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Characterizations of Rayleigh Distribution Based on Order Statistics and Record Values

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Abstract. In this paper, some new results on characterizations of Rayleigh distribution based on order statistics and record values have been established. The conditional expectation (also known as mean residual life) of a distribution plays important roles in modeling and analysis of life time data. Some results on characterizations based on mean residual life have also been established.

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1. Introduction

Many researchers have studied the characterizations of probability distributions. For example, Su and Huang [12] studied the characterizations based on conditional expectations. Recently, Nanda [10] studied the characterizations through the expected values of failure rate and mean residual life functions of a nonnegative absolutely continuous random variable X. The problems of characterizations based on record values and order statistics started in late sixties by Tata [13], and followed in seventies by Nagaraja [9] and Ahsanullah [1]. For further development and various characterizations of probability distributions based on record values and order statistics, the interested readers are referred to Arnold *et al.* [6], Rao and Shanbhag [11], Ahsanullah [2, 3, 4], Bairamov *et al.* [7], Yanev *et al.* [14], and Ahsanullah and Aliev [5], among others. In this paper, some new results on characterizations of Rayleigh distribution based on order statistics and record values have been established.

Rayleigh Distribution: A random variable X is said to have a Rayleigh distribution if its probability density function (pdf) f is given by

$$f(x) = \begin{cases} 2cxe^{-cx^2}, & x > 0, c > 0, \\ 0, & otherwise, \end{cases} \tag{1.1}$$

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with the corresponding cumulative distribution function (cdf) F is given by $F(x) = 1 - e^{-cx^2}$, where $c > 0$ is known as the scale parameter of Rayleigh distribution. For detailed treatment on Rayleigh distribution, the interested readers are referred to Johnson *et al.* [8].

Record Values: Suppose that $(X_n)_{n \geqslant 1}$ is a sequence of independent and identically distributed $(i.i.d.)$ random variables $(rv\text{'}s)$ with cdf F. Let $Y_n = \max(\min)\{X_j \mid 1 \leqslant j \leqslant n\}$, for $n \geqslant 1$. We say X_j is an upper (lower) record value of $\{X_n \mid n \geqslant 1\}$, if $Y_j > (<) Y_{j-1}$, $j > 1$. By definition X_1 is an upper as well as a lower record value. The indices at which the upper record values occur are given by the record times $\{U(n), n \geqslant 1\}$, where $U(n) = \min\{j \mid j > U(n-1), X_j > X_{U(n-1)}, n > 1\}$ and $U(1) = 1$. Many properties of the upper record value sequence can be expressed in terms of the cumulative hazard rate function $R(x) = -\ln\overline{F}(x)$, where $\overline{F}(x) = 1 - F(x)$, $0 < \overline{F}(x) < 1$. We will denote $X_{U(n)}$ by $X(n)$ and $F_n(x)$ as the cdf of $X(n)$ for $n \geqslant 1$. We have

$$F_n(x) = \int_{-\infty}^{x} \frac{(R(u))^{n-1}}{\Gamma(n)} dF(u), \quad -\infty < x < \infty,$$

from which it is easy to see that

$$F_n(x) = 1 - \overline{F}(x) \sum_{j=0}^{n-1} \frac{(R(x))^j}{\Gamma(j+1)},$$

that is,

$$F_n(x) = \overline{F}(x) \sum_{j=0}^{n-1} \frac{(R(x))^j}{\Gamma(j+1)}. \tag{1.2}$$

We assume F(x) is absolutely continuous with respect to Lebesgue measure and denote $f_n(x)$ as the pdf of $X(n)$, where

$$f_n(x) = \frac{(R(x))^{n-1}}{\Gamma(n)} f(x), \quad -\infty < x < \infty. \tag{1.3}$$

From (1.2) and (1.3) it is easy to show that $\overline{F}_n(x) - \overline{F}_{n-1}(x) = \overline{F}(x)\frac{f_n(x)}{f(x)}$. For details on record values, see Ahsanullah [3]. Using Equation (1.3), the pdf and cdf of the nth record value $X(n)$ from Rayleigh(c) distribution are, respectively, given by

$$f_n(x) = \frac{2c^n x^{2n-1} e^{-cx^2}}{\Gamma(n)}, \quad n = 1, 2, 3, \ldots, \tag{1.4}$$

and

$$F_n(x) = \frac{\gamma(n, cx^2)}{\Gamma(n)}, n = 1, 2, 3, \ldots, \tag{1.5}$$

where $x > 0, c > 0$, and $\gamma(\alpha, z) = \int_0^z t^{\alpha-1} dt, \alpha > 0$, denotes incomplete gamma function. The kth moment of the nth record value $X(n)$ with the pdf (1.4) is given by

$$E[X^k(n)] = \frac{\Gamma(n + \frac{2}{k})}{\Gamma(n)} c^{-\frac{k}{2}}. \tag{1.6}$$

In this paper, some new results on characterizations of Rayleigh distribution based on order statistics and record values have been established. The organization of this paper is as follows. Section 2 contains characterizations based on conditional expectations. In Section

3, some new results based on order statistics have been established. Section 4 contains characterizations based on record values.

2. Characterizations based on conditional expectations

We first prove the following two lemmas.

Lemma 2.1. *Case (i) Let X be an absolutely continuous (with respect to Lebesgue measure) random variable with cumulative distribution function $F(x)$ with $F(0)=0$ and $F(x)>0$ for all $x>0$. If for any non-negative number s, any $t>0$ and $c>0$,*

$$\frac{\int_t^\infty (1-F(x))^s dx}{(1-F(t))^s} = \frac{1}{2}\sqrt{\frac{\pi}{cs}}(1-\operatorname{erf}((\sqrt{cs})t))e^{cst^2},$$

where $\operatorname{erf}(x)=\int_0^x \frac{2}{\sqrt{\pi}}e^{-t^2}dt$ *denotes the error function, then* $F(x)=1-e^{-cx^2}$, $x \geqslant 0$, $c>0$.

Proof. We have

$$\int_t^\infty (1-F(x))^s dx = \frac{1}{2}\sqrt{\frac{\pi}{cs}}(1-F(t))^s(1-\operatorname{erf}((\sqrt{cs})t))e^{cst^2}. \tag{2.1}$$

Noting $\frac{d}{dt}(\frac{1}{2}\sqrt{\frac{\pi}{cs}}(1-\operatorname{erf}((\sqrt{cs})t))e^{cst^2}) = -1+(\sqrt{\pi cs})te^{cst^2}(1-\operatorname{erf}((\sqrt{cs})t))$, and differentiating both sides of (2.1) with respect to t, we obtain

$$\begin{aligned}-(1-F(t))^s = &-\frac{1}{2}\sqrt{\frac{\pi}{cs}}s(1-F(t))^{s-1}(1-\operatorname{erf}((\sqrt{cs})t))f(t)\\ &+(1-F(t))^s(-1+(\sqrt{\pi cs})te^{cst^2}(1-\operatorname{erf}((\sqrt{cs})t))).\end{aligned}$$

On simplifying, we have from the above equation

$$\frac{f(t)}{1-F(t)} = 2ct \tag{2.2}$$

On integrating (2.2) with respect to t and using the boundary conditions $F(0)=0$ and $F(\infty)=1$, we have

$$F(x)=1-e^{-cx^2}, \quad x \geqslant 0, \quad c>0.$$ ∎

Case (ii) Let X be an absolutely continuous (with respect to Lebesgue measure) random variable with cumulative distribution function $F(x)$ with $F(0)=0$ and $F(x)>0$ for all $x>0$. If, for any non-negative number s, any $t>0$, $k \geqslant 2$ and $c>0$,

$$\begin{aligned}&\frac{\int_t^\infty (2k-1)x^{2k-2}(1-F(x))^s dx}{(1-F(t))^s}\\ &=\sum_{j=0}^{k-2}\frac{(2k-1)!!}{(2cs)^{j+1}(2k-3-2j)!!}t^{2k-3-2j}+\frac{(2k-1)!!\sqrt{cs\pi}}{(2cs)^k}(1-\operatorname{erf}((\sqrt{cs})t))e^{cst^2},\end{aligned} \tag{2.3}$$

where $(2k-1)!! = 1.3.5...(2k-1)$, *then* $F(x)=1-e^{cx^2}$, $x \geq 0$, $c>0$.

Proof. We have from (2.3)

$$\begin{aligned}\int_t^\infty (2k-1)x^{2k-2}(1-F(x))^s dx = (1-F(t))^s\Bigg[&\sum_{j=0}^{k-2}\frac{(2k-1)!!}{(2cs)^{j+1}(2k-3-2j)!!}t^{2k-3-2j}\\ &+\frac{(2k-1)!!\sqrt{cs\pi}}{(2cs)^k}(1-\operatorname{erf}((\sqrt{cs})t))e^{cst^2}\Bigg].\end{aligned} \tag{2.4}$$

Noting

$$\frac{(2k-1)!!\sqrt{cs\pi}}{(2cs)^k}\frac{d}{dt}(1-\operatorname{erf}((\sqrt{cs})t))e^{cst^2})$$
$$=\frac{(2k-1)!!}{(2cs)^{k-1}}\left(-1+(\sqrt{\pi cs})te^{cst^2}(1-\operatorname{erf}((\sqrt{cs})t))\right),$$

and

$$\frac{d}{dt}\left[\sum_{j=0}^{k-2}\frac{(2k-1)!!}{(2cs)^{j+1}(2k-3-2j)!!}t^{2k-3-2j}\right]$$
$$=\frac{d}{dt}\left[\sum_{j=0}^{k-2}\frac{(2k-1)!!}{(2cs)^{k-1}}t+\sum_{j=0}^{k-3}\frac{(2k-1)!!}{(2cs)^{j+1}(2k-3-2j)!!}t^{2k-3-2j}\right]$$
$$=\frac{d}{dt}\left[\sum_{j=0}^{k-2}\frac{(2k-1)!!}{(2cs)^{k-1}}+\sum_{j=0}^{k-3}(2k-3-2j)\frac{(2k-1)!!}{(2cs)^{j+1}(2k-3-2j)!!}t^{2k-3-2j-1}\right]$$

we have, on simplification,

$$\frac{d}{dt}\left[\sum_{j=0}^{k-2}\frac{(2k-1)!!}{(2cs)^{j+1}(2k-3-2j)!!}t^{2k-3-2j}+\frac{(2k-1)!!\sqrt{cs\pi}}{(2cs)^k}(1-\operatorname{erf}((\sqrt{cs})t))e^{cst^2}\right]$$
$$=\sum_{j=0}^{k-3}(2k-3-2j)\frac{(2k-1)!!}{(2cs)^{j+1}(2k-3-2j)!!}t^{2k-3-2j-1}$$
$$+\frac{(2k-1)!!}{(2cs)^{k-1}}(\sqrt{cs\pi})te^{cst^2}(1-\operatorname{erf}((\sqrt{cs})t)).$$

Thus differentiating both sides of (2.4) with respect to t and using the above equation, we obtain

$$-(2k-1)t^{2k-2}(1-F(t))^s$$
$$=(1-F(t))^s\left[\sum_{j=0}^{k-3}(2k-3-2j)\frac{(2k-1)!!}{(2cs)^{j+1}(2k-3-2j)!!}t^{2k-3-2j-1}\right.$$
$$\left.+\frac{(2k-1)!!}{(2cs)^{k-1}}e^{cst^2}t\sqrt{\pi cs}(1-\operatorname{erf}(t\sqrt{cs}))\right]$$
$$-s(1-F(t))^{s-1}f(t)\left[\frac{(2k-1)!!}{(2cs)^{k-1}}t+\sum_{j=0}^{k-3}\frac{(2k-1)!!}{(2cs)^{j+1}(2k-3-2j)!}t^{2k-3-2j}\right.$$
$$\left.+\frac{(2k-1)!!\sqrt{cs\pi}}{(2cs)^k}(1-\operatorname{erf}((\sqrt{cs})t))e^{cst^2}\right],$$

from which we have

$$\frac{sf(t)}{1-F(t)}\left[\frac{(2k-1)!!}{(2cs)^{k-1}}t+\sum_{j=0}^{k-3}\frac{(2k-1)!!}{(2cs)^{j+1}(2k-3-2j)!!}t^{2k-3-2j}\right.$$
$$\left.+\frac{(2k-1)!!\sqrt{cs\pi}}{(2cs)^k}(1-\operatorname{erf}((\sqrt{cs})t))e^{cst^2}\right]$$

$$= (2k-1)t^{2k-2} + \sum_{j=0}^{k-3}(2k-3-2j)\frac{(2k-1)!!}{(2cs)^{j+1}(2k-3-2j)!!}t^{2k-3-2j-1}$$
$$+ \frac{(2k-1)!!}{(2cs)^{k-1}}e^{cst^2}\sqrt{\pi cs}(1-\operatorname{erf}((\sqrt{cs})t)).$$

In the above equation, noting that

$$(2k-1)t^{2k-2} + \sum_{j=0}^{k-3}(2k-3-2j)\frac{(2k-1)!!}{(2cs)^{j+1}(2k-3-2j)!!}t^{2k-3-2j-1}$$
$$= \frac{2cs(2k-1)!!}{2cs(2k-3)!!}t^{2k-2} + \sum_{j=0}^{k-3}\frac{(2k-1)!!}{(2cs)^{j+1}(2k-5-2j)!!}t^{2k-3-2j-1}$$
$$= \frac{2cs(2k-1)!!}{2cs(2k-3)!!}t^{2k-2} + 2cst\sum_{j=1}^{k-2}\frac{(2k-1)!!}{(2cs)^{j+1}(2k-3-2j)!!}t^{2k-3-2j}$$
$$= \frac{2cs(2k-1)!!}{2cs(2k-3)!!}t^{2k-2} + \frac{(2k-1)!!}{(2cs)^{k-2}}t^2 + 2cst\sum_{j=1}^{k-3}\frac{(2k-1)!!}{(2cs)^{j+1}(2k-3-2j)!!}t^{2k-3-2j}$$
$$= \frac{(2k-1)!!}{(2cs)^{k-2}}t^2 + 2cst\sum_{j=0}^{k-3}\frac{(2k-1)!!}{(2cs)^{j+1}(2k-3-2j)!!}t^{2k-3-2j}$$
$$= 2cst\left[\frac{(2k-1)!!}{(2cs)^{k-1}}t + \sum_{j=0}^{k-3}\frac{(2k-1)!!}{(2cs)^{j+1}(2k-3-2j)!!}t^{2k-3-2j}\right],$$

we obtain, on simplification, the following equation

$$\frac{sf(t)}{1-F(t)}\left[\frac{(2k-1)!!}{(2cs)^{k-1}}t + \sum_{j=0}^{k-3}\frac{(2k-1)!!}{(2cs)^{j+1}(2k-3-2j)!!}t^{2k-3-2j}\right.$$
$$\left. + \frac{(2k-1)!!\sqrt{cs\pi}}{(2cs)^k}(1-\operatorname{erf}((\sqrt{cs})t))e^{cst^2}\right]$$
$$= 2cst\left[\frac{(2k-1)!!}{(2cs)^{k-1}}t + \sum_{j=0}^{k-3}\frac{(2k-1)!!}{(2cs)^{j+1}(2k-3-2j)!!}t^{2k-3-2j}\right.$$
$$\left. + \frac{(2k-1)!!\sqrt{cs\pi}}{(2cs)^k}(1-\operatorname{erf}((\sqrt{cs})t))e^{cst^2}\right].$$

Hence

$$\frac{f(t)}{1-F(t)} = 2ct.$$

Thus on integrating the above equation with respect to t and using the boundary conditions $F(0) = 0$ and $F(\infty) = 1$, we have

$$F(x) = 1 - e^{-cx^2}, \quad x \geqslant 0,\ c > 0$$

This completes the proof of Lemma 2.1. ∎

Lemma 2.2. *Let X be an absolutely continuous (with respect to Lebesgue measure) random variable with cumulative distribution function* $F(x)$ *with* $F(0) = 0$ *and* $F(x) > 0$ *for all*

$x > 0$. If for any nonnegative number s, any $t > 0$, $k \geqslant 2$ and $c > 0$,

$$\frac{\int_t^\infty 2kx^{2k-1}(1-F(x))^s dx}{(1-F(x))^s} = \sum_{j=1}^{k} \frac{k^{(j)}t^{2k-2j}}{(cs)^i}, \tag{2.5}$$

where $k^{(j)} = k(k-1)...(k-j+1)$, then $F(x) = 1-e^{-cx^2}$, $x \geqslant 0, c > 0$.

Proof. We have from (2.5)

$$\int_t^\infty 2kx^{2k-1}(1-F(x))^s dx = (1-F(t))^s \left(\sum_{j=1}^{k} \frac{k^{(j)}t^{2k-2j}}{(cs)^j}\right) \tag{2.6}$$

Differentiating both sides of (2.6) with respect to t, we obtain

$$-2kt^{2k-1}(1-F(t))^s = -s(1-F(t))^{s-1}f(t)\sum_{j=1}^{k} \frac{k^{(j)}t^{2k-2j}}{(cs)^j} + (1-F(t))^s \sum_{j=1}^{k-1} \frac{2(k-j)k^{(j)}t^{2k-2j-1}}{(cs)^j},$$

that is,

$$\begin{aligned} s(1-F(t))^{s-1}f(t)\sum_{j=1}^{k} \frac{k^{(j)}t^{2k-2j}}{(cs)^j} &= (1-F(t))^s \sum_{j=1}^{k-1} \frac{2(k-j)k^{(j)}t^{2k-2j-1}}{(cs)^j} + 2kt^{2k-1}(1-F(t))^s \\ &= (1-F(t))^s 2t \left(\sum_{j=1}^{k-1} \frac{(k-j)k^{(j)}t^{2k-2j-2}}{(cs)^j} + kt^{2k-2}\right) \\ &= (1-F(t))^s 2cst \left(\sum_{j=1}^{k} \frac{k^{(j)}t^{2k-2j}}{(cs)^j}\right). \end{aligned}$$

Simplifying the above equation, we have

$$\frac{f(t)}{1-F(t)} = 2ct.$$

Thus on integrating the above equation with respect to t and using the boundary conditions $F(0) = 0$ and $F(\infty) = 1$, we have

$$F(x) = 1 - e^{-cx^2}, \quad x \geqslant 0, \ c > 0.$$

This completes the proof of Lemma 2.2. ∎

Theorem 2.1. *Let X be a nonnegative random variable with absolutely continuous (with respect to Lebesgue measure) distribution function $F(x)$ with $F(0) = 0$, $F(x) > 0$ for all $x > 0$ and finite $E(X^{2n})$, for some fixed $n \geqslant 1$. Then X has a Rayleigh distribution with $F(x) = 1 - e^{-cx^2}$, $x \geqslant 0$, $c > 0$, iff $E(X^{2n} \mid X > t) = \sum_{i=0}^{n} \frac{n^{(i)}}{c^{(i)}} t^{2(n-i)}$, where $n^{(i)} = n(n-1)...(n-i+1)$, $n^{(0)} = 1$.*

Proof. It is easy to show that if

$$F(x) = 1 - e^{-c\,x^2}, \quad x \geqslant 0, \quad c > 0,$$

then

$$E(X^{2n}\,|X > t) = \sum_{i=0}^{n} \frac{n^{(i)}}{c^i} t^{2(n-i)}.$$

We will prove here the "only if" condition. Suppose

$$E(X^{2n}\,|X > t) = \sum_{i=0}^{n} \frac{n^{(i)}}{c^i} t^{2(n-i)}$$

then we have

$$E(X^{2n}|X > t) = \frac{\int_t^\infty x^{2n} f(x)dx}{1 - F(t)} = t^{2n} + \frac{\int_t^\infty 2nx^{2n-1}(1 - F(x))dx}{1 - F(t)}. \tag{2.7}$$

Hence, using Lemma 2.2 in Equation (2.7), the "only if" condition easily follows. ∎

Theorem 2.2. *Let X be a nonnegative random variable with absolutely continuous (with respect to Lebesgue measure) distribution function* $F(x)$ *with* $F(0) = 0$ *and* $F(x) > 0$ *for all* $x > 0$ *and finite* $E(X^{2n+1})$ *for some fixed* $n \geqslant 1$*. Then X has a Rayleigh distribution with* $F(x) = 1 - e^{-cx^2}$, $x \geqslant 0$, $c > 0$, *iff*

$$E(X^{2n-1}|X > t) = \sum_{j=0}^{n-1} \frac{(2n-1)!!}{(2n-1-2j)!!(2c)^j} t^{2n-1-2j} + \frac{(2n-1)!!}{(2c)^k} \sqrt{c\pi}(1 - \mathrm{erf}((\sqrt{c})t))e^{ct^2},$$

where $(2n-1)!! = 1.3...(2n-1)$, $n \geqslant 1$.

Proof. It is easy to show that if

$$F(x) = 1 - e^{-cx^2},\ x \geq 0,\ c > 0,$$

then

$$E(X^{2n-1}|X > t) = \sum_{j=0}^{n-1} \frac{(2n-1)!!}{(2n-1-2j)!!(2c)^j} t^{2n-1-2j} + \frac{(2n-1)!!}{(2c)^k} \sqrt{c\pi}(1 - \mathrm{erf}((\sqrt{c})t))e^{ct^2}.$$

We will prove here the "only if" condition. Suppose

$$E(X^{2n-1}|X > t) = \sum_{j=0}^{n-1} \frac{(2n-1)!!}{(2n-1-2j)!!(2c)^j} t^{2n-1-2j} + \frac{(2n-1)!!}{(2c)^k} \sqrt{c\pi}(1 - \mathrm{erf}((\sqrt{c})t))e^{ct^2}.$$

Then we have

$$E(X^{2n+1}|X > t) = \frac{\int_t^\infty x^{2n+1} f(x)dx}{1 - F(t)} dx = t^{2n+1} + \frac{\int_t^\infty (2n+1)x^{2n}(1 - F(x))dx}{1 - F(t)}. \tag{2.8}$$

Using Lemma 2.1 in Equation (2.8), the "only if" condition easily follows. ∎

3. Characterizations based on order statistics

In this section, we establish some results based on order statistics.

Theorem 3.1. *Let X be a nonnegative random variable with absolutely continuous (with respect to Lebesgue measure) distribution function $F(x)$ with $F(0) = 0$ and $F(x) > 0$ for all $x > 0$ and finite $E(X^2)$. Then X has the Rayleigh distribution with $F(x) = 1 - e^{-c\,x^2}, \quad x \geqslant 0$, $c > 0$, iff*

$$E(X_{i,n}^{2m}|X_{i-1,n} = t) = \sum_{j=0}^{m} \frac{m!}{(m-j)!}\left(\frac{1}{c(n-j+1)}\right)^j j^{2(m-j)},$$

for some fixed $n \geq 1,\ m \geq 1,$ where $X_{i,n}$ is the ith order statistics in a sample of size n.

Proof. Suppose that $F(x) = 1 - e^{-cx^2}$, $c > 0$, $x > 0$. Then, it can easily be seen, after integration, that

$$\begin{aligned} E(X_{i,n}^{2m}|X_{i-1,n} = t) &= t^{2m} + \frac{\int_t^\infty 2mx^{m-1}e^{-(n-i+1)cx^2}dx}{e^{-(n-i+1)ct^2}} \\ &= \sum_{j=0}^{m} \frac{m!}{(m-j)!}\left(\frac{1}{c(n-i+1)}\right)^j t^{2(m-j)}. \end{aligned}$$

Suppose that

$$E(X_{i,n}^{2m}|X_{i-1,n} = t) = \sum_{j=0}^{m} \frac{m!}{(m-j)!}\left(\frac{1}{c(n-i+1)}\right)^j t^{2(m-j)}.$$

Since

$$E(X_{i,n}^{2m}|X_{i-1,n} = t) = t^{2m} + \frac{\int_t^\infty 2mx^{m-1}(1-F(x))^{n-i+1}dx}{(1-F(t))^{n-i+1}},$$

therefore, we have

$$\int_t^\infty 2mx^{2m-1}(1-F(x))^{n-i+1}dx = [(1-F(t))^{n-i+1}]\sum_{j=1}^{m} \frac{m!}{(m-j)!}\left(\frac{1}{c(n-i+1)}\right)^j t^{2(m-j)}.$$

The result easily follows from Lemma 2.2. ∎

Remark 3.1. Let X be a nonnegative random variable with absolutely continuous (with respect to Lebesgue measure) distribution function $F(x)$ with $F(0) = 0$ and $F(x) > 0$ for all $x > 0$ and finite $E(X^2)$.Then X has a Rayleigh distribution with $F(x) = 1 - e^{-cx^2}, \quad c > 0,\ x \geqslant 0$, iff $E(X_{i,n}^2|X_{i-1,n} = t) = t^2 + \frac{1}{(n-i+1)c}$, where $X_{i,n}$ is the ith order statistics in a sample of size n.

Proof. The proof easily follows by taking $m = 1$ in Theorem 3.1. ∎

4. Characterization based on record values

In this section, some results based on record values have been established.

4.1. We first prove the following theorem based on conditional expectation.

Theorem 4.1. *Let X be a nonnegative random variable with absolutely continuous (with respect to Lebesgue measure) distribution function* $F(x)$ *with* $F(0) = 0$ *and* $F(x) > 0$ *for all* $x > 0$. *Assume that* $E(X_{U(n+1)})$ *is finite. Then X has a Rayleigh distribution with* $F(x) = 1 - e^{-x^2}, \quad x \geqslant 0$, *iff*

$$E(X_{U(n+1)}|X_{U(n)} = t) = t + \sqrt{\tfrac{\pi}{2}}e^{t^2} - \sqrt{\tfrac{\pi}{2}}e^{t^2}\operatorname{erf}\left(\sqrt{2}t\right), \quad \textit{for some fixed} \quad n \geqslant 1.$$

Proof. Since

$$E(X_{U(n+1)}|X_{U(n)} = t) = t + \frac{\int_t^\infty (1 - F(x))dx}{1 - F(t)}.$$

the proof follows from Theorem 2.2. ∎

Remark 4.1. Since the conditional pdf of $\left(X_{U(n+1)}|X_{U(n)} = t\right)$ is the same as the conditional pdf of $(X|X > t)$, the characterizations (see Ahsanullah [3]) using $E\left(X^{2m}_{U(n+1)}|X_{U(n)} = t\right)$ and $E\left(X^{2m-1}_{U(n+1)}|X_{U(n)} = t\right)$ are same as given in Theorems 2.1 and 2.2, respectively.

4.2. It is well known that if X has a Rayleigh distribution with $F(x) = 1 - e^{-c\,x^2}$, $c > 0$, $x \geqslant 0$, then $Y = X^2$ has an exponential distribution with $F_Y(y) = 1 - e^{-c\,y}$. Further, applying Theorem 8.4.1, page 256, Ahsanullah [3], it is easy to see that, if X be a nonnegative random variable with the Rayleigh distribution, the we have $X^2_{U(n)} \overset{d}{=} X_1^2 + \ldots + X_n^2$, $n \geqslant 1$, where $X_{U(n)}$ is the *nth* upper record and $X_1, \ldots, X_n$ are independent copies of X's. Also see Ahsanullah [4]. Thus, by the definition of the Erlang distribution, it follows that $X^2_{U(n)}$ is distributed as Erlang which is defined as follows: A random variable is said to be distributed as Erlang if its pdf is given by $f_{c,n}(x) = \frac{1}{\Gamma(n)} c^n x^{n-1} e^{-cx}$, where $c > 0$, $x \geqslant 0$, and $n > 0$ is an integer.

Theorem 4.2. *Let X be a nonnegative random variable with absolutely continuous (with respect to Lebesgue measure) distribution function* $F(x)$ *with* $F(0) = 0$ *and* $F(x) > 0$ *for all* $x > 0$ *and finite* $E(X^2)$. *Then X has a Rayleigh distribution with* $F(x) = 1 - e^{-cx^2}$, $c > 0$, $x \geqslant 0$, *iff* $X^2_{U(n)}$ *is distributed as Erlang, for some fixed* $n \geqslant 1$.

Proof. The "if condition" is known. We will prove here the "only if" condition. If $X^2_{U(n)}$ is distributed as Erlang, then

$$\int_0^x \frac{1}{\Gamma(n)}(R(u))^{n-1} f(u)du = \int_0^x \frac{1}{\Gamma(n)}(c)^n u^{n-1} e^{-cu^2} du$$

for all $x \geq 0$.

We can rewrite the above equality as

$$\int_0^{R(x)} \frac{1}{\Gamma(n)} t^{n-1} e^{-t} dt = \int_0^{cu^2} \frac{1}{\Gamma(n)} t^{n-1} e^{-t} dt$$

for all $x \geq 0$.

Thus, we have $R(x) = cx^2$, for almost all x, $x \geq 0$. That is, $F(x) = 1 - e^{-cx^2}$, $c > 0$, $x \geqslant 0$. ∎

Theorem 4.3. *Let X be a nonnegative random variable with absolutely continuous (with respect to Lebesgue measure) distribution function $F(x)$ with $F(0) = 0$ and $F(x) > 0$, for all $x > 0$. Assume that $E(X_{U(n+1)})$ is finite. Then X has a Rayleigh distribution with $F(x) = 1 - e^{-x^2}$, $x \geqslant 0$, iff $E(X_{U(n+1)}|X_{U(n)} = t) = t + \sqrt{\frac{\pi}{2}}e^{t^2} - \sqrt{\frac{\pi}{2}}e^{t^2}\operatorname{erf}\left(\sqrt{2}t\right)$, for some fixed $n \geqslant 1$.*

Proof. Since

$$E(X_{U(n+1)}|X_{U(n)} = t) = t + \frac{\int_t^\infty (1 - F(x))dx}{1 - F(t)},$$

the proof follows from Theorem 2.2. ∎

Remark 4.2. Since the conditional pdf of $\left(X_{U(n+1)}|X_{U(n)} = t\right)$ is $f_{n+1|n}(x|t) = \frac{f(x)}{1-F(t)}$, $x > t$, which is exactly equal to the conditional pdf of $(X|X > t)$, that is, $f_{n+1|n}(x|t) = \frac{f(x)}{1-F(t)}$, $x > t$, see, for example, Ahsanullah [3], the following results related to the characterizations using $E\left(X^{2m}_{U(n+1)}|X_{U(n)} = t\right)$ and $E\left(X^{2m-1}_{U(n+1)}|X_{U(n)} = t\right)$ are same as given in Theorems 2.1 and 2.2 above, that is,

(i) $E(X^{2m}_{U(n+1)}|X_{U(n)} = t) = \sum_{j=0}^{m} m^{(j)}(\frac{1}{c})^j t^{2(m-j)}$. where $m^{(j)} = m(m-1)...(m-j+1)$, $m^{(0)} = 1$, and

(ii) $E(X^{2m-1}_{U(n+1)}|X_{U(n)} = t) = \sum_{j=0}^{m-1} \frac{(2m-1)!!}{(2m-1-2j)!}(\frac{1}{c})^j t^{(2m-1-2j)} + \frac{(2m-1)!!}{(2c)^m}\sqrt{c\pi}(1 - \operatorname{erf}((\sqrt{c})t))e^{ct^2}$.

The proofs of the above remarks (i) and (ii) are similar to the proofs of Theorems 2.1 and 2.2, respectively.

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