

## Almost Periodic Solution of a Population Model: via Spectral Radius of Matrix

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**Abstract.** In the present paper, some new results on the existence and uniqueness of almost periodic solution ( $\omega$ -periodic solution) are obtained for a delayed population model. The method is based on combining matrix's spectral theory with the generalized Banach fixed point theory, which is different from the method employed in the literature. Due to employing the matrix's spectral theory, the existence and stability conditions are given in terms of spectral radius of explicit matrices. The obtained sufficient conditions are much different from the conditions given by the algebraic inequalities. Our new results generalize the previous results in the literature.

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### 1. Introduction

#### 1.1. History

Kirlinger [6] proposed a single species Logarithmic model which takes the form of

$$(1.1) \quad \dot{N}(t) = N(t)[a - b \ln N(t) - c \ln N(t - \tau)].$$

Then Li [7] studied the non-autonomous case of system (1.1). Based on the coincidence degree theory, sufficient conditions are obtained for the existence of positive periodic solutions. Liu [9] generalized system (1.1) to a periodic Logarithmic population model with multispecies:

$$(1.2) \quad \dot{N}_i(t) = N_i(t) \left[ r_i(t) - \sum_{j=1}^n a_{ij}(t) \ln N_j(t) - \sum_{j=1}^n b_{ij}(t) \ln N(t - \tau_{ij}(t)) \right], \quad i = 1, 2, \dots, n.$$

By using coincidence degree theory and constructing Lyapunov functional, he obtained some sufficient conditions which guarantee the existence, uniqueness and stability of the positive periodic solution of the system (1.2).

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On the other hand, ecosystem in the real world is continuously distributed by unpredictable forces which can result in changes in the biological parameters such as survival rates. Of practical interest in ecology is the question of whether or not an ecosystem can withstand those unpredictable disturbances which persist for a finite period of time. In the language of control variables, we call the disturbance functions as control variables. It has been proved that some species can be permanent while the other will go to extinction under certain conditions in the multi-species population systems. In order to search for certain schemes (such as harvesting procedure) to ensure these systems coexist, some feedback control variables were introduced to these systems. (For more details on the feedback controls, one could refer to [5, 11]). Motivated by [1, 3, 11], Wang and Shi [10] considered a multispecies Logarithmic population model with feedback controls as follows.

$$(1.3) \quad \begin{cases} \dot{N}_i(t) = N_i(t) \left[ r_i(t) - \sum_{j=1}^n a_{ij}(t) \ln N_j(t) - \sum_{j=1}^n b_{ij}(t) \ln N_j(t - \tau_{ij}(t)) \right. \\ \quad \left. - \sum_{j=1}^n c_{ij}(t) \int_{-\infty}^t K_{ij}(t-s) \ln N_j(s) ds - d_i(t) u_i(t) - e_i(t) u_i(t - \sigma_i(t)) \right], \\ \dot{u}_i(t) = -\alpha_i(t) u_i(t) + \beta_i(t) \ln N_i(t) + \gamma_i(t) \ln N_i(t - \delta_i(t)), \quad i = 1, 2, \dots, n, \end{cases}$$

where  $u_i$ ,  $i = 1, 2, \dots, n$  denote indirect feedback control variables. Some criteria were established for the existence and stability of the  $\omega$ -periodic solution of system (1.3).

## 1.2. Motivations

Most of works mentioned above considered population systems under the effects of a periodically varying environment by assuming the parameters with periodic coefficients. However, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. If we consider the effects of the environmental factors, the assumption of almost periodicity is more realistic, more important and more general. Therefore, the aim of this paper is to obtain some new and interesting sufficient conditions for the existence and uniqueness of *almost periodic solution* of system (1.3).

## 1.3. Differences from previous work

There are three differences between the present paper and previous work.

- i) This paper considers the existence and uniqueness of *almost periodic solution*, while [10] reported the periodic solution. Note that periodic function is a special case of *almost periodic function*.
- ii) The approaches employed in this paper are based on combining matrix's spectral theory with the generalized Banach fixed point theory, while the method used in [10] is essentially based on the contraction mapping principle.
- iii) Due to employing the matrix's spectral theory, the existence and stability conditions are given in terms of spectral radius of explicit matrices, which are much different from the conditions given by the algebraic inequalities. In fact, to guarantee the existence of  $\omega$ -periodic solution, the authors in [10] make an essential assumption as follows.

**Theorem 1.1.** *If there exists positive real numbers  $h_i, i = 1, 2, \dots, n$  such that*

$$a_{ii}(t) > h_i^{-1} \sum_{j=1, j \neq i}^n h_j a_{ij}(t) + h_i^{-1} \sum_{j=1}^n h_j [b_{ij}(t) + c_{ij}(t)] + T_i(t) + \Delta_i(t), \quad i = 1, 2, \dots, n,$$

*then (1.3) has a unique  $\omega$ -periodic solution, where  $T_i(t) = d_i(t)(\Phi 1)(t)$ ,  $\Delta_i(t) = e_i(t)(\Phi_i 1)(t - \sigma_i(t))$ , and  $\Phi_i$  is defined by*

$$(\Phi_i \ln N_i)(t) = \int_t^{t+\omega} [\beta_i(s) \ln N_i(s) + \gamma_i(s) \ln N_i(s - \delta_i(s))] \frac{\exp\{\int_t^s \alpha_i(\xi) d\xi\}}{(\exp\{\int_0^\omega \alpha_i(\xi) d\xi\} - 1)} ds.$$

Different from Theorem 1.1 in [10], to guarantee the existence of almost periodic solution ( $\omega$ -periodic solution), the essential assumption in this paper is that  $\rho(\mathcal{K}) < 1$  ( $\mathcal{K}$ ) is defined in Theorem 2.1.

### 1.4. Notations and standard assumptions

We use  $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$  to denote a column vector,  $\mathcal{D} = (d_{ij})_{n \times n}$  is an  $n \times n$  matrix,  $\mathcal{D}^T$  denotes the transpose of  $\mathcal{D}$ , and  $E_n$  is the identity matrix of size  $n$ . A matrix or vector  $\mathcal{D} > 0$  means that all entries of  $\mathcal{D}$  are greater than zero, likewise for  $\mathcal{D} \geq 0$ . For matrices or vectors  $\mathcal{D}$  and  $E$ ,  $\mathcal{D} > E$  ( $\mathcal{D} \geq E$ ) means that  $\mathcal{D} - E > 0$  ( $\mathcal{D} - E \geq 0$ ). We also denote the spectral radius of the matrix  $\mathcal{D}$  by  $\rho(\mathcal{D})$ .

If  $v = (v_1, v_2, \dots, v_n)^T \in \mathbb{R}^n$ , then we have a choice of vector norms in  $\mathbb{R}^n$ , for instance  $\|v\|_1$ ,  $\|v\|_2$  and  $\|v\|_\infty$  are the commonly used norms, where

$$\|v\|_1 = \sum_{j=1}^n |v_j|, \quad \|v\|_2 = \left\{ \sum_{j=1}^n |v_j|^2 \right\}^{\frac{1}{2}}, \quad \|v\|_\infty = \max_{1 \leq i \leq n} |v_i|.$$

We recall the following norms of matrices induced by respective vector norms. For instance if  $\mathcal{A} = (a_{ij})_{n \times n}$ , the norm of the matrix  $\|\mathcal{A}\|$  induced by a vector norm  $\|\cdot\|$  is defined by

$$\|\mathcal{A}\| = \sup_{v \in \mathbb{R}^n, v \neq 0} \frac{\|\mathcal{A}v\|}{\|v\|} = \sup_{\|v\|=1} \|\mathcal{A}v\| = \sup_{\|v\| \leq 1} \|\mathcal{A}v\|.$$

In particular one can show that  $\|\mathcal{A}\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|$  (column norm),  $\|\mathcal{A}\|_2 = [\lambda_{\max}(\mathcal{A}^T \mathcal{A})]^{1/2} = [\max. \text{ eigenvalue of } \mathcal{A}^T \mathcal{A}]^{1/2}$ ,  $\|\mathcal{A}\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$  (row norm).

If  $f(t)$  is almost periodic, then

$$m(f) = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T f(t) dt.$$

Throughout this paper, we assume that

- (H<sub>1</sub>)  $r_i(t), a_{ij}(t), b_{ij}(t), c_{ij}(t), \alpha_i(t), \beta_i(t), \gamma_i(t), d_i(t), e_i(t), i, j = 1, 2, \dots, n$  are continuous, real-valued nonnegative almost periodic functions on  $\mathbb{R}$  such that
- (H<sub>2</sub>) Assume that the kernels  $K_{ij}(\cdot), i, j = 1, 2, \dots, n$  are nonnegative continuous functions defined on  $[0, \infty)$  satisfying  $\int_0^{+\infty} K_{ij}(s) ds = 1$ .
- (H<sub>3</sub>)  $\tau_{ij}(t), \sigma_i(t)$  and  $\delta_i(t)$  are nonnegative, continuously differentiable and almost periodic functions on  $t \in \mathbb{R}$ . Moreover,  $\dot{\tau}_{ij}(t), \dot{\sigma}_i(t)$  and  $\dot{\delta}_i(t)$  are all uniformly continuous on  $\mathbb{R}$  with  $\inf_{t \in \mathbb{R}} \{1 - \dot{\tau}_{ij}(t)\} > 0, \inf_{t \in \mathbb{R}} \{1 - \dot{\sigma}_i(t)\} > 0$  and  $\inf_{t \in \mathbb{R}} \{1 - \dot{\delta}_i(t)\} > 0$ .

System (1.3) is supplemented with the initial value conditions:

$$N_i(s) = \varphi_{N_i}(s) \geq 0, \quad s \in (-\infty, 0], \quad \varphi_{N_i}(0) > 0, \quad \sup_{s \in (-\infty, 0]} \varphi_{N_i}(s) < +\infty,$$

$$u_i(s) = \varphi_{u_i}(s) \geq 0, \quad s \in [-\sigma, 0], \quad \sigma = \max_{t \in \mathbb{R}} \sigma_i(t), \quad \varphi_{u_i}(0) > 0, \quad i = 1, 2, \dots, n,$$

where  $\varphi_{N_i}(s)$  and  $\varphi_{u_i}(s)$  denote the real-valued continuous functions defined on  $(-\infty, 0]$  and  $[-\sigma_i, 0]$ , respectively. It is not difficult to see that there exists a positive solution  $z(t) = (N_1(t), N_2(t), \dots, N_n(t), u_1(t), u_2(t), \dots, u_n(t))$  of system (1.3) satisfying the initial value condition.

### 2. Existence and uniqueness

This section is to obtain some new and interesting sufficient conditions for the existence and uniqueness of almost periodic solution ( $\omega$ -periodic solution) for system (1.3). For convenience, we introduce some definitions and lemmas which will be used in the following.

**Definition 2.1.** Let  $f(t) : \mathbb{R} \rightarrow \mathbb{R}^n$  be continuous in  $t$ .  $f(t)$  is said to be almost periodic on  $\mathbb{R}$ , if for any  $\varepsilon > 0$ , the set  $T(f, \varepsilon) = \{\delta : |f(t + \delta) - f(t)| < \varepsilon, \forall t \in \mathbb{R}\}$  is relatively dense, i.e., for  $\forall \varepsilon > 0$ , it is possible to find a real number  $l = l(\varepsilon) > 0$ , for any interval with length  $l(\varepsilon)$ , there exists a number  $\delta = \delta(\varepsilon)$  in this interval such that  $|f(t + \delta) - f(t)| < \varepsilon$ , for  $\forall t \in \mathbb{R}$ , (see [4, 8]).

**Definition 2.2.** Let  $z \in \mathbb{R}^n$  and  $Q(t)$  be a  $n \times n$  continuous matrix defined on  $\mathbb{R}$ . The linear system

$$(2.1) \quad \frac{dz}{dt} = Q(t)z(t)$$

is said to admit an exponential dichotomy on  $\mathbb{R}$  if there exists constants  $k, \lambda > 0$ , projection  $P$  and the fundamental matrix  $Z(t)$  of (2.1) satisfying

$$\|Z(t)PZ^{-1}(s)\| \leq ke^{-\lambda(t-s)}, \quad \text{for } t \geq s, \quad \|Z(t)(I - P)Z^{-1}(s)\| \leq ke^{-\lambda(s-t)}, \quad \text{for } t \leq s.$$

**Lemma 2.1.** If the linear system (2.1) admits an exponential dichotomy, then almost periodic system

$$(2.2) \quad \frac{dz}{dt} = Q(t)z + g(t)$$

has a unique almost periodic solution  $z(t)$ , and

$$z(t) = \int_{-\infty}^t Z(t)PZ^{-1}(s)g(s)ds - \int_t^{+\infty} Z(t)(I - P)Z^{-1}(s)g(s)ds, \quad (\text{see [2, 8]}).$$

**Lemma 2.2.** Let  $a_i(t)$  be an almost periodic function on  $\mathbb{R}$  and  $m(a_i) > 0$ . Then the system

$$\frac{dz}{dt} = \text{diag}(-a_1(t), -a_2(t), \dots, -a_n(t))z(t)$$

admits an exponential dichotomy, (see [2, 8]).

**Remark 2.1.** In this case, the projector  $P = E_n$ .  $E_n$  is the identity matrix of the size  $n$ . Therefore, in this case, system (2.2) has a unique almost periodic solution  $z(t)$  which can be represented as

$$z(t) = \int_{-\infty}^t Z(t)Z^{-1}(s)g(s)ds$$

$$= \left( \int_{-\infty}^t \exp\left(-\int_s^t a_1(u)du\right)g_1(s)ds, \dots, \int_{-\infty}^t \exp\left(-\int_s^t a_n(u)du\right)g_n(s)ds \right)^T.$$

**Lemma 2.3.** Let  $m$  be a positive integer and  $B$  be an Banach space. If the mapping  $\mathcal{T}^m : B \rightarrow B$  is a contraction mapping, then  $\mathcal{T} : B \rightarrow B$  has exactly one fixed point in  $B$ , where  $\mathcal{T}^m = \mathcal{T}(\mathcal{T}^{m-1})$ .

In view of  $(H_1)$ ,  $m(\alpha_i) > 0$ . By Remark 2.1, the following preliminary result follows immediately.

**Lemma 2.4.**  $(N_1(t), \dots, N_n(t), u_1(t), \dots, u_n(t))^T$  is an almost periodic solution of system (1.3) if and only if it is an almost periodic solution of

$$(2.3) \quad \begin{cases} \dot{N}_i(t) = N_i(t) \left[ r_i(t) - \sum_{j=1}^n a_{ij}(t) \ln N_j(t) - \sum_{j=1}^n b_{ij}(t) \ln N_j(t - \tau_{ij}(t)) \right. \\ \quad \left. - \sum_{j=1}^n c_{ij}(t) \int_{-\infty}^t K_{ij}(t-s) \ln N_j(s) ds - d_i(t)u_i(t) - e_i(t)u_i(t - \sigma_i(t)) \right], \\ u_i(t) = \int_{-\infty}^t \exp\left(-\int_s^t \alpha_i(\xi)d\xi\right) \left[ \beta_i(s) \ln N_i(s) + \gamma_i(s) \ln N_i(s - \delta_i(s)) \right] ds, \\ \quad := (\Phi_i \ln N_i)(t), \quad i = 1, 2, \dots, n. \end{cases}$$

**Remark 2.2.** By Lemma 2.4, the existence problem of almost periodic solutions of (1.3) is equivalent to the existence problem of almost periodic solutions of the following system

$$(2.4) \quad \begin{aligned} \dot{N}_i(t) = N_i(t) & \left[ r_i(t) - \sum_{j=1}^n a_{ij}(t) \ln N_j(t) - \sum_{j=1}^n b_{ij}(t) \ln N_j(t - \tau_{ij}(t)) \right. \\ & - \sum_{j=1}^n c_{ij}(t) \int_{-\infty}^t K_{ij}(t-s) \ln N_j(s) ds \\ & \left. - d_i(t)(\Phi_i \ln N_i)(t) - e_i(t)(\Phi_i \ln N_i)(t - \sigma_i(t)) \right], \quad i = 1, 2, \dots, n. \end{aligned}$$

In what follows, we use the following notations:

$$T_i(t) = (\Phi_i 1)(t), \quad \Delta_i(t) = (\Phi_i 1)(t - \sigma_i(t)).$$

Now we are in a position to state our main results on the existence and uniqueness of almost periodic solution for system (1.3).

**Theorem 2.1.** In addition to  $(H_1)$ – $(H_3)$ , if we further suppose that

$(H_4)$   $\rho(\mathcal{K}) < 1$ , where  $\mathcal{K} = (\Gamma_{ij})_{n \times n}$ ,  $\Gamma_{ii}$  and  $\Gamma_{ij}(i \neq j)$  are defined by

$$\Gamma_{ii} = \int_{-\infty}^t \exp\left(-\int_s^t a_{ii}(\xi)d\xi\right) \vartheta_{ii}(s)ds, \quad i = 1, 2, \dots, n,$$

$$\Gamma_{ij} = \int_{-\infty}^t \exp\left(-\int_s^t a_{ii}(\xi)d\xi\right) \vartheta_{ij}(s)ds, \quad i \neq j, \quad i, j = 1, 2, \dots, n,$$

$$\begin{aligned} \vartheta_{ii}(s) &= b_{ii}(s) + c_{ii}(s) + T_i(s) + \Delta_i(s), \quad i = 1, 2, \dots, n, \\ \vartheta_{ij}(s) &= a_{ij}(s) + b_{ij}(s) + c_{ij}(s), \quad i \neq j \quad i, j = 1, 2, \dots, n. \end{aligned}$$

Then system (1.3) has a unique positive almost periodic solution.

*Proof.* Making the change of variables

$$(2.5) \quad N_i(t) = \exp\{x_i(t)\}, \quad i = 1, 2, \dots, n.$$

Then system (2.4) can be written as

$$(2.6) \quad \begin{aligned} \dot{x}_i(t) &= -a_{ii}(t)x_i(t) - \sum_{j=1, j \neq i}^n a_{ij}(t)x_j(t) - \sum_{j=1}^n b_{ij}(t)x_j(t - \tau_{ij}(t)) \\ &\quad - \sum_{j=1}^n c_{ij}(t) \int_{-\infty}^t K_{ij}(t-s)x_j(s)ds \\ &\quad - d_i(t)(\Phi_i x_i)(t) - e_i(t)(\Phi_i x_i)(t - \sigma_i(t)) + r_i(t), \quad i = 1, 2, \dots, n. \end{aligned}$$

One would see that if system (2.6) has an almost periodic solution  $(x_1^*(t), x_2^*(t), \dots, x_n^*(t))^T$  then  $(N_1^*(t), N_2^*(t), \dots, N_n^*(t))^T = (e^{x_1^*(t)}, e^{x_2^*(t)}, \dots, e^{x_n^*(t)})^T$  is an almost periodic solution of (2.4). By Lemma 2.4,  $(e^{x_1^*(t)}, e^{x_2^*(t)}, \dots, e^{x_n^*(t)}, u_1^*(t), u_2^*(t), \dots, u_n^*(t))^T$  is an almost periodic solution of (1.3), where  $u_i^*(t) = \int_{-\infty}^t \exp(-\int_s^t \alpha_i(\xi)d\xi) [\beta_i(s)x_i^*(s) + \gamma_i(s)x_i^*(s - \delta_i(s))] ds$ .

Therefore, to prove Theorem 2.1, it suffices to show that system (2.6) has a unique almost periodic solution. To this end, we define

$$B = \left\{ \phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t))^T \mid \phi(t) \text{ is a continuous almost periodic function} \right\}.$$

Obviously,  $B$  is a Banach space with the norm  $\|\phi\| = \max_{1 \leq i \leq n} \sup_{t \in \mathbb{R}} |x_i(t)|$ .

For any  $\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t))^T \in B$ , we consider the following almost periodic system

$$(2.7) \quad \begin{aligned} \dot{x}_i(t) &= -a_{ii}(t)x_i(t) - \sum_{j=1, j \neq i}^n a_{ij}(t)\phi_j(t) - \sum_{j=1}^n b_{ij}(t)\phi_j(t - \tau_{ij}(t)) \\ &\quad - \sum_{j=1}^n c_{ij}(t) \int_{-\infty}^t K_{ij}(t-s)\phi_j(s)ds \\ &\quad - d_i(t)(\Phi_i \phi_i)(t) - e_i(t)(\Phi_i \phi_i)(t - \sigma_i(t)) + r_i(t), \quad i = 1, 2, \dots, n. \end{aligned}$$

In view of  $(H_1)$ ,  $m(a_{ii}) > 0$ . By Lemma 2.2, the linear part of system (2.7)

$$(2.8) \quad \dot{x}_i(t) = -a_{ii}(t)x_i(t), \quad i = 1, 2, \dots, n$$

possesses an exponential dichotomy. Therefore, from Remark 2.1, system (2.7) has a unique almost periodic solution  $x^\phi(t)$ , which can be represented as

$$(2.9) \quad x^\phi(t) = (x_1^\phi(t), x_2^\phi(t), \dots, x_n^\phi(t))^T = \begin{pmatrix} \int_{-\infty}^t \exp(-\int_s^t a_{11}(\xi)d\xi) g_1^\phi(s) ds \\ \int_{-\infty}^t \exp(-\int_s^t a_{22}(\xi)d\xi) g_2^\phi(s) ds \\ \dots \\ \int_{-\infty}^t \exp(-\int_s^t a_{nn}(\xi)d\xi) g_n^\phi(s) ds \end{pmatrix}$$

where

$$(2.10) \quad \begin{aligned} g_i^\phi(s) = & - \sum_{j=1, j \neq i}^n a_{ij}(s)\phi_j(s) - \sum_{j=1}^n b_{ij}(s)\phi_j(s - \tau_{ij}(s)) \\ & - \sum_{j=1}^n c_{ij}(s) \int_{-\infty}^s K_{ij}(s - \nu)\phi_j(\nu) d\nu \\ & - d_i(s)(\Phi_i\phi_i)(s) - e_i(s)(\Phi_i\phi_i)(s - \sigma_i(s)) + r_i(s), \quad i = 1, 2, \dots, n. \end{aligned}$$

Now define a mapping  $\mathcal{T} : B \rightarrow B$  by setting

$$\mathcal{T}\phi(t) = Z^\phi(t), \quad \text{for any } \phi \in B.$$

For any  $\phi, \psi \in B$ , it follows from (2.9) that

$$(2.11) \quad \begin{aligned} & |(\mathcal{T}(\phi) - \mathcal{T}(\psi))| \\ & = \left( |(\mathcal{T}(\phi(t)) - \mathcal{T}(\psi(t)))_1|, |(\mathcal{T}(\phi(t)) - \mathcal{T}(\psi(t)))_2|, \dots, |(\mathcal{T}(\phi(t)) - \mathcal{T}(\psi(t)))_n| \right)^T \\ & \leq \left( \begin{array}{l} \int_{-\infty}^t \exp\left(-\int_s^t a_{11}(\xi)d\xi\right) |g_1^\phi(s) - g_1^\psi(s)| ds \\ \int_{-\infty}^t \exp\left(-\int_s^t a_{22}(\xi)d\xi\right) |g_2^\phi(s) - g_2^\psi(s)| ds \\ \dots, \\ \int_{-\infty}^t \exp\left(-\int_s^t a_{nn}(\xi)d\xi\right) |g_n^\phi(s) - g_n^\psi(s)| ds \end{array} \right) \end{aligned}$$

On the other hand, it follows from (2.10) that

$$\begin{aligned} & |g_i^\phi(s) - g_i^\psi(s)| \\ & \leq \sum_{j=1, j \neq i}^n a_{ij}(s) |\phi_j(s) - \psi_j(s)| + \sum_{j=1}^n b_{ij}(s) |\phi_j(s - \tau_{ij}(s)) - \psi_j(s - \tau_{ij}(s))| \\ & \quad + \sum_{j=1}^n c_{ij}(s) \int_{-\infty}^s K_{ij}(s - \nu) |\phi_j(\nu) - \psi_j(\nu)| d\nu \\ & \quad + d_i(s) |(\Phi_i\phi_i)(s) - (\Phi_i\psi_i)(s)| + e_i(s) |(\Phi_i\phi_i)(s - \sigma_i(s)) - (\Phi_i\psi_i)(s - \sigma_i(s))| \\ & \leq \sum_{j=1, j \neq i}^n a_{ij}(s) \sup_{t \in \mathbb{R}} |\phi_j(s) - \psi_j(s)| + \sum_{j=1}^n b_{ij}(s) \sup_{t \in \mathbb{R}} |\phi_j(t) - \psi_j(t)| \\ & \quad + \sum_{j=1}^n c_{ij}(s) \int_{-\infty}^s K_{ij}(s - \nu) \sup_{t \in \mathbb{R}} |\phi_j(t) - \psi_j(t)| d\nu \\ & \quad + \int_{-\infty}^s \exp\left(-\int_\tau^s \alpha_i(\xi)d\xi\right) \times \left[ \beta_i(\tau) \sup_{t \in \mathbb{R}} |\phi_j(t) - \psi_j(t)| + \gamma_i(\tau) \sup_{t \in \mathbb{R}} |\phi_j(t) - \psi_j(t)| \right] d\tau \\ & \quad + \int_{-\infty}^{s - \delta_i(s)} \exp\left(-\int_\tau^{s - \delta_i(s)} \alpha_i(\xi)d\xi\right) \times \left[ \beta_i(\tau) \sup_{t \in \mathbb{R}} |\phi_i(t) - \psi_i(t)| + \gamma_i(\tau) \sup_{t \in \mathbb{R}} |\phi_i(t) - \psi_i(t)| \right] d\tau \\ & = \left[ \sum_{j=1, j \neq i}^n a_{ij}(s) + \sum_{j=1}^n b_{ij}(s) + \sum_{j=1}^n c_{ij}(s) \int_0^\infty K_{ij}(u) du \right] \sup_{t \in \mathbb{R}} |\phi_j(t) - \psi_j(t)| \\ & \quad + (\Phi_i 1)(s) \sup_{t \in \mathbb{R}} |\phi_i(t) - \psi_i(t)| + (\Phi_i 1)(s - \delta_i(s)) \sup_{t \in \mathbb{R}} |\phi_i(t) - \psi_i(t)| \end{aligned}$$

$$\begin{aligned}
 &= \left[ \sum_{j=1, j \neq i}^n a_{ij}(s) + \sum_{j=1}^n b_{ij}(s) + \sum_{j=1}^n c_{ij}(s) + T_i(s) + \Delta_i(s) \right] \sup_{t \in \mathbb{R}} |\phi_j(t) - \psi_j(t)| \\
 &= \left[ b_{ii}(s) + c_{ii}(s) + T_i(s) + \Delta_i(s) \right] \sup_{t \in \mathbb{R}} |\phi_i(t) - \psi_i(t)| \\
 &\quad + \sum_{j=1, j \neq i}^n \left[ a_{ij}(s) + b_{ij}(s) + c_{ij}(s) \right] \sup_{t \in \mathbb{R}} |\phi_j(t) - \psi_j(t)| \\
 &= \vartheta_{ii}(s) \sup_{t \in \mathbb{R}} |\phi_i(t) - \psi_i(t)| + \sum_{j=1, j \neq i}^n \vartheta_{ij}(s) \cdot \sup_{t \in \mathbb{R}} |\phi_j(t) - \psi_j(t)|,
 \end{aligned}$$

where  $\vartheta_{ii}(s), i = 1, 2, \dots, n$  and  $\vartheta_{ij}(s) (i \neq j), i, j = 1, 2, \dots, n$  are defined in Theorem 2.1. Thus, the above inequality implies,

$$\begin{aligned}
 &\int_{-\infty}^t \exp\left(-\int_s^t a_{ii}(\xi) d\xi\right) |g_i^\phi(s) - g_i^\psi(s)| ds \\
 &\leq \int_{-\infty}^t \exp\left(-\int_s^t a_{ii}(\xi) d\xi\right) \vartheta_{ii}(s) ds \cdot \sup_{t \in \mathbb{R}} |\phi_i(t) - \psi_i(t)| \\
 (2.12) \quad &\quad + \sum_{j=1, j \neq i}^n \int_{-\infty}^t \exp\left(-\int_s^t a_{ii}(\xi) d\xi\right) \vartheta_{ij}(s) ds \cdot \sup_{t \in \mathbb{R}} |\phi_j(t) - \psi_j(t)| \\
 &= \Gamma_{ii} \sup_{t \in \mathbb{R}} |\phi_i(t) - \psi_i(t)| + \sum_{j=1, j \neq i}^n \Gamma_{ij} \sup_{t \in \mathbb{R}} |\phi_j(t) - \psi_j(t)|,
 \end{aligned}$$

where  $\Gamma_{ii}, i = 1, 2, \dots, n$  and  $\Gamma_{ij} (i \neq j), i, j = 1, 2, \dots, n$  are defined in Theorem 2.1. Therefore, it follows from (2.11) and (2.12) that

$$\begin{aligned}
 & \left( |(\mathcal{T}(\phi(t)) - \mathcal{T}(\psi(t)))_1|, |(\mathcal{T}(\phi(t)) - \mathcal{T}(\psi(t)))_2|, \dots, |(\mathcal{T}(\phi(t)) - \mathcal{T}(\psi(t)))_n| \right)^T \\
 & \leq \begin{pmatrix} \Gamma_{11} \sup_{t \in \mathbb{R}} |\phi_1(t) - \psi_1(t)| + \sum_{j=1, j \neq 1}^n \Gamma_{1j} \sup_{t \in \mathbb{R}} |\phi_j(t) - \psi_j(t)| \\ \Gamma_{22} \sup_{t \in \mathbb{R}} |\phi_2(t) - \psi_2(t)| + \sum_{j=1, j \neq 2}^n \Gamma_{2j} \sup_{t \in \mathbb{R}} |\phi_j(t) - \psi_j(t)| \\ \dots \\ \Gamma_{nn} \sup_{t \in \mathbb{R}} |\phi_n(t) - \psi_n(t)| + \sum_{j=1, j \neq n}^n \Gamma_{nj} \sup_{t \in \mathbb{R}} |\phi_j(t) - \psi_j(t)| \end{pmatrix} \\
 & = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \dots & \Gamma_{1n} \\ \Gamma_{21} & \Gamma_{22} & \dots & \Gamma_{2n} \\ \dots & \dots & \dots & \dots \\ \Gamma_{n1} & \Gamma_{n2} & \dots & \Gamma_{nn} \end{pmatrix}_{n \times n} \begin{pmatrix} \sup_{t \in \mathbb{R}} |\phi_1(t) - \psi_1(t)| \\ \sup_{t \in \mathbb{R}} |\phi_2(t) - \psi_2(t)| \\ \dots \\ \sup_{t \in \mathbb{R}} |\phi_n(t) - \psi_n(t)| \end{pmatrix}_{n \times 1} \\
 & = \mathcal{H} \left( \sup_{t \in \mathbb{R}} |\phi_1(t) - \psi_1(t)|, \sup_{t \in \mathbb{R}} |\phi_2(t) - \psi_2(t)|, \dots, \sup_{t \in \mathbb{R}} |\phi_n(t) - \psi_n(t)| \right)^T \\
 & = \mathcal{H} \left( \sup_{t \in \mathbb{R}} |(\phi(t) - \psi(t))_1|, \sup_{t \in \mathbb{R}} |(\phi(t) - \psi(t))_2|, \dots, \sup_{t \in \mathbb{R}} |(\phi(t) - \psi(t))_n| \right)^T.
 \end{aligned}$$



Note that (2.13) holds for all  $t \in \mathbb{R}$ . Consequently, we have

$$(2.14) \quad \begin{pmatrix} \sup_{t \in \mathbb{R}} |(\mathcal{T}(\phi(t)) - \mathcal{T}(\psi(t)))_1| \\ \sup_{t \in \mathbb{R}} |(\mathcal{T}(\phi(t)) - \mathcal{T}(\psi(t)))_2| \\ \dots \\ \sup_{t \in \mathbb{R}} |(\mathcal{T}(\phi(t)) - \mathcal{T}(\psi(t)))_n| \end{pmatrix} \leq \mathcal{K} \begin{pmatrix} \sup_{t \in \mathbb{R}} |(\phi(t) - \psi(t))_1| \\ \sup_{t \in \mathbb{R}} |(\phi(t) - \psi(t))_2| \\ \dots \\ \sup_{t \in \mathbb{R}} |(\phi(t) - \psi(t))_n| \end{pmatrix}.$$

Let  $m$  be a positive integer. Then it follows from (2.14) that

$$(2.15) \quad \begin{aligned} & \begin{pmatrix} \sup_{t \in \mathbb{R}} |(\mathcal{T}^m(\phi(t)) - \mathcal{T}^m(\psi(t)))_1| \\ \sup_{t \in \mathbb{R}} |(\mathcal{T}^m(\phi(t)) - \mathcal{T}^m(\psi(t)))_2| \\ \dots \\ \sup_{t \in \mathbb{R}} |(\mathcal{T}^m(\phi(t)) - \mathcal{T}^m(\psi(t)))_n| \end{pmatrix} \\ &= \begin{pmatrix} \sup_{t \in \mathbb{R}} |(\mathcal{T}(\mathcal{T}^{m-1}(\phi(t))) - \mathcal{T}(\mathcal{T}^{m-1}(\psi(t))))_1| \\ \sup_{t \in \mathbb{R}} |(\mathcal{T}(\mathcal{T}^{m-1}(\phi(t))) - \mathcal{T}(\mathcal{T}^{m-1}(\psi(t))))_2| \\ \dots \\ \sup_{t \in \mathbb{R}} |(\mathcal{T}(\mathcal{T}^{m-1}(\phi(t))) - \mathcal{T}(\mathcal{T}^{m-1}(\psi(t))))_n| \end{pmatrix} \\ &\leq \mathcal{K} \begin{pmatrix} \sup_{t \in \mathbb{R}} |(\mathcal{T}^{m-1}(\phi(t)) - \mathcal{T}^{m-1}(\psi(t)))_1| \\ \sup_{t \in \mathbb{R}} |(\mathcal{T}^{m-1}(\phi(t)) - \mathcal{T}^{m-1}(\psi(t)))_2| \\ \dots \\ \sup_{t \in \mathbb{R}} |(\mathcal{T}^{m-1}(\phi(t)) - \mathcal{T}^{m-1}(\psi(t)))_n| \end{pmatrix} \leq \dots \\ &\leq \mathcal{K}^m \begin{pmatrix} \sup_{t \in \mathbb{R}} |(\phi(t) - \psi(t))_1| \\ \sup_{t \in \mathbb{R}} |(\phi(t) - \psi(t))_2| \\ \dots \\ \sup_{t \in \mathbb{R}} |(\phi(t) - \psi(t))_n| \end{pmatrix}. \end{aligned}$$

since  $\rho(\mathcal{K}) < 1$ , we obtain

$$\lim_{m \rightarrow +\infty} \mathcal{K}^m = 0$$

which implies that there exists a positive integer  $\tilde{N}$  and a positive constant  $r_0 < 1$  such that

$$(2.16) \quad \mathcal{K}^{\tilde{N}} = (h_{ij})_{n \times n}, \quad \text{and} \quad \sum_{j=1}^n h_{ij} \leq r_0, \quad i = 1, 2, \dots, n.$$

In view of (2.15) and (2.16), we have

$$\begin{aligned} |(\mathcal{T}^{\tilde{N}}(\phi) - \mathcal{T}^{\tilde{N}}(\psi))_i| &\leq \sum_{j=1}^n h_{ij} \sup_{t \in \mathbb{R}} |\phi_j(t) - \psi_j(t)| \\ &\leq \max_{1 \leq i \leq n} \sup_{t \in \mathbb{R}} |\phi_i(t) - \psi_i(t)| \sum_{j=1}^n h_{ij} \leq r_0 \|\phi - \psi\| \end{aligned}$$

for all  $i = 1, 2, \dots, n$ . It follows that

$$\|\mathcal{T}^{\tilde{N}}(\phi) - \mathcal{T}^{\tilde{N}}(\psi)\| = \max_{1 \leq i \leq n} |(\mathcal{T}^{\tilde{N}}(\phi) - \mathcal{T}^{\tilde{N}}(\psi))_i| \leq r\|u - v\|.$$

This implies the mapping  $\mathcal{T}^{\tilde{N}} : B \rightarrow B$  is a contraction mapping. By Lemma 2.3,  $\mathcal{T}$  has a unique fixed point  $x^*(t)$  in  $B$ . Thus, system (2.6) has a unique almost periodic solution  $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))^T$ , then  $(N_1^*(t), N_2^*(t), \dots, N_n^*(t))^T = (e^{x_1^*(t)}, e^{x_2^*(t)}, \dots, e^{x_n^*(t)})^T$  is the unique almost periodic solution of (2.4). Therefore, by Lemma 2.4,  $(e^{x_1^*(t)}, e^{x_2^*(t)}, \dots, e^{x_n^*(t)}, u_1^*(t), u_2^*(t), \dots, u_n^*(t))^T$  is the unique almost periodic solution of (1.3). The proof of Theorem 2.1 is complete. ■

The next result is concerned with the existence and uniqueness of  $\omega$ -periodic solution of system (1.3). To this end, we assume that

- (H5)  $r_i(t), a_{ij}(t), b_{ij}(t), c_{ij}(t), \alpha_i(t), \beta_i(t), \gamma_i(t), d_i(t), e_i(t), i, j = 1, 2, \dots, n$  are continuous, real-valued, nonnegative  $\omega$ -periodic functions on  $\mathbb{R}$  such that  $\int_0^\omega a_{ii}(t)dt > 0, \int_0^\omega \alpha_i(t)dt > 0$ .

**Theorem 2.2.** *Suppose that (H2)–(H5) hold, system (1.3) has a unique positive  $\omega$ -periodic solution.*

*Sketch of Proof.* By Lemma 2.4 and (2.6), we see that to prove Theorem 2.2, it suffices to show that system (2.6) has a unique  $\omega$ -periodic solution. To this end, we define

$$\tilde{B} = \left\{ \phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t))^T \mid \phi(t) \text{ is a continuous and } \phi_i(t + \omega) = \phi_i(t) \right\}.$$

Obviously,  $\tilde{B}$  is a Banach space with the norm  $\|\phi\| = \max_{1 \leq i \leq n} \sup_{t \in [0, \omega]} |\phi_i(t)|$ .

For any  $\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t))^T \in \tilde{B}$ , we consider the following  $\omega$ -periodic system

$$\begin{aligned} \dot{x}_i(t) = & -a_{ii}(t)x_i(t) - \sum_{j=1, j \neq i}^n a_{ij}(t)\phi_j(t) - \sum_{j=1}^n b_{ij}(t)\phi_j(t - \tau_{ij}(t)) \\ & - \sum_{j=1}^n c_{ij}(t) \int_{-\infty}^t K_{ij}(t-s)\phi_j(s)ds \\ & - d_i(t)(\Phi_i\phi_i)(t) - e_i(t)(\Phi_i\phi_i)(t - \sigma_i(t)) + r_i(t), \quad i = 1, 2, \dots, n. \end{aligned} \tag{2.17}$$

In view of (H5),  $m(a_{ii}) > 0$ . By Lemma 2.2, the linear part of system (2.17)

$$\dot{x}_i(t) = -a_{ii}(t)x_i(t), \quad i = 1, 2, \dots, n \tag{2.18}$$

possesses an exponential dichotomy. Therefore, from Remark 2.1, system (2.17) has a unique  $\omega$ -periodic solution  $x^\phi(t)$ , which can be represented as

$$x^\phi(t) = (x_1^\phi(t), x_2^\phi(t), \dots, x_n^\phi(t))^T = \begin{pmatrix} \int_{-\infty}^t \exp\left(-\int_s^t a_{11}(\xi)d\xi\right)g_1^\phi(s)ds \\ \int_{-\infty}^t \exp\left(-\int_s^t a_{22}(\xi)d\xi\right)g_2^\phi(s)ds \\ \dots \\ \int_{-\infty}^t \exp\left(-\int_s^t a_{nn}(\xi)d\xi\right)g_n^\phi(s)ds \end{pmatrix}$$

where  $g_i^\phi(s)$  is defined in (2.10). Define a mapping  $\mathcal{T} : \tilde{B} \rightarrow \tilde{B}$  by setting

$$\mathcal{T}\phi(t) = Z^\phi(t), \quad \text{for any } \phi \in \tilde{B}.$$

The rest of proof is similar to that of Theorem 2.1. ■

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