Approximation of Common Fixed Points of Family of Asymptotically Nonexpansive Mappings

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Abstract. In this paper, we introduce *a new iteration process* for approximation of common fixed point of countably infinite family of *nonself* asymptotically nonexpansive mappings in uniformly convex Banach spaces, and prove weak and strong convergence of our iteration process to a common fixed point of these operators. Our theorems extend, generalize and unify many recently announced results. Our iteration process, corollaries and methods of proof are of independent interest.

1. Introduction

Let K be a nonempty subset of a real normed linear space E. A self-mapping $T : K \to K$ is called *asymptotically nonexpansive* if there exists a sequence $\{\mu_n\}_{n\geq 1} \subset [1,\infty)$, $\lim_{n\to\infty} \mu_n = 1$ such that for all $x,y \in K$

$$\|T^n x - T^n y\| \leq \mu_n \|x - y\|, \quad \text{for all} \quad n \in \mathbb{N}. \tag{1.1}$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [10] as an important generalization of the class of nonexpansive mappings, where a self mapping $T : K \to K$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \text{for all} \quad x,y \in K. \tag{1.2}$$

Goebel and Kirk [10] proved that if K is a nonempty bounded closed convex subset of a real uniformly convex Banach space and T is *self* asymptotically nonexpansive mapping of K, then T has a fixed point.

Iterative methods for approximating fixed points of self asymptotically nonexpansive mappings have been studied by various authors using the *Mann iteration scheme* (see e.g., [15]) or the *Ishikawa iteration scheme* (see e.g., [11]).

Bose [3], proved that if K is a bounded closed convex nonempty subset of a uniformly convex Banach space E satisfying *Opial's condition* (that is, for all sequences $\{x_n\}_{n\geq 1}$ in E such that $\{x_n\}_{n\geq 1}$ converges weakly to some $x \in E$, the inequality $\liminf_{n\to\infty} \|x_n - y\| > \liminf_{n\to\infty} \|x_n - x\|$ holds for all $y \neq x$; (see e.g., [18]) and $T : K \to K$ is an asymptotically nonexpansive mapping, then the sequence $\{T^n x\}_{n\geq 1}$ converges weakly to a fixed point of T provided that T is *asymptotically regular* at $x \in K$, that is, $\lim_{n\to\infty} \|T^n x - T^{n+1} x\| = 0$. Pasty [21] and also Xu [37] proved that the requirement that E satisfies Opial's condition can be replaced by the condition that E has a *Fréchet differentiable norm* (that is, for all $x \in U := \{x \in E : \|x\| = 1\}$, $\lim_{t\to 0} (\|x + ty\| - \|x\|)/t$ exists and is attained uniformly in $y \in U$). Furthermore, Tan and Xu [33,34] later proved that the asymptotic regularity of T at x can be weakened to the weak asymptotic regularity of T at x, that is, $w - \lim_{n\to\infty}(T^n x - T^{n+1} x) = 0$.

In [23, 24], Schu introduced a modified Mann iteration process to approximate fixed points of *self* asymptotically nonexpansive mappings. More precisely, he proved the following theorems:

Theorem 1.1. [23, Theorem 1.5, p. 409] *Let H be a Hilbert space, K a bounded closed convex and nonempty subset of H. Let $T : K \to K$ be a completely continuous asymptotically nonexpansive mapping with sequence $\{\mu_n\}_{n\geq 1} \subset [1, +\infty)$; $\lim_{n\to\infty} \mu_n = 1$, and $\sum_{n=1}^{\infty}(\mu_n^2 - 1) < \infty$. Let $\{\alpha_n\}_{n\geq 1}$ be a real sequence in $[0, 1]$ satisfying the condition $\varepsilon \leq \alpha_n \leq 1 - \varepsilon$ for all $n \geq 1$ and for some $\varepsilon > 0$. Then, the sequence $\{x_n\}$ generated from arbitrary $x_1 \in K$ by $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n$, $n \geq 1$, converges strongly to some fixed point of T.*

Theorem 1.2. [24, Theorem 2.1, p.156] *Let E be a uniformly convex Banach space satisfying Opial's condition and let K be a bounded closed convex and nonempty subset of E. Let $T : K \to K$ be an asymptotically nonexpansive mapping with sequence $\{\mu_n\}_{n\geq 1} \subset [1, +\infty)$, $\lim_{n\to\infty} \mu_n = 1$, and $\sum_{n=1}^{\infty}(\mu_n^2 - 1) < \infty$. Let $\{\alpha_n\}_{n\geq 1}$ be a real sequence in $[0, 1]$ satisfying the condition $0 < a \leq \alpha_n \leq b < 1$, for all $n \geq 1$, for some constants a, b. Then, the sequence $\{x_n\}$ generated from arbitrary $x_1 \in K$ by $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n$, $n \geq 1$, converges weakly to some fixed point of T.*

In [22], Rhoades extended Theorem 1.1 to uniformly convex Banach space using a modified Ishikawa iteration method. Osilike and Aniagbosor [19], proved that the theorems of Schu and Rhoades remain true without the boundedness condition imposed on K, provided that $F(T) = \{x \in K : Tx = x\} \neq \emptyset$.

Existence theorems for common fixed points of certain families of nonlinear mappings have been established by various authors (see e.g., [2,4,5,13,14]).

Within the past 30 years or so, research on iterative approximation of common fixed points of families of nonexpansive mappings and generalizations of nonlinear nonexpansive mappings surged. Considerable research efforts have been devoted to developing iterative methods for approximating common fixed points (when they exist) of finite families of these class of mappings (see e.g., [1,6,9,12,17,26,28,29] and references therein).

Using an Ishikawa-type scheme [11], Takahashi and Tamura [32] proved strong and weak convergence of a sequence defined by

$$(1.3) \qquad x_{n+1} = \alpha_n S(\beta_n T x_n + (1 - \beta_n)x_n) + (1 - \alpha_n)x_n$$

to a common fixed point of a pair of nonexpansive mappings T and S. In [25], Shahzad and Udomene established necessary and sufficient conditions for convergence of Ishikawa-type iteration sequences involving two asymptotically quasi-nonexpansive mappings to a

common fixed point of the mappings in arbitrary real Banach spaces. They also established a sufficient condition for the convergence of the Ishikawa-type iteration sequences involving two uniformly continuous asymptotically quasi-nonexpansive mappings to a common fixed point of the mappings in real uniformly convex Banach spaces.

Let K be a nonempty closed convex subset of a Hilbert space H and let $\{T_n\}_{n \geq 1}$ be countably infinite family of asymptotically nonexpansive mappings of K into itself and let $\{\beta_{n,j} : n, j \in \mathbb{N}, 1 \leq j \leq n\}$ be a sequence of real numbers such that $0 \leq \beta_{n,j} \leq 1$ for $n, j \in \mathbb{N}$ with $n \geq j$. Motivated by the W-mapping introduced in [27, 30, 31] for nonexpansive mappings, Nakajo *et al.* [16] introduced (for each $n \in \mathbb{N}$) the following mapping of K into itself:

$$U_{n,n} = \beta_{n,n}T_n^n + (1 - \beta_{n,n})I,$$
$$U_{n,n-1} = \beta_{n,n-1}T_{n-1}^n U_{n,n} + (1 - \beta_{n,n-1})I,$$
$$\vdots$$
$$U_{n,j} = \beta_{n,j}T_j^n U_{n,j+1} + (1 - \beta_{n,j})I,$$
$$\vdots$$
$$U_{n,2} = \beta_{n,2}T_2^n U_{n,3} + (1 - \beta_{n,2})I,$$
$$\text{(1.4)} \qquad W_n = U_{n,1} = \beta_{n,1}T_1^n U_{n,2} + (1 - \beta_{n,1})I.$$

Such a mapping W_n is called the modified W-mapping generated by $T_n, T_{n-1}, ..., T_2, T_1$ and $\beta_{n,n}, \beta_{n,n-1}, ..., \beta_{n,2}, \beta_{n,1}$.

Very recently, Nakajo *et al.* [16] obtained strong convergence theorem for approximation of common fixed point of countably infinite family of asymptotically nonexpansive self-mappings in the frame work of Hilbert spaces by the *so called hybrid method* using the modified W-mapping.

In all the above results, the operator T remains a *self-mapping* of a nonempty closed convex subset K of either Hilbert space or uniformly convex Banach space. If, however, the domain, $D(T)$, of T is a *proper subset* of E and T maps $D(T)$ into E (which is the case in several applications), then the iteration processes studied by these authors, and their modifications may fail to be well defined.

In [8], Chidume, Ofoedu and Zegeye introduced the class of *nonself* asymptotically non-expansive mappings.

A subset K of E is said to be a *retract* of E if there exists a continuous map $P : E \rightarrow K$ such that $Px = x$ for all $x \in K$. Every closed convex nonempty subset of a uniformly convex Banach space is a retract. A map $P : E \rightarrow E$ is said to be a *retraction* if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P.

The following definition was given in [8].

Definition 1.1. *Let E be a real normed linear space, K a nonempty subset of E. Let $P : E \rightarrow K$ be the nonexpansive retraction of E onto K. A nonself map $T : K \rightarrow E$ is said to be* asymptotically nonexpansive *if there exists a sequence $\{\mu_n\}_{n \geq 1} \subset [1, +\infty)$, $\mu_n \rightarrow 1$ as $n \rightarrow \infty$, such that the following inequality holds:*

$$\text{(1.5)} \qquad \|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq \mu_n \|x - y\|, \quad \forall x, y \in K, n \in \mathbb{N}.$$

The operator T is called uniformly *L*-Lipschitzian *if there exists $L > 0$ such that*

$$(1.6) \qquad \|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|, \quad \forall x, y \in K, \, n \in \mathbb{N}.$$

Let K be a closed convex nonempty subset of a real uniformly convex Banach space E, the following iteration scheme was introduced and studied in [8]:

$$(1.7) \qquad x_1 \in K, \quad x_{n+1} = P\left((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n\right), \quad n \geq 1,$$

where $\{\alpha_n\}_{n \geq 1}$ is a sequence in $(0, 1)$ satisfying appropriate conditions.

Observe that if T is a self-mapping, the operator P becomes identity mapping so that (1.5) coincides with (1.1). Moreover, (1.7) reduces to a Mann-type iteration scheme.

Wang [36] used a scheme similar to (1.3) and the definition of Chidume *et al.* [8] to prove strong and weak convergence theorems for a pair of *nonself asymptotically nonexpansive mappings*. More precisely, he proved the following theorems.

Theorem 1.3. [36] *Let K be a nonempty closed convex subset of uniformly convex Banach space E. Suppose $T_1, T_2 : K \to E$ are two nonself asymptotically nonexpansive mappings with sequences $\{\mu_n\}_{n \geq 1}$ and $\{l_n\}_{n \geq 1} \subset [1, +\infty)$ such that $\sum_{n=1}^{\infty}(\mu_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$. Let $\{x_n\}_{n \geq 1}$ be generated by*

$$(1.8) \qquad \begin{cases} x_1 \in K, \\ x_{n+1} = P\left((1 - \alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n\right) \\ y_n = P\left((1 - \alpha_n)x_n + \alpha_n T_2(PT_2)^{n-1}x_n\right), \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[\varepsilon, 1 - \varepsilon]$ for some $\varepsilon \in (0, 1)$. If one of T_1 and T_2 is completely continuous, and $F(T_1) \cap F(T_2) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.

Theorem 1.4. [36] *Let K, T_1, T_2, $\{\mu_n\}$, $\{l_n\}$ and $\{x_n\}$ be as in Theorem 1.3. If one of T_1 and T_2 is semicompact, and $F(T_1) \cap F(T_2) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.*

Theorem 1.5. [36] *Let K, T_1, T_2, $\{\mu_n\}$, $\{l_n\}$ and $\{x_n\}$ be as in Theorem 1.3. If E satisfies Opial condition, and $F(T_1) \cap F(T_2) \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2.*

It is our aim in this paper to introduce *a new iteration process* for approximation of a common fixed point of countably infinite family of *nonself* asymptotically nonexpansive mappings in uniformly convex Banach spaces; and prove weak and strong convergence of our iteration process to a common fixed point of these mappings. Our theorems extend, generalize and unify many recently announced results, especially those of the authors mentioned above. Our iteration process, corollaries and methods of proof are of independent interest.

2. Preliminaries

Let E be a real normed linear space. The *modulus of convexity* of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$(2.1) \qquad \delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : \|x\| = \|y\| = 1, \quad \varepsilon = \|x - y\|\right\}.$$

The space E is called *uniformly convex* if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be *demiclosed* at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $x_n \rightharpoonup x \in D(T)$ and $Tx_n \to p$ then $Tx = p$.

A mapping $T : D(T) \subseteq E \to E$ is said to be *semicompact* if, for any bounded sequence $\{x_n\}_{n\geq 1}$ in $D(T)$ such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that $\{x_{n_k}\}_{k\geq 1}$ converges strongly to some $x^* \in D(T)$.

An operator T is said to be *completely continuous* if for every bounded sequence $\{x_n\}_{n\geq 1}$ in the domain, $D(T)$, of T, there exists a subsequence $\{x_{n_j}\}_{j\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that the sequence $\{Tx_{n_j}\}_{j\geq 1}$ converges to some element of the range, $R(T)$, of T.

In the sequel, we shall make use of the following lemmas.

Lemma 2.1. *(see e.g., [35]) Let $\{\lambda_n\}_{n\geq 1}$ and $\{\sigma_n\}_{n\geq 1}$ be a sequence of nonnegative real numbers such that $\lambda_{n+1} \leq \lambda_n + \sigma_n$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$; then $\lim_{n\to\infty} \lambda_n$ exists. Moreover, if there exists a subsequence $\{\lambda_{n_j}\}_{j\geq 1}$ of $\{\lambda_n\}_{n\geq 1}$ such that $\lambda_{n_j} \to 0$ as $j \to \infty$, then $\lambda_n \to 0$ as $n \to \infty$.*

Lemma 2.2. *(see e.g., [37]) Let E be a real uniformly convex Banach space. Let $r > 0$, then there exists a continuous strictly increasing function $g : [0, +\infty) \to [0, +\infty)$, $g(0) = 0$ such that for all $x, y \in B_r(0)$, the following inequalities hold:*

$$(2.2) \qquad \|\alpha x + (1-\alpha)y\|^2 \leq \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)g(\|x-y\|).$$

Lemma 2.3. ([8, Theorem 3.4], Demiclosedness principle for nonself-maps). *Let E be a uniformly convex Banach space, K a nonempty closed convex nonexpansive retract of E, and let $T : K \to E$ be a nonself asymptotically nonexpansive mapping with a sequence $\{\mu_n\}_{n\geq 1} \subset [1, +\infty)$ and $\mu_n \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed at zero.*

3. Main results

In this section, we give new definitions and prove our main theorems. We start as follows:

Let K be a nonempty closed convex nonexpansive retract of a real normed linear space E with P as nonexpansive retraction. Let $T_m : K \to E$, $m = 1, 2, ...$ be a countably infinite family of nonself asymptotically nonexpansive mappings and $\{\beta_{n,m}\}_{n,m\in\mathbb{N}} \subset (0, 1)$. Then, for the family $\{T_m\}_{m\geq 1}$, the modified W-mapping generated by $T_n, T_{n-1}, ..., T_2, T_1$ and $\beta_{n,n}, \beta_{n,n-1}, ..., \beta_{n,2}, \beta_{n,1}$ for all $n \in \mathbb{N}$ is given by

$$U_{n,n} = P\left(\beta_{n,n}T_n(PT_n)^{n-1} + (1 - \beta_{n,n})I\right),$$

$$U_{n,n-1} = P\left(\beta_{n,n-1}T_{n-1}(PT_{n-1})^{n-1}U_{n,n} + (1 - \beta_{n,n-1})I\right),$$

$$\vdots$$

$$U_{n,j} = P\left(\beta_{n,j}T_j(PT_j)^{n-1}U_{n,j+1} + (1 - \beta_{n,j})I\right),$$

$$\vdots$$

$$U_{n,2} = P\left(\beta_{n,2}T_2(PT_2)^{n-1}U_{n,3} + (1 - \beta_{n,2})I\right),$$

$$(3.1) \qquad W_n = U_{n,1} = P\left(\beta_{n,1}T_1(PT_1)^{n-1}U_{n,2} + (1 - \beta_{n,1})I\right).$$

Observe that in the case of self mappings, P becomes the identity map so that in this case, (1.4) and (3.1) coincide.

Lemma 3.1. *Let K be a nonempty closed convex nonexpansive retract of a uniformly convex real Banach space E with P as a nonexpansive retraction. Let $\{T_m\}_{m\geq 1}$ be a countably infinite family of nonself asymptotically nonexpansive mappings of K into E with sequence $\{\mu_{m,n}\} \subset [1,+\infty)$, $\lim_{n\to\infty}\mu_{m,n}=1$ such that $F = \bigcap_{m=1}^{\infty}F(T_m) \neq \emptyset$ and $\{\beta_{n,j} : n,j \in \mathbb{N}, 1 \leq j \leq n\}$ be a sequence of real numbers such that $0 < \varepsilon \leq \beta_{n,j} \leq (1-\varepsilon) < 1$, for some $\varepsilon \in (0,1)$. Let W_n be the modified W-mapping generated by $T_n, T_{n-1}, ..., T_2, T_1$ and $\beta_{n,n}, \beta_{n,n-1}, ..., \beta_{n,2}, \beta_{n,1}$ as in (3.1). Let*

(3.2)
$$\begin{aligned}
\gamma_{n,j} = {} & \beta_{n,j}(\mu_{j,n}^2-1) + \beta_{n,j}\beta_{n,j+1}\mu_{j,n}^2(\mu_{j+1,n}^2-1) \\
& + ... + \beta_{n,j}\beta_{n,j+1}...\beta_{n,n-1}\mu_{j,n}^2\mu_{j+1,n}^2\cdots\mu_{n-2,n}^2(\mu_{n-1,n}^2-1) \\
& + \beta_{n,j}\beta_{n,j+1}...\beta_{n,n}\mu_{j,n}^2\mu_{j+1,n}^2\cdots\mu_{n-1,n}^2(\mu_{n,n}^2-1),
\end{aligned}$$

for all $n \in \mathbb{N}$ and $j = 1,2,...,n$. Then, the following hold:

(1) *$\|W_n x - W_n y\|^2 \leq (1+\gamma_{n,1})\|x-y\|^2$ for all $n \in \mathbb{N}$, $x,y \in K$;*

(2) *if $\lim_{n\to\infty}\gamma_{n,1} = 0$, then for every bounded sequence $\{z_n\}$ in K such that*

$$\lim_{n\to\infty}\|z_{n+1}-z_n\| = 0 = \lim_{n\to\infty}\|W_n z_n - z_n\|,$$

we have $\lim_{n\to\infty}\|z_n - T_m(PT_m)^{n-1}z_n\| = 0 = \lim_{n\to\infty}\|z_n - T_m z_n\|$ for all $m \in \mathbb{N}$; moreover, we have $\omega_\omega(x_n) \subset F$;

(3) *if $\lim_{n\to\infty}\gamma_{n,1} = 0$, then $F = \bigcap_{n=1}^{\infty}F(W_n)$.*

Proof. (1) For $x,y \in K$, we obtain from (3.1) that

$$\begin{aligned}
\|U_{n,n}x - U_{n,n}y\|^2 &\leq \beta_{n,n}\|T_n(PT_n)^{n-1}x - T_n(PT_n)^{n-1}y\|^2 + (1-\beta_{n,n})\|x-y\|^2 \\
&\leq \beta_{n,n}\mu_{n,n}^2\|x-y\|^2 + (1-\beta_{n,n})\|x-y\|^2 \\
&= \|x-y\|^2 + \beta_{n,n}(\mu_{n,n}^2-1)\|x-y\|^2,
\end{aligned}$$
(3.3)

which implies that

$$\begin{aligned}
&\|U_{n,n-1}x - U_{n,n-1}y\|^2 \\
&\leq \beta_{n,n-1}\mu_{n-1,n}^2\|U_{n,n}x - U_{n,n}y\|^2 + (1-\beta_{n,n-1})\|x-y\|^2 \\
&\leq \beta_{n,n-1}\mu_{n-1,n}^2\left(\|x-y\|^2 + \beta_{n,n}(\mu_{n,n}^2-1)\|x-y\|^2\right) + (1-\beta_{n,n-1})\|x-y\|^2 \\
&= \|x-y\|^2 + \left(\beta_{n,n-1}(\mu_{n-1,n}^2-1) + \beta_{n,n-1}\beta_{n,n}\mu_{n-1,n}^2(\mu_{n,n}^2-1)\right)\|x-y\|^2.
\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}
\|U_{n,k}x - U_{n,k}y\|^2 \leq {} & \|x-y\|^2 + \big(\beta_{n,k}(\mu_{k,n}^2-1) + \beta_{n,k}\beta_{n,k+1}\mu_{k,n}^2(\mu_{k+1,n}^2-1) \\
& + ... + \beta_{n,k}\beta_{n,k+1}...\beta_{n,n-1}\mu_{k,n}^2\mu_{k+1,n}^2\cdots\mu_{n-2,n}^2(\mu_{n-1,n}^2-1) \\
& + \beta_{n,k}\beta_{n,k+1}...\beta_{n,n}\mu_{k,n}^2\mu_{k+1,n}^2\cdots\mu_{n-2,n}^2\mu_{n-1,n}^2(\mu_{n,n}^2-1)\big)\|x-y\|^2 \\
= {} & (1+\gamma_{n,k})\|x-y\|^2
\end{aligned}$$

for $k = 1,2,...,n$. In particular, for $k = 1$, we have that

$$\|W_n x - W_n y\|^2 \leq (1+\gamma_{n,1})\|x-y\|^2.$$

(2) Let $\{z_n\}_{n\geq 1}$ be a bounded sequence in K such that

$$\lim_{n\to\infty}\|z_{n+1}-z_n\| = 0 = \lim_{n\to\infty}\|W_n z_n - z_n\|.$$

Then, for $x^* \in F$, we have (using Lemma 2.2) that

$$
\begin{aligned}
\|W_n z_n - x^*\|^2 &= \left\| P\big(\beta_{n,1} T_1 (PT_1)^{n-1} U_{n,2} z_n + (1-\beta_{n,1})z_n\big) - Px^* \right\|^2 \\
&\leq \left\| \big(\beta_{n,1} T_1 (PT_1)^{n-1} U_{n,2} z_n + (1-\beta_{n,1})z_n\big) - x^* \right\|^2 \\
&\leq \beta_{n,1} \|T_1 (PT_1)^{n-1} U_{n,2} z_n - x^*\|^2 + (1-\beta_{n,1})\|z_n - x^*\|^2 \\
&\quad - \beta_{n,1}(1-\beta_{n,1}) g\big(\|T_1 (PT_1)^{n-1} U_{n,2} z_n - z_n\|\big) \\
&\leq (1+\gamma_{n,1})\|z_n - x^*\|^2 - \varepsilon^2 g\big(\|T_1 (PT_1)^{n-1} U_{n,2} z_n - z_n\|\big).
\end{aligned}
$$

Thus, from this inequality and the hypothesis, we get that

$$
\begin{aligned}
&\varepsilon^2 g\big(\|T_1 (PT_1)^{n-1} U_{n,2} z_n - z_n\|\big) \\
&\leq \|z_n - x^*\|^2 - \|W_n z_n - x^*\|^2 + \gamma_{n,1}\|z_n - x^*\|^2 \\
&= (\|z_n - x^*\| - \|W_n z_n - x^*\|)(\|z_n - x^*\| + \|W_n z_n - x^*\|) + \gamma_{n,1}\|z_n - x^*\|^2 \\
&\leq (\|z_n - W_n z_n\| + \gamma_{n,1})M_0 \to 0 \quad \text{as} \quad n \to \infty
\end{aligned}
$$

for some $M_0 > 0$. Thus, by the properties of g, we obtain that

$$
(3.4) \qquad\qquad \lim_{n\to\infty} \|T_1 (PT_1)^{n-1} U_{n,2} z_n - z_n\| = 0.
$$

Moreover,

$$
\begin{aligned}
\|z_n - x^*\|^2 &\leq \big(\|z_n - T_1 (PT_1)^{n-1} U_{n,2} z_n\| + \|T_1 (PT_1)^{n-1} U_{n,2} z_n - x^*\|\big)^2 \\
&= \|z_n - T_1 (PT_1)^{n-1} U_{n,2} z_n\| \big(\|z_n - T_1 (PT_1)^{n-1} U_{n,2} z_n\| \\
&\quad + 2\|T_1 (PT_1)^{n-1} U_{n,2} z_n - x^*\|\big) + \|T_1 (PT_1)^{n-1} U_{n,2} z_n - x^*\|^2 \\
&\leq \|z_n - T_1 (PT_1)^{n-1} U_{n,2} z_n\| M_1 + \mu_{1,n}^2 \|U_{n,2} z_n - x^*\|^2 \ \text{(for some } M_1 > 0\text{)} \\
&\leq \|z_n - T_1 (PT_1)^{n-1} U_{n,2} z_n\| M_1 + \mu_{1,n}^2 \big[\beta_{n,2}\|T_2 (PT_2)^{n-1} U_{n,3} z_n - x^*\|^2 \\
&\quad + (1-\beta_{n,2})\|z_n - x^*\|^2 - \beta_{n,2}(1-\beta_{n,2}) g\big(\|T_2 (PT_2)^{n-1} U_{n,3} z_n - z_n\|\big)\big] \\
&\leq \|z_n - T_1 (PT_1)^{n-1} U_{n,2} z_n\| M_1 + \mu_{1,n}^2 \big[\beta_{n,2}\mu_{2,n}^2 \|U_{n,3} z_n - x^*\|^2 \\
&\quad + (1-\beta_{n,2})\|z_n - x^*\|^2 - \beta_{n,2}(1-\beta_{n,2}) g\big(\|T_2 (PT_2)^{n-1} U_{n,3} z_n - z_n\|\big)\big] \\
&\leq \|z_n - T_1 (PT_1)^{n-1} U_{n,2} z_n\| M_1 + \mu_{1,n}^2 \big[\beta_{n,2}\mu_{2,n}^2 (1+\gamma_{n,3})\|z_n - x^*\|^2 \\
&\quad + (1-\beta_{n,2})\|z_n - x^*\|^2 - \varepsilon^2 g\big(\|T_2 (PT_2)^{n-1} U_{n,3} z_n - z_n\|\big)\big].
\end{aligned}
$$

Thus, we obtain from this inequality that

$$
\begin{aligned}
&g\big(\|T_2 (PT_2)^{n-1} U_{n,3} z_n - z_n\|\big) \\
&\leq \frac{1}{\mu_{1,n}^2 \varepsilon^2} \big\{ \|z_n - T_1 (PT_1)^{n-1} U_{n,2} z_n\| M_1 + \mu_{1,n}^2 \big[\beta_{n,2}\mu_{2,n}^2 (1+\gamma_{n,3}) \\
&\quad \times \|z_n - x^*\|^2 + (1-\beta_{n,2})\|z_n - x^*\|^2\big] - \|z_n - x^*\|^2 \big\} \\
&\leq \frac{1}{\mu_{1,n}^2 \varepsilon^2} \big\{ \|z_n - T_1 (PT_1)^{n-1} U_{n,2} z_n\| M_1 + \mu_{1,n}^2 \big[\beta_{n,2}\mu_{2,n}^2 (1+\gamma_{n,3}) \\
&\quad \times \|z_n - x^*\|^2 + (1-\beta_{n,2})\mu_{2,n}^2 (1+\gamma_{n,3})\|z_n - x^*\|^2\big] - \|z_n - x^*\|^2 \big\}
\end{aligned}
$$

$$= \frac{1}{\mu_{1,n}^2 \varepsilon^2} \Big\{ \|z_n - T_1 (PT_1)^{n-1} U_{n,2} z_n \| M_1$$

$$+ \mu_{1,n}^2 \mu_{2,n}^2 (1 + \gamma_{n,3}) \|z_n - x^*\|^2 - \|z_n - x^*\|^2 \Big\}$$

(3.5) $$\leq \frac{1}{\mu_{1,n}^2 \varepsilon^2} \Big[\|z_n - T_1 (PT_1)^{n-1} U_{n,2} z_n \| + \big(\mu_{1,n}^2 \mu_{2,n}^2 (1 + \gamma_{n,3}) - 1 \big) \Big] M,$$

for some constant $M > 0$. Thus, from (3.4), the fact that $\lim_{n \to \infty} \gamma_{n,3} = 0; \lim_{n \to \infty} \mu_{1,n} = 1 = \lim_{n \to \infty} \mu_{2,n}$ and the properties of g, we obtain from (3.5) that

(3.6) $$\lim_{n \to \infty} \|T_2 (PT_2)^{n-1} U_{n,3} z_n - z_n \| = 0.$$

Hence, we obtain by induction that

(3.7) $$\lim_{n \to \infty} \|T_m (PT_m)^{n-1} U_{n,m+1} z_n - z_n \| = 0 \quad \text{for all} \quad m \in \mathbb{N}.$$

Now, since

$$\|z_n - T_m (PT_m)^{n-1} z_n \|$$

$$\leq \|z_n - T_m (PT_m)^{n-1} U_{n,m+1} z_n \| + \|T_m (PT_m)^{n-1} U_{n,m+1} z_n - T_m (PT_m)^{n-1} z_n \|$$

$$\leq \|z_n - T_m (PT_m)^{n-1} U_{n,m+1} z_n \| + \mu_{m,n} \|U_{n,m+1} z_n - z_n \|$$

$$\leq \|z_n - T_m (PT_m)^{n-1} U_{n,m+1} z_n \| + \mu_{m,n} \beta_{n,m+1} \|T_{m+1} (PT_{m+1})^{n-1} U_{n,m+2} z_n - z_n \|$$

(3.8) $$\leq \|z_n - T_m (PT_m)^{n-1} U_{n,m+1} z_n \| + (1 - \varepsilon) \mu_{m,n} \|T_{m+1} (PT_{m+1})^{n-1} U_{n,m+2} z_n - z_n \|,$$

we have from (3.7) that

(3.9) $$\|z_n - T_m (PT_m)^{n-1} z_n \| \to 0 \quad as \quad n \to \infty.$$

Thus, for all $m, n \in \mathbb{N}$, we get

$$\|T_m z_n - z_n \| \leq \|T_m z_n - T_m (PT_m)^n z_n \| + \|T_m (PT_m)^n z_n - T_m (PT_m)^n z_{n+1} \|$$

$$+ \|T_m (PT_m)^n z_{n+1} - z_{n+1} \| + \|z_{n+1} - z_n \|$$

$$\leq \mu_{m,1} \|z_n - T_m (PT_m)^{n-1} z_n \| + (1 + \mu_{m,n+1}) \|z_{n+1} - z_n \|$$

(3.10) $$+ \|T_m (PT_m)^n z_{n+1} - z_{n+1} \|.$$

So, using (3.9), we obtain from (3.10) and the hypothesis that

(3.11) $$\lim_{n \to \infty} \|z_n - T_m z_n \| = 0 \quad \text{for all} \quad m \in \mathbb{N}.$$

Moreover, by Lemma 2.3, we have that $\omega_\omega(x_n) \subset F$.

(3) It suffices to show that $\bigcap_{n=1}^\infty F(W_n) \subset F$, since $F \subset \bigcap_{n=1}^\infty F(W_n)$ is obvious. So, let $z^* \in \bigcap_{n=1}^\infty F(W_n)$, and $z_n = z^*$ for all $n \in \mathbb{N}$, then by (2), we have that $\|z^* - T_m z^* \| = 0$ for all $m \in \mathbb{N}$, that is $z^* = T_m z^*$ for all $m \in \mathbb{N}$. This completes that proof.   ∎

Remark 3.1. In the sequel, $\{ \beta_{n,j} : n, j \in \mathbb{N}, 1 \leq j \leq n \}$, $\{ \mu_{m,n} \} \subset [1, +\infty)$, and $\{ \gamma_{n,j} : n, j \in \mathbb{N}, 1 \leq j \leq n \}$ are as indicated in Lemma 3.1.

Lemma 3.2. *Let K be a closed convex nonempty nonexpansive retract of a real Banach space E with P as nonexpansive retraction. Let $T_m : K \to E$, $m = 1, 2, \ldots$ be countably infinite family of nonself asymptotically nonexpansive mappings such that $F = \bigcap_{m=1}^\infty F(T_m) \neq \emptyset$. Let W_n be the modified W-mapping generated by $T_n, T_{n-1}, \ldots, T_2, T_1$ and $\beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,2}, \beta_{n,1}$*

for all $n \in \mathbb{N}$ as in (3.1). Let $\{\alpha_n\}_{n\geq 1} \subset (0,1)$. From arbitrary $x_1 \in K$, let the sequence $\{x_n\}_{n\geq 1}$ be iteratively generated by

$$(3.12) \qquad\qquad x_{n+1} = (1-\alpha_n)x_n + \alpha_n W_n x_n \quad \text{for all} \quad n \in \mathbb{N}.$$

Suppose that $\sum_{n=1}^{\infty} \gamma_{n,1} < \infty$, then $\{x_n\}_{n\geq 1}$ is bounded and $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F$.

Proof. Let $x^* \in F$, then from (3.12) we have that

$$\|x_{n+1} - x^*\| = \|(1-\alpha_n)x_n + \alpha_n W_n x_n - x^*\| \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|W_n x_n - x^*\|$$

$$\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n(1+\gamma_{n,1})\|x_n - x^*\| \leq (1+\gamma_{n,1})\|x_n - x^*\|$$

$$(3.13) \qquad\qquad \leq exp\left(\sum_{i=1}^{n} \gamma_{i,1}\right)\|x_1 - x^*\| \leq exp\left(\sum_{i=1}^{\infty} \gamma_{i,1}\right)\|x_1 - x^*\|.$$

So, from (3.13), we get that $\{x_n\}_{n\geq 1}$ is bounded and by Lemma 2.1 we conclude that $\lim_{n\to\infty} \|x_n - x^*\|$ exists. This completes the proof. ∎

Lemma 3.3. *Let E be a real uniformly convex Banach space, K a closed convex nonempty retract of E with P as nonexpansive retraction. Let $T_m : K \to E$, $m = 1,2,...$ be countably infinite family of nonself asymptotically nonexpansive mappings such that $F = \bigcap_{m=1}^{\infty} F(T_m) \neq \emptyset$. Let W_n be the modified W-mapping generated by $T_n, T_{n-1}, ..., T_2, T_1$ and $\beta_{n,n}, \beta_{n,n-1}, ..., \beta_{n,2}, \beta_{n,1}$ for all $n \in \mathbb{N}$ as in (3.1). Let $\{\alpha_n\}_{n\geq 1} \subset (0,1)$ such that $\delta \leq (1-\alpha_n) \leq (1-\delta)$, for some $\delta \in (0,1)$. From arbitrary $x_1 \in K$, let the sequence $\{x_n\}_{n\geq 1}$ be iteratively generated by (3.12). Suppose that $\sum_{n=1}^{\infty} \gamma_{n,1} < \infty$, then for all $u, v \in F$, $\lim_{n\to\infty} \|tx_n + (1-t)u - v\|$ exists for all $t \in [0,1]$.*

Proof. Let $a_n(t) = \|tx_n + (1-t)u - v\|$. Then, $\lim_{n\to\infty} a_n(0) = \|u - v\|$ and from Lemma 3.2, $\lim_{n\to\infty} a_n(1) = \|x_n - v\|$ exists. It remains to show that $\lim_{n\to\infty} a_n(t)$ exists for $t \in (0,1)$. To do this, define

$$G_n x = (1-\alpha_n)x + \alpha_n W_n x \quad \text{for all} \quad x \in K.$$

Then, the proof follows as in the proof of [20, Lemma 3]. ∎

Lemma 3.4. *Let E be a real uniformly convex Banach space which has a Fréchet differentiable norm, K a closed convex nonempty retract of E with P as nonexpansive retraction. Let $T_m : K \to E$, $m = 1,2,...$ be countably infinite family of nonself asymptotically nonexpansive mappings such that $F = \bigcap_{m=1}^{\infty} F(T_m) \neq \emptyset$. Let W_n be the modified W-mapping generated by $T_n, T_{n-1}, ..., T_2, T_1$ and $\beta_{n,n}, \beta_{n,n-1}, ..., \beta_{n,2}, \beta_{n,1}$ for all $n \in \mathbb{N}$ as in (3.1). Let $\{\alpha_n\}_{n\geq 1} \subset (0,1)$ such that $\delta \leq (1-\alpha_n) \leq (1-\delta)$, for some $\delta \in (0,1)$. From arbitrary $x_1 \in K$, let the sequence $\{x_n\}_{n\geq 1}$ be iteratively generated by (3.12). Then, for all $u, v \in F$, the limit $\lim_{n\to\infty} \langle x_n, j(u-v)\rangle$ exists. Furthermore, if $\omega_w(x_n)$ denotes the set of weak subsequential limits of $\{x_n\}_{n\geq 1}$, then $\langle x^* - y^*, j(u-v)\rangle = 0$, $\forall u, v \in F(T)$ and $\forall x^*, y^* \in \omega_w(x_n)$.*

Proof. This follows basically as in the proof of [20, Lemma 4] using Lemma 3.3 instead of [20, Lemma 3]. This completes the proof. ∎

Theorem 3.1. *Let E be a uniformly convex real Banach space E which has a Fréchet differentiable norm. Let K be a closed convex nonempty retract of E with P as nonexpansive retraction. Let $T_m : K \to E$, $m = 1,2,...$ be countably infinite family of nonself asymptotically nonexpansive mappings such that $F = \bigcap_{m=1}^{\infty} F(T_m) \neq \emptyset$. Let W_n be the modified W-mapping*

generated by $T_n, T_{n-1}, ..., T_2, T_1$ and $\beta_{n,n}, \beta_{n,n-1}, ..., \beta_{n,2}, \beta_{n,1}$ for all $n \in \mathbb{N}$ as in (3.1). Let $\{\alpha_n\}_{n \geq 1} \subset (0,1)$ be such that $\delta \leq 1 - \alpha_n \leq 1 - \delta$, for some $\delta \in (0,1)$. From arbitrary $x_1 \in K$, let the sequence $\{x_n\}_{n \geq 1}$ be iteratively generated by (3.12). Suppose that $\sum_{n=1}^{\infty} \gamma_{n,1} < \infty$, then $\{x_n\}_{n \geq 1}$ converges weakly to some element of F.

Proof. Let $x^* \in F$. Since by Lemma 2.1, $\{x_n\}_{n \geq 1}$ is bounded, we obtain from (3.12) and Lemma 2.2 that

$$\|x_{n+1} - x^*\|^2 = \|(1 - \alpha_n)x_n + \alpha_n W_n x_n - x^*\|^2$$
$$\leq (1 - \alpha_n)\|x_n - x^*\|^2 + \alpha_n \|W_n x_n - x^*\|^2 - \alpha_n(1 - \alpha_n)g(\|x_n - W_n x_n\|)$$
$$\leq (1 - \alpha_n)\|x_n - x^*\|^2 + \alpha_n(1 + \gamma_{n,1})\|x_n - x^*\|^2 - \delta^2 g(\|x_n - W_n x_n\|)$$
$$(3.14) \qquad \leq \|x_n - x^*\|^2 + \gamma_{n,1} M_1 - \delta^2 g(\|x_n - W_n x_n\|),$$

for some constant $M_1 > 0$. Thus,

$$\delta^2 g(\|x_n - W_n x_n\|) \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \gamma_{n,1} M_1.$$

So, properties of g and existence of the limit of the sequence $\{\|x_n - x^*\|\}_{n \geq 1}$ give that

$$(3.15) \qquad \lim_{n \to \infty} \|x_n - W_n x_n\| = 0.$$

Furthermore, we obtain from the recursion formula (3.12) that

$$(3.16) \qquad \|x_{n+1} - x_n\| = \alpha_n \|x_n - W_n x_n\| \leq \|x_n - W_n x_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

So, using (3.15) and (3.16), we obtain from Lemma 3.1 that $\omega_\omega(x_n) \subset F$. Moreover, reflexivity of E implies the existence of a subsequence $\{x_{n_j}\}_{j \geq 1}$ of $\{x_n\}_{n \geq 1}$ which converges weakly to some $u^* \in K$. We also obtain from Lemma 3.1 that $\lim_{j \to \infty} \|x_{n_j} - T_m x_{n_j}\| = 0$ and hence $u^* = T_m u^*$ for all $m \in \mathbb{N}$ (since by Lemma 2.3, $(I - T_m)$ is demiclosed at 0 for all $m \in \mathbb{N}$). We now show that $\{x_n\}_{n \geq 1}$ converges weakly to u^*. Suppose that $\{x_{n_p}\}_{p \geq 1}$ is another subsequence of $\{x_n\}_{n \geq 1}$ which converges weakly to some v^*. Since v^* must be in $F(T_m)$ for all $m \in \mathbb{N}$, then by Lemma 3.4, we have that $u^* = v^*$. Hence, $\omega_\omega(x_n)$ is a singleton, so that $\{x_n\}_{n \geq 1}$ converges weakly to some element of $F = \bigcap_{m=1}^{\infty} F(T_m)$. This completes the proof. ∎

If in Theorem 3.1, we have that T_m is nonexpansive for all $m = 1, 2, ...$, then we get the following corollary.

Corollary 3.1. *Let E be a uniformly convex real Banach space E which has a Fréchet differentiable norm. Let K be a closed convex nonempty retract of E with P as nonexpansive retraction. Let $T_m : K \to E$, $m = 1, 2, ...$ be countably infinite family of nonself nonexpansive mappings such that $F = \bigcap_{m=1}^{\infty} F(T_m) \neq \emptyset$. Let W_n be the modified W-mapping generated by $T_n, T_{n-1}, ..., T_2, T_1$ and $\beta_{n,n}, \beta_{n,n-1}, ..., \beta_{n,2}, \beta_{n,1}$ for all $n \in \mathbb{N}$ as in (3.1) Let $\{\alpha_n\}_{n \geq 1} \subset (0,1)$ be such that $\delta \leq 1 - \alpha_n \leq 1 - \delta$, for some $\delta \in (0,1)$. From arbitrary $x_1 \in K$, let the sequence $\{x_n\}_{n \geq 1}$ be iteratively generated by (3.12). Then, $\{x_n\}_{n \geq 1}$ converges weakly to some element of F.*

Proof. Since $\mu_{j,n} \equiv 1$ for all $n, j \in N$, we get that $\gamma_{n,1} = 0$ for each $n \in N$ and hence $\sum_{n \geq 1} \gamma_{n,1} = 0$. Therefore, the conclusion follows from Theorem 3.1. ∎

Remark 3.2. If in the definition of the modified W-mapping, the family $T_m : K \to K$, $m = 1, 2, ...$ are self mappings, then the nonexpansive retraction P becomes identity operator on K. Thus, we have the following corollary.

Corollary 3.2. *Let K be a closed convex nonempty subset of a uniformly convex real Banach space E which has a Fréchet differentiable norm. Let $T_m : K \to K$, $m = 1, 2, \dots$ be countably infinite family of asymptotically nonexpansive self-map such that $F = \bigcap_{m=1}^{\infty} F(T_m) \neq \emptyset$. Let W_n be the modified W-mapping generated by $T_n, T_{n-1}, \dots, T_2, T_1$ and $\beta_{n,n}, \beta_{n,n-1}, \dots, \beta_{n,2}, \beta_{n,1}$ for all $n \in \mathbb{N}$ as in (1.4). Let $\{\alpha_n\}_{n \geq 1} \subset (0,1)$ be such that $\delta \leq 1 - \alpha_n \leq 1 - \delta$, for some $\delta \in (0,1)$. From arbitrary $x_1 \in K$, let the sequence $\{x_n\}_{n \geq 1}$ be iteratively generated by (3.12) with $P = I$, identity mapping on K. Suppose that $\sum_{n=1}^{\infty} \gamma_{n,1} < \infty$, then $\{x_n\}_{n \geq 1}$ converges weakly to some element of F.*

Proof. Follows from Theorem 3.1. ∎

If in Theorem 3.1, we have that $\{T_m, m = 1, 2, \dots, r\}$ is finite family of asymptotically nonexpansive mappings, we define W_n as follows: Let $T_m : K \to E$, $m = 1, 2, \dots, r$ be a finite family of nonself asymptotically nonexpansive mappings and $\{\beta_{n,m}\}_{n \in \mathbb{N}, m = 1, 2, \dots, r} \subset (0,1)$. Then, for the family $\{T_m\}_{m = 1, 2, \dots, r}$, the modified W-mapping generated by $T_r, T_{r-1}, \dots, T_2, T_1$ and $\beta_{n,r}, \beta_{n,r-1}, \dots, \beta_{n,2}, \beta_{n,1}$ is given by:

(1) for $n \leq r$, we take W_n as in (3.1);

(2) for $n \geq r + 1$, we have the following:

$$U_{n,r+1} = I, \quad \text{identity mapping on } K,$$
$$U_{n,r} = P\big(\beta_{n,r} T_r (PT_r)^{n-1} U_{n,r+1} + (1 - \beta_{n,r})I\big),$$
$$U_{n,r-1} = P\big(\beta_{n,r-1} T_{r-1} (PT_{r-1})^{n-1} U_{n,r} + (1 - \beta_{n,r-1})I\big),$$
$$\vdots$$
$$U_{n,j} = P\big(\beta_{n,j} T_j (PT_j)^{n-1} U_{n,j+1} + (1 - \beta_{n,j})I\big),$$
$$\vdots$$
$$U_{n,2} = P\big(\beta_{n,2} T_2 (PT_2)^{n-1} U_{n,3} + (1 - \beta_{n,2})I\big),$$

(3.17) $$W_n = U_{n,1} = P\big(\beta_{n,1} T_1 (PT_1)^{n-1} U_{n,2} + (1 - \beta_{n,1})I\big).$$

Now, we state a corollary for finite family of nonself asymptotically nonexpansive mappings.

Corollary 3.3. *Let E be a uniformly convex real Banach space E which has a Fréchet differentiable norm. Let K be a closed convex nonempty retract of E with P as nonexpansive retraction. Let $T_m : K \to E$, $m = 1, 2, \dots, r$ be finite family of nonself asymptotically nonexpansive mappings such that $F = \bigcap_{m=1}^{r} F(T_m) \neq \emptyset$. Let W_n be the modified W-mapping generated by $T_r, T_{r-1}, \dots, T_2, T_1$ and $\beta_{n,r}, \beta_{n,r-1}, \dots, \beta_{n,2}, \beta_{n,1}$ for all $n \in \mathbb{N}$ as in (3.17) such that $\sum_{n \geq 1}(\mu_{m,n}^2 - 1) < \infty$, for $m = 1, 2, \dots, r$. Let $\{\alpha_n\}_{n \geq 1} \subset (0,1)$ be such that $\delta \leq 1 - \alpha_n \leq 1 - \delta$, for some $\delta \in (0,1)$. From arbitrary $x_1 \in K$, let the sequence $\{x_n\}_{n \geq 1}$ be iteratively generated by (3.12). Then $\{x_n\}_{n \geq 1}$ converges weakly to some element of F.*

Proof. Take $T_i = I$, identity operator on K, with $\mu_{i,n} = 1$ and $\beta_{n,i} = \alpha \in (\varepsilon, 1 - \varepsilon)$ for $i \geq r + 1$ in Theorem 3.1. Then, since $\sum_{n \geq 1}(\mu_{m,n}^2 - 1) < \infty$ for $m = 1, 2, \dots, r$, we have that $\sum \gamma_{n,1} < \infty$. Thus, the conclusion follows from Theorem 3.1. ∎

Now, we give strong convergence theorems.

Theorem 3.2. *Let K be a closed convex nonempty retract of a uniformly convex real Banach space E with P as nonexpansive retraction. Let $T_m : K \to E$, $m = 1, 2, \dots$ be countably infinite family of nonself asymptotically nonexpansive mappings such that $F = \bigcap_{m=1}^{\infty} F(T_m) \neq \emptyset$. Let W_n be the modified W-mapping generated by $T_n, T_{n-1}, \dots, T_2, T_1$ and $\beta_{n,n}, \beta_{n,n-1}, \dots, \beta_{n,2}, \beta_{n,1}$ for all $n \in \mathbb{N}$ as in (3.1). Let $\{\alpha_n\}_{n \geq 1} \subset (0, 1)$ be such that $\delta \leq 1 - \alpha_n \leq 1 - \delta$, for some $\delta \in (0, 1)$. From arbitrary $x_1 \in K$, let the sequence $\{x_n\}_{n \geq 1}$ be iteratively generated by (3.12). Suppose that $\sum_{n=1}^{\infty} \gamma_{n,1} < \infty$ and that one member of the family $\{T_m\}_{m \geq 1}$ is either semicompact or completely continuous, then $\{x_n\}_{n \geq 1}$ converges strongly to some element of F.*

Proof. For fixed $r \in \mathbb{N}$, let T_r be semicompact. Since $\lim_{n \to \infty} \|x_n - T_r x_n\| = 0$, there exists a subsequence say $\{x_{n_k}\}_{k \geq 1}$ of $\{x_n\}_{n \geq 1}$ that converges strongly to some point $z^* \in K$. Thus, by continuity of T_r, we have that $T_r z^* = z^*$. Furthermore, since T_m is continuous for all $m \in \mathbb{N}$ and $\lim_{k \to \infty} \|z_{n_k} - T_m z_{n_k}\| = 0$, $m = 1, 2, \dots$, we have that $z^* \in F$. Hence, we obtain from Lemma 3.2 that $\{x_n\}_{n \geq 1}$ converges strongly to $z^* \in F$.

If, on the other hand, one member of the family $\{T_m\}_{m \geq 1}$, say T_q (for some $q \in \mathbb{N}$) is completely continuous, then since $\{x_n\}_{n \geq 1}$ is bounded, we have that $\{T_q x_n\}$ is bounded. Thus, there exists a subsequence $\{T_q x_{n_j}\}_{j \geq 1}$ of $\{T_q x_n\}_{n \geq 1}$ which converges to some x^*. Since by Lemma 3.1, $\lim_{j \to \infty} \|x_{n_j} - T_q x_{n_j}\| = 0$, we have that $\lim_{j \to \infty} x_{n_j} = x^*$. Furthermore, we have that $x^* \in F$ since $\lim_{j \to \infty} \|x_{n_j} - T_m x_{n_j}\| = 0$ and T_m is continuous for all $m \in \mathbb{N}$. Finally, since $\lim_{j \to \infty} \|x_n - x^*\|$ exists by Lemma 3.2, we obtain that $\{x_n\}_{n \geq 1}$ converges strongly to $x^* \in F$. This completes the proof. ∎

If in Theorem 3.2, we have that T_m is nonexpansive for $m = 1, 2, \dots$, then we get the following corollary.

Corollary 3.4. *Let K be a closed convex nonempty nonexpansive retract of uniformly convex real Banach space E with P as a nonexpansive retraction. Let $T_m : K \to E$, $m = 1, 2, \dots$ be countably infinite family of nonself nonexpansive mappings such that $F = \bigcap_{m=1}^{\infty} F(T_m) \neq \emptyset$. Let W_n be the modified W-mapping generated by $T_n, T_{n-1}, \dots, T_2, T_1$ and $\beta_{n,n}, \beta_{n,n-1}, \dots, \beta_{n,2}, \beta_{n,1}$ for all $n \in \mathbb{N}$ as in (3.1). Let $\{\alpha_n\}_{n \geq 1} \subset (0, 1)$ be such that $\delta \leq 1 - \alpha_n \leq 1 - \delta$, for some $\delta \in (0, 1)$. From arbitrary $x_1 \in K$, let the sequence $\{x_n\}_{n \geq 1}$ be iteratively generated by (3.12). Suppose that one member of the family $\{T_m\}_{m \geq 1}$ is either semicompact or completely continuous, then $\{x_n\}_{n \geq 1}$ converges strongly to some element of F.*

For finite family of nonself asymptotically nonexpansive mappings, we have the following corollary.

Corollary 3.5. *Let K be a closed convex nonempty nonexpansive retract of uniformly convex real Banach space E with P as a nonexpansive retraction. Let $T_m : K \to E$, $m = 1, 2, \dots, r$ be finite family of nonself asymptotically nonexpansive mappings such that $F = \bigcap_{m=1}^{r} F(T_m) \neq \emptyset$. Let W_n be the modified W-mapping generated by $T_1, T_2, \dots, T_r$ and $\beta_{n,r}, \beta_{n,r-1}, \dots, \beta_{n,2}, \beta_{n,1}$ for all $n \in \mathbb{N}$ as in (3.17) such that $\sum_{n \geq 1} (\mu_{m,n}^2 - 1) < \infty$ for $m = 1, 2, \dots, r$. Let $\{\alpha_n\}_{n \geq 1} \subset (0, 1)$ be such that $\delta \leq 1 - \alpha_n \leq 1 - \delta$, for some $\delta \in (0, 1)$. From arbitrary $x_1 \in K$, let the sequence $\{x_n\}_{n \geq 1}$ be iteratively generated by (3.12). Suppose that one member of the family $\{T_m\}_{m \geq 1}$ is either semicompact or completely continuous, then $\{x_n\}_{n \geq 1}$ converges strongly to some element of F.*

If in the definition of the modified W-mapping, the family $T_m : K \to K$, $m = 1, 2, \dots$ are self mappings, then we have the following corollary.

Corollary 3.6. *Let K be a nonempty closed convex subset of a uniformly convex real Banach space. Let $T_m : K \to K$, $m = 1, 2, \ldots$ be countably infinite family of self asymptotically nonexpansive mappings such that $F = \bigcap_{m=1}^{\infty} F(T_m) \neq \emptyset$. Let W_n be the modified W-mapping generated by $T_n, T_{n-1}, \ldots, T_2, T_1$ and $\beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,2}, \beta_{n,1}$ for all $n \in \mathbb{N}$ as in (1.4). Let $\{\alpha_n\}_{n \geq 1} \subset (0, 1)$ be such that $\delta \leq 1 - \alpha_n \leq 1 - \delta$, for some $\delta \in (0, 1)$. From arbitrary $x_1 \in K$, let the sequence $\{x_n\}_{n \geq 1}$ be iteratively generated by (3.12). Suppose that $\sum_{n=1}^{\infty} \gamma_{n,1} < \infty$; and that one member of the family $\{T_m\}_{m \geq 1}$ is either semicompact or completely continuous, then $\{x_n\}_{n \geq 1}$ converges strongly to some element of F.*

Proof. The proof follows as in the proof of Theorem 3.2. ∎

Remark 3.3. All the theorems and corollaries obtained in this paper are also valid for the *so-called* Ishikawa-type iteration method with parameters $\{\alpha_n\}_{n \geq 1}$ and $\{\beta_n\}_{n \geq 1}$. Under the conditions on $\{\alpha_n\}_{n \geq 1}$ and $\{\beta_n\}_{n \geq 1}$ in which they are independent, there is no further generality obtained in using the cumbersome Ishikawa-type scheme rather than the scheme considered in this paper given by the recurrence relation (3.12). In fact, in this case, setting $\beta_n \equiv 0$ for all $n \in \mathbb{N}$, the so-called Ishikawa-type algorithm reduces to the Mann-type scheme considered in this paper.

Remark 3.4. Theorem 3.1 extends and generalizes Theorem 1 of Osilike and Aniagbosor [19] and Theorem 1.5 to the more general case of countably infinite family of nonself asymptotically nonexpansive mappings with Opial's condition dispensed. Moreover, Theorem 3.2 extends Theorems 1.3, 1.4, the corresponding results of Rhoades [22], Osilike and Aniagbosor [19] and those of many other authors to the more general case of countably infinite family of nonself asymptotically nonexpansive mappings. No boundedness assumption is imposed on K. Furthermore, the results of [7–9] are generalized and improved upon. We remark that Theorem 3.1 is applicable in L_p spaces $1 < p < +\infty$, while 1.5 and the corresponding results of [19] are not.

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