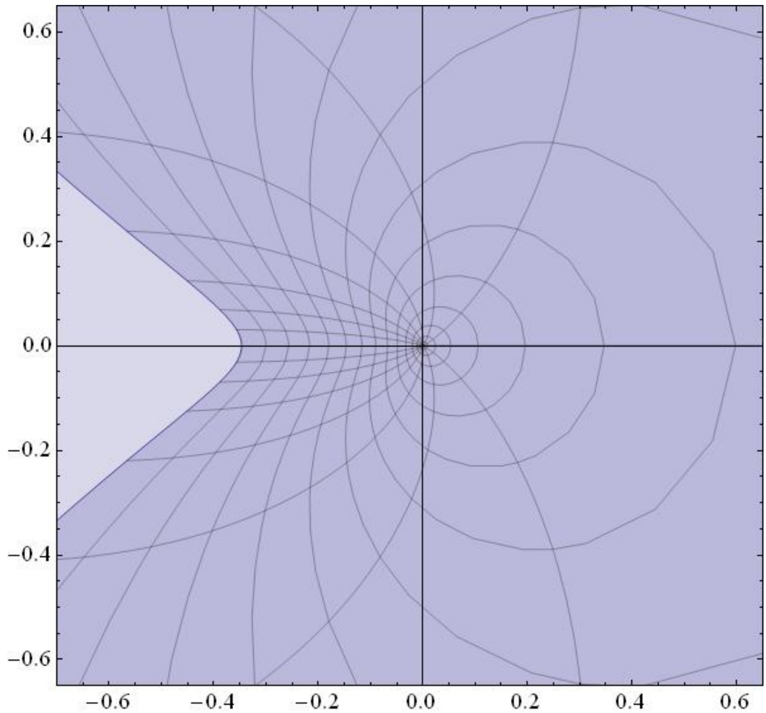

Figure 8. Graph of $f(z) = -|z|^{2(p-1)}\dfrac{\log(1-z)}{1-z}$ for $p = 2$

Similarly, we have

$$F_{\overline{z}} = |z|^{2(p-2)}z[(p-1)G + \overline{z}G_{\overline{z}}] + K_{\overline{z}}.$$

Using the last two equations, we see that

$$\begin{aligned} J_F &= |F_z|^2 - |F_{\overline{z}}|^2 \\ &= |z|^{2(2p-3)}\{|(p-1)G + zG_z|^2 - |(p-1)G + \overline{z}G_{\overline{z}}|^2\} + J_K \\ &\quad +2|z|^{2(p-2)}\mathrm{Re}\left(\overline{z}[(p-1)G + zG_z]\overline{K_z} - z[(p-1)G + \overline{z}G_{\overline{z}}]\overline{K_{\overline{z}}}\right) \\ &= |z|^{2(2p-3)}\{2(p-1)\mathrm{Re}\left[\overline{G}(zG_z - \overline{z}G_{\overline{z}})\right] + |z|^2 J_G\} + J_K \\ &\quad +2|z|^{2(p-2)}\left[(p-1)\mathrm{Re}\left[\overline{G}(zK_z - \overline{z}K_{\overline{z}})\right] + |z|^2\mathrm{Re}\left[G_z\overline{K_z} - G_{\overline{z}}\overline{K_{\overline{z}}}\right]\right] \\ &= |z|^{2(2p-3)}\left\{2(p-1)|G|^2\mathrm{Re}\left(\frac{zG_z - \overline{z}G_{\overline{z}}}{G}\right) + |z|^2 J_G\right\} + J_K \\ &\quad +2|z|^{2(p-2)}\left[(p-1)\mathrm{Re}\left\{\overline{G}(zK_z - \overline{z}K_{\overline{z}})\right\} + |z|^2\mathrm{Re}\left\{G_z\overline{K_z} - G_{\overline{z}}\overline{K_{\overline{z}}}\right\}\right]. \end{aligned}$$

The hypotheses "G being starlike" and "K being sense-preserving" imply that for $z \in \mathbb{D}\backslash\{0\}$,

$$\mathrm{Re}\left(\frac{zG_z - \overline{z}G_{\overline{z}}}{G}\right) > 0,\ J_G > 0,\ \text{ and }\ J_K > 0.$$

Using (1.3), we deduce that $J_F(z) > 0$ for $z \neq 0$. Since $J_F(0) = J_K(0)$, we see that $J_F > 0$ in $\mathbb{D}$ which implies that F is locally univalent. ∎

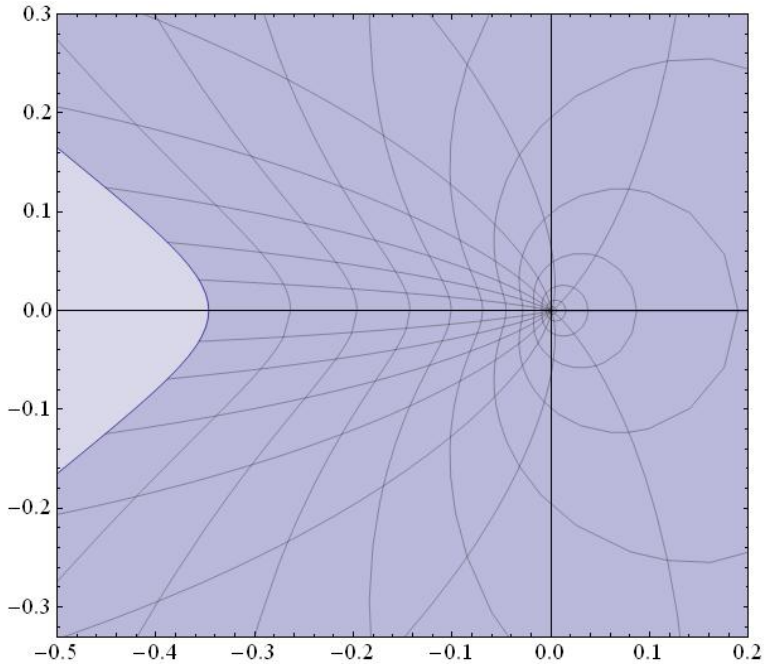

Figure 9. Graph of $f(z) = -|z|^{2(p-1)}\dfrac{\log(1-z)}{1-z}$ for $p = 3$

3.2. Proof of Corollary 1.1

Proceeding exactly as in the proof of Theorem 1.3 with $K = 0$ (so that $K_z = K_{\overline{z}} = 0$), the last formula in the proof of Theorem 1.3 for J_F takes the form

$$J_F = 2(p-1)|z|^{2(2p-3)}|G|^2\mathrm{Re}\left(\frac{zG_z - \overline{z}G_{\overline{z}}}{G}\right) + |z|^{4(p-1)}J_G.$$

Since G is orientation preserving and starlike in $\mathbb{D}$ and $p > 1$, it follows that $F(z) \neq 0$ in $\mathbb{D}\setminus\{0\}$ and the last relation gives $J_F(z) > 0$ for $z \in \mathbb{D}\setminus\{0\}$. Moreover, as

$$zF_z = |z|^{2(p-1)}[(p-1)G + zG_z] \quad \text{and} \quad \overline{z}F_{\overline{z}} = |z|^{2(p-1)}[(p-1)G + \overline{z}G_{\overline{z}}],$$

it follows that

$$\frac{zF_z - \overline{z}F_{\overline{z}}}{F} = \frac{zG_z - \overline{z}G_{\overline{z}}}{G}$$

and therefore,

$$\mathrm{Re}\left(\frac{DF(z)}{F(z)}\right) > 0 \quad \text{for all } z \in \mathbb{D}\setminus\{0\}.$$

Hence, F is starlike in $\mathbb{D}$. ∎

4. Starlikeness

Lemma 4.1. Let F be a log-p-harmonic mapping of $\mathbb{D}$ of the form $F(z)=G(z)^{|z|^{2(p-1)}}$, where G is a nonvanishing logharmonic mapping and $G(0)=1$. If $\log G$ is starlike in $\mathbb{D}$, then $J_{\log F}(z)>0$ for $z\in\mathbb{D}\backslash\{0\}$ and $J_{\log F}(0)=0$.

Proof. Obviously, it suffices to consider the case $p>1$. Let $F(z)=G(z)^{|z|^{2(p-1)}}$. Simple calculations give

$$\frac{F_z(z)}{F(z)}=|z|^{2(p-2)}\left[(p-1)\overline{z}\log G(z)+|z|^2\frac{G_z(z)}{G(z)}\right]$$

and similarly

$$\frac{F_{\overline{z}}(z)}{F(z)}=|z|^{2(p-2)}\left[(p-1)z\log G(z)+|z|^2\frac{G_{\overline{z}}(z)}{G(z)}\right].$$

Using the last two relations, it follows that

$$\begin{aligned} J_F &= |F_z|^2-|F_{\overline{z}}|^2\\ &= |F|^2|z|^{4(p-2)}\left[\left|(p-1)\overline{z}\log G+|z|^2\frac{G_z}{G}\right|^2-\left|(p-1)z\log G+|z|^2\frac{G_{\overline{z}}}{G}\right|^2\right]\\ &= |F|^2|z|^{4(p-2)}\left[\frac{|z|^4}{|G|^2}J_G+2(p-1)|z|^2\mathrm{Re}\left(z\overline{\log G}\frac{G_z}{G}-\overline{z}\overline{\log G}\frac{G_{\overline{z}}}{G}\right)\right]\\ &= |F|^2|z|^{4(p-2)}\left[\frac{|z|^4}{|G|^2}J_G+2(p-1)|z|^2|\log G|^2\mathrm{Re}\left(\frac{zG_z-\overline{z}G_{\overline{z}}}{G\log G}\right)\right]. \end{aligned}$$

Since G is sense-preserving and $\log G$ is starlike, we deduce that

$$J_{\log F}(z)=\left|\frac{F_z}{F}\right|^2-\left|\frac{F_{\overline{z}}}{F}\right|^2>0$$

for $z\in\mathbb{D}\backslash\{0\}$ and obviously, $J_{\log F}(0)=0$. ∎

4.1. Proof of Theorem 1.5

Obviously, it suffices to consider the case $p>1$. Let

$$F^*(z)=|z|^{2(p-1)}g(z),$$

where $g=\log G$.

Since g is starlike, it follows that g and F^* are zero only at $z=0$ and, in addition, Definition 1.1 and argument principle imply that g is univalent in $\mathbb{D}$. Elementary calculations yield

$$\mathrm{Re}\left(\frac{zF^*_z-\overline{z}F^*_{\overline{z}}}{F^*}\right)=\mathrm{Re}\left(\frac{zg_z-\overline{z}g_{\overline{z}}}{g}\right)>0$$

when $z\neq 0$. This shows the starlikeness of F^*. Since by Lemma 4.1, $J_{F^*}(z)>0$ for $z\in\mathbb{D}\backslash\{0\}$ and $J_{F^*}(0)=0$, we see that F^* is univalent on each $|z|=r$ for $r\in(0,1)$.

In order to prove the univalence of F^*, suppose that there are two distinct points z_1, $z_2\in\mathbb{D}$ such that $F^*(z_1)=F^*(z_2)$. Then $|z_1|\neq|z_2|$. Without loss of generality, we assume that $|z_1|<|z_2|$. Then

$$\frac{g(z_1)}{g(z_2)}=\frac{|z_2|^{2(p-1)}}{|z_1|^{2(p-1)}}>1,$$

which implies $g(\mathbb{D}_{|z_2|}) \subset g(\mathbb{D}_{|z_1|})$. This is the desired contradiction. The arbitrariness of z_1 and z_2 shows that $\log F$ is univalent in $\mathbb{D}$. ∎

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