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The Independence Number of $\Gamma(\mathbb{Z}_{p^n}(x))$

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Abstract. The zero-divisor graph of a commutative ring with unity (say R) is a graph whose vertices are the nonzero zero-divisors of this ring, where two distinct vertices are adjacent when their product is zero. This graph is denoted by $\Gamma(R)$. In this paper, we study the structure of the zero-divisor graph $\Gamma(\mathbb{Z}_{p^n}(x))$ where p is an *odd* prime number, $\mathbb{Z}_{p^n}$ is the set of integers modulo p^n, and $\mathbb{Z}_{p^n}(x) = \{a+bx : a,b \in \mathbb{Z}_{p^n}$ and $x^2 = 0\}$. We find the Independence number of $\Gamma(\mathbb{Z}_{p^n}(x))$.

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1. Introduction

Throughout this paper, all rings are commutative with identity. For a ring R, let $R(x) = R[X]/(X^2)$, where $x = \overline{X}$ and X is an indeterminant over R. Clearly, $R(x) = \{a+bx \,|\, x^2 = 0, x \notin R\}$. For a ring R, let $Z(R)$ be the set of zero-divisors of R and let $Z^*(R) = Z(R) \setminus \{0\}$. Zero-divisor graph was first introduced by Beck [7], where Beck was mainly interested in graph coloring. In his work, for a ring R, $Z(R)$ was taken to be the vertex set and distinct vertices x and y are adjacent if $xy = 0$. In a subsequent work, Anderson and Livingston [5] modified the definition. In their definition, the vertex set is taken to be $Z^*(R)$ and distinct vertices x and y are adjacent if $xy = 0$. The zero-divisor graph of R, $\Gamma(Z^*(R))$, is usually written $\Gamma(R)$. The definition of zero-divisor graph given by Anderson and Livingston is the one that has been used in the recent literature. In this paper we will also use their definition. Much work has been done on zero-divisor graphs, and the reader is advised to consult [2, 5, 6, 10] for more details. Some researchers generalized the idea to commutative semigroups, see [8, 9]. Others worked on the noncommutative case, where they studied a directed graph related to the zero-divisors of noncommutative rings. For more information see [12, 13, 15]. Recently, some researchers worked on graphs whose vertex set is the set of all non-zero and non-unit elements, see [3, 4].

An independent set in a graph Γ is a subset I of the vertex set of Γ such that no two vertices of I are adjacent, i.e., the induced subgraph on I is discrete. The independence

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number of Γ, denoted by $Indep(\Gamma)$, is defined as the maximum of the set of cardinalities of independent sets of vertices of Γ. In this paper, we study the structure of the graph $\Gamma(\mathbb{Z}_{p^n}(x))$, where p is an *odd* prime number, $\mathbb{Z}_{p^n}$ is the set of integers modulo p^n. We find the independence number of $\Gamma(\mathbb{Z}_{p^n}(x))$.

2. The structure of $\Gamma(\mathbb{Z}_{p^n}(x))$

The goal of this section is to describe the adjacency rules among vertices of $\Gamma(\mathbb{Z}_{p^n}(x))$. Part of this was given in [1], [11] and [14]. The following lemma characterizes the zero-divisors of $R(x)$. The proof of this lemma is easy and direct, thus it will be omitted here.

Lemma 2.1. *Let R be a commutative ring. Then $a+bx$ is a zero-divisor in $R(x)$ if and only if a is a zero-divisor in R.*

According to Lemma 2.1, one can characterize the zero-divisors of $\mathbb{Z}_{p^n}(x)$ by finding the zero-divisors of $\mathbb{Z}_{p^n}$. But the zero-divisors of $\mathbb{Z}_{p^n}$ are the set of all integers in $\mathbb{Z}_{p^n}$ that are divisible by p. Hence the set of the zero-divisors of $\mathbb{Z}_{p^n}(x)$ is $\{a+bx : a,b \in \mathbb{Z}_{p^n} \text{ and } \gcd(a,p) > 1\}$. We want to discuss how the vertices of $\Gamma(\mathbb{Z}_{p^n}(x))$ are adjacent. To do that we divide the set of vertices of $\Gamma(\mathbb{Z}_{p^n}(x))$ into three types. The first type consists of the $n-1$ sets: $S_p, S_{p^2}, \ldots, S_{p^{n-1}}$ where $S_{p^i} = \{sp^i : \gcd(s,p^{n-i}) = 1\}$. The use of Euler's phi-function gives the sizes of the S_{p^i}'s and one will get $|S_{p^i}| = p^{n-i} - p^{n-(i+1)}$ for $1 \le i \le n-1$. Note that S_{p^0} is the set of units in $\mathbb{Z}_{p^n}$, i.e. the set of elements in $\mathbb{Z}_{p^n}$ that are not divisible by p. The second type consists of n sets: $E_{p^0}, E_{p^1}, \ldots, E_{p^{n-1}}$ where $E_{p^i} = \{bx : b \in S_{p^i}\}$. We have $|E_{p^i}| = |S_{p^i}| = p^{n-i} - p^{n-(i+1)}$ for $1 \le i \le n-1$ and $|E_{p^0}| = p^n - p^{n-1}$. The third type consists of $n(n-1)$ sets and these sets are: S_{p^i,p^j}, $i \in \{1,2,\ldots,n-1\}$ and $j \in \{0,1,\ldots,n-1\}$ where $S_{p^i,p^j} = \{sp^i + tp^jx : \gcd(s,p^{n-i}) = 1 \text{ and } \gcd(t,p^{n-j}) = 1\}$. Again the use of Euler's phi-function gives the sizes of the S_{p^i,p^j}'s and one will get $|S_{p^i,p^j}| = (p^{n-i} - p^{n-(i+1)})(p^{n-j} - p^{n-(j+1)})$ for $1 \le i \le n-1$ and $0 \le j \le n-1$.

We state how the elements of these types are adjacent. Each element of S_{p^i} is adjacent to all the elements of $S_{p^{i'}}$ if $i+i' \ge n$. Also, each element of S_{p^i} is adjacent to all the elements of $E_{p^{i'}}$ if $i+i' \ge n$. Again, each element of S_{p^i} is adjacent to all the elements of $S_{p^{i'},p^{j'}}$ if $i+i' \ge n$ and $i+j' \ge n$. For the E_{p^i}'s, each element of E_{p^i} is adjacent to all the elements of $E_{p^{i'}}$ for any i' and hence $\bigcup_{i=0}^{n-1} E_{p^i}$ forms a complete subgraph of $\Gamma(\mathbb{Z}_{p^n}(x))$. Again, each element of E_{p^i} is adjacent to all the elements of $S_{p^{i'},p^{j'}}$ if $i+i' \ge n$. Observe that no conditions are required on j'.

Finally, we want to discuss how the elements of the third type are adjacent to each other. To do that, consider the two sets S_{p^i,p^j} and S_{p^k,p^m}. Observe that a necessary condition in order that some elements in the set S_{p^i,p^j} are adjacent to some elements in the set S_{p^k,p^m} is that $i+k \ge n$. Now consider the following subcases where $i+k \ge n$.

(1) If $i+m \ge n$ and $k+j \ge n$, then it is clear that each element of S_{p^i,p^j} is adjacent to all the elements of S_{p^k,p^m}.
(2) Suppose that $i+m \ge n$ and $k+j < n$ and take $y_1 \in S_{p^i,p^j}$ and $y_2 \in S_{p^k,p^m}$, say $y_1 = a_1p^i + b_1p^jx$ and $y_2 = a_2p^k + b_2p^mx$ where $\gcd(a_1,p^{n-i}) = \gcd(b_1,p^{n-j}) = \gcd(a_2,p^{n-k}) = \gcd(b_2,p^{n-m}) = 1$. We have $y_1 \cdot y_2 = a_1a_2p^{i+k} + a_1b_2p^{i+m}x + a_2b_1p^{k+j}x = a_2b_1p^{k+j}x \neq 0 \pmod{p^n}$. Hence no element of S_{p^i,p^j} is adjacent

to any element of S_{p^k,p^m}. Similarly, if $i+m<n$ and $k+j\geq n$, then no element of S_{p^i,p^j} is adjacent to any element of S_{p^k,p^m}.

(3) Suppose that $i+m<n$ and $k+j<n$ with $i+m\neq k+j$. We assume that $i+m<k+j$. Take $y_1\in S_{p^i,p^j}$ and $y_2\in S_{p^k,p^m}$, say $y_1=a_1p^i+b_1p^jx$ and $y_2=a_2p^k+b_2p^mx$ where $\gcd(a_1,p^{n-i})=\gcd(b_1,p^{n-j})=\gcd(a_2,p^{n-k})=\gcd(b_2,p^{n-m})=1$. We have $y_1\cdot y_2=a_1a_2p^{i+k}+a_1b_2p^{i+m}x+a_2b_1p^{k+j}x=p^{i+m}(a_1b_2+a_2b_1p^{k+j-i-m})x$ $(\bmod\ p^n)$. If $p^{i+m}(a_1b_2+a_2b_1p^{k+j-i-m})=0\ (\bmod\ p^n)$, then $p^{n-(i+m)}$ divides $(a_1b_2+a_2b_1p^{k+j-i-m})$ and hence p divides a_1b_2. So we get p divides a_1 or p divides b_2. But this is impossible because $\gcd(a_1,p^{n-i})=\gcd(b_2,p^{n-m})=1$. Thus $y_1\cdot y_2\neq 0\ (\bmod\ p^n)$. Hence no element of S_{p^i,p^j} is adjacent to any element of S_{p^k,p^m}. Similarly, if $i+m<n$ and $k+j<n$ with $i+m>k+j$, then no element of S_{p^i,p^j} is adjacent to any element of S_{p^k,p^m}.

(4) Suppose that $i+m<n$ and $k+j<n$ with $i+m=k+j$. In this case, we show that some elements of S_{p^i,p^j} are adjacent to some elements of S_{p^k,p^m}. To explain that, take $y_1\in S_{p^i,p^j}$ and $y_2\in S_{p^k,p^m}$, say $y_1=a_1p^i+b_1p^jx$ and $y_2=a_2p^k+b_2p^mx$ where $\gcd(a_1,p^{n-i})=\gcd(b_1,p^{n-j})=\gcd(a_2,p^{n-k})=\gcd(b_2,p^{n-m})=1$. We have $y_1\cdot y_2=a_1a_2p^{i+k}+a_1b_2p^{i+m}x+a_2b_1p^{k+j}x=p^{i+m}(a_1b_2+a_2b_1)x\ (\bmod\ p^n)$. If $a_1b_2+a_2b_1=0\ (\bmod\ p^{n-i-m})$, then y_1 is adjacent to y_2 and otherwise y_1 and y_2 are not adjacent. For instance, if $a_1=a_2=1$, $b_1=p^{n-i-m}+1$, and $b_2=p^{n-i-m}-1$, then $a_1b_2+a_2b_1=0\ (\bmod\ p^{n-i-m})$ and hence y_1 and y_2 are adjacent. On the other hand, if $a_1=a_2=b_1=b_2=1$, then $a_1b_2+a_2b_1=1\ (\bmod\ p^{n-i-m})$ and hence y_1 and y_2 are not adjacent.

Now, we look more closely in the last subcase. For S_{p^i,p^j} and S_{p^k,p^m}, suppose that $i+k\geq n$ and $i+m=k+j<n$. We decompose S_{p^i,p^j} into $p^{n-(k+j)}-p^{n-(k+j)-1}$ mutually disjoint subsets. These subsets are $X_t=\{rp^i+p^j(sp^{n-k-j}+\overline{rt})x$, where $1\leq r\leq p^{n-i}$ with $\gcd(r,p^{n-i})=1$, $0\leq s\leq p^k-1\}$ and $1\leq t\leq p^{n-(k+j)}$ with $\gcd(t,p^{n-(k+j)})=1$. Observe that the set $\{r:1\leq r\leq p^{n-i}$ and $\gcd(r,p^{n-i})=1\}$ with multiplication is the group $(U(p^{n-i}),\cdot)$ and the set $\{t:1\leq t\leq p^{n-(k+j)}$ and $\gcd(t,p^{n-(k+j)})=1\}$ with multiplication is the group $(U(p^{n-(k+j)}),\cdot)$. The bar on rt means that we are taking rt modulo $p^{n-(k+j)}$. Observe that when t runs over all the elements of $U(p^{n-(k+j)})$ and $r\in U(p^{n-i})$ then $\overline{rt}$ runs over all the elements of $U(p^{n-(k+j)})$. Hence $S_{p^i,p^j}=\bigcup_{t\in U(p^{n-(k+j)})}X_t$ and the size of each X_t is $p^k(p^{n-i}-p^{n-i-1})$. Let X be the set $\{X_t:t\in U(p^{n-(k+j)})\}$. Then we have $|X|=p^{n-(k+j)}-p^{n-(k+j)-1}$.

Similarly, $S_{p^k,p^m}=\bigcup_{t\in U(p^{n-(i+m)})}Y_t$ where $Y_t=\{r'p^k+p^m(s'p^{n-i-m}+\overline{r't})x:\ r'\in U(p^{n-k})$ and $0\leq s'\leq p^i-1\}$ and $|Y_t|=p^i(p^{n-k}-p^{n-k-1})$. Let Y be the set $\{Y_t:t\in U(p^{n-(i+m)})\}$. Then we have $|Y|=p^{n-(i+m)}-p^{n-(i+m)-1}$.

Since $i+m=k+j$, we get $|X|=|Y|$. We want to see when the elements of X_t are adjacent to the elements of $Y_{t'}$ where $t,t'\in U(p^{n-(k+j)})$. Suppose that $y\in X_t$ and $y'\in Y_{t'}$, say $y=rp^i+p^j(sp^{n-(j+k)}+\overline{rt})x$ and $y'=r'p^k+p^m(s'p^{n-(j+k)}+\overline{r't'})x$. We get $y\cdot y'=rr'p^{i+k}+(rs'p^n+r'sp^n+r\overline{r't'}p^{i+m}+r'\overline{rt}p^{k+j})x$. Since $i+k\geq n$ and $i+m=k+j$, we get $y\cdot y'=(r\overline{r't'}+r'\overline{rt})p^{k+j}x=((\overline{r}+up^{n-(k+j)})\overline{r't'}+(\overline{r'}+u'p^{n-(k+j)})\overline{rt})p^{k+j}x=(\overline{rr't'}+\overline{r'rt})p^{k+j}x\ (\bmod\ p^n)=\overline{rr'}(\bar{t}+\overline{t'})p^{k+j}x\ (\bmod\ p^n)$, where u and u' are nonnegative integers. Since $r\in U(p^{n-i})$ and $r'\in U(p^{n-k})$, we get $y\cdot y'=0\ (\bmod\ p^n)$ if and only if $p^{n-(k+j)}$ divides

$t+t'$. Hence, y and y' are adjacent if and only if $p^{n-(k+j)}$ divides $t+t'$. So, each element of X_t is adjacent to all the elements of $Y_{t'}$ if and only if $p^{n-(k+j)}$ divides $t+t'$. Thus each element of X_t is adjacent to all the elements of $Y_{p^{n-(k+j)}-t}$, where $t \in U(p^{n-(k+j)})$.

Let us see how the elements of S_{p^u,p^v} are adjacent to each other when $u \geq \lfloor (n+1)/2 \rfloor$ and $u+v \leq n-1$. As above, $S_{p^u,p^v} = \bigcup_{t \in U(p^{n-(u+v)})} W_t$ with $W_t = \{rp^u + p^v(sp^{n-u-v} + \overline{rt})x$, where $r \in U(p^{n-u})$, $0 \leq s \leq p^u - 1\}$ and $|W_t| = p^i(p^{n-u} - p^{n-v-1})$. As explained above, each element of W_t is adjacent to all the elements of $W_{t'}$ if and only if $p^{n-(u+v)}$ divides $t+t'$. Thus each element of W_t is adjacent to all the elements of $W_{p^{n-(u+v)}-t}$ and no element of W_t is adjacent to any element of $S_{p^u,p^v} - W_{p^{n-(u+v)}-t}$ where $t \in U(p^{n-(u+v)})$. Hence the induced subgraph on each pair W_t, $W_{p^{n-(u+v)}-t}$, where $t \in U(p^{n-(u+v)})$, is isomorphic to the complete bipartite graph $K_{(p^{n-u}-p^{n-u-1})p^u,(p^{n-u}-p^{n-u-1})p^u} = K_{(p-1)p^{n-1},(p-1)p^{n-1}}$. So, the induced subgraph on S_{p^u,p^v} is isomorphic to $(p^{n-(u+v)} - p^{n-(u+v)-1})/2$ disjoint copies of $K_{(p-1)p^{n-1},(p-1)p^{n-1}}$. We summarize this in the following lemma.

Lemma 2.2. *For S_{p^i,p^j} and S_{p^k,p^m}, suppose that $i+k \geq n$ and $i+m = k+j < n$. Consider $\{X_t : t \in U(p^{n-(k+j)})\}$ and $\{Y_t : t \in U(p^{n-(i+m)})\}$, where X_t and Y_t are defined above. Then every element of X_t is adjacent to all the elements of $Y_{t'}$ if and only if $p^{n-(k+j)}$ divides $t_1 + t_1'$. Moreover, suppose that $2u \geq n$ and $u+v \leq n-1$ hold in S_{p^u,p^v}. Then the induced subgraph on S_{p^u,p^v} is isomorphic to $(p^{n-(u+v)} - p^{n-(u+v)-1})/2$ disjoint copies of the complete bipartite graph $K_{(p-1)p^{n-1},(p-1)p^{n-1}}$.*

The previous work gives an algorithm to construct the zero divisor graph $\Gamma(\mathbb{Z}_{p^n}(x))$. The following example explains how to construct the zero divisor graph $\Gamma(\mathbb{Z}_{5^7}(x))$.

Example 2.1. Consider the zero divisor graph $\Gamma(\mathbb{Z}_{5^7}(x))$. For $1 \leq i \leq 6$, each element y of S_{5^i} is adjacent to all the elements of $\bigcup_{j=7-i}^{6} \bigcup_{t=7-i}^{6} (S_{5^j} \cup E_{5^j} \cup S_{5^j,5^t}) - \{y\}$. Also, for $0 \leq i \leq 6$, each element of E_{5^i} is adjacent to all the elements of $\bigcup_{j=7-i}^{6} S_{5^j} \cup \bigcup_{j=0}^{6} E_{5^j} \cup \bigcup_{j=7-i}^{6} \bigcup_{t=0}^{6} S_{5^j,5^t}$. Now, we want to determine the adjacency between the elements of the $S_{5^i,5^j}$'s. To do that take the pair $S_{5^i,5^j}$, $S_{5^k,5^m}$ where $1 \leq i,k \leq 6$ and $0 \leq j,m \leq 6$. We have three cases to consider. Assuming $i+k \geq 7$,

Case 1) $i+m \geq 7$ and $j+k \geq 7$. In this case, each element of $S_{5^i,5^j}$ is adjacent to all the elements of $S_{5^k,5^m}$.

Case 2) $i+m \geq 7$, $j+k < 7$ or $i+m < 7$, $j+k \geq 7$. In this case, no element of $S_{5^i,5^j}$ is adjacent to any element of $S_{5^k,5^m}$.

Case 3) $i+m$ and $j+k$ are both less than 7. Here there are two subcases. (i) $i+m$ is not equal to $j+k$. In this subcase, no element of $S_{5^i,5^j}$ is adjacent to any element of $S_{5^k,5^m}$. (ii) $i+m = j+k$. In this subcase, we use Lemma 2.2 to determine the adjacency between the elements of $S_{5^i,5^j}$ and $S_{5^k,5^m}$. For instance, take $S_{5^3,5^2}$ and $S_{5^4,5^3}$. According to Lemma 2.2 we divide the set $S_{5^3,5^2}$ into $5^{7-(3+3)} - 5^{7-(3+3)-1} = 5-1 = 4$ subsets. These subsets are $X_t = \{r5^3 + 5^2(s5^1 + \overline{rt})x : r \in U(5^4)$ and $0 \leq s \leq 5^4 - 1\}$, where $t \in U(5)$ and $|X_t| = 5^4(5^4 - 5^3)$. Also, we divide $S_{5^4,5^3}$ into $5^{7-(3+3)} - 5^{7-(3+3)-1} = 5-1 = 4$ subsets. These subsets are $Y_{t'} = \{r'5^4 + 5^3(s'5^1 + \overline{r't'})x : r' \in U(5^3)$ and $0 \leq s' \leq 5^3 - 1\}$, where $t' \in U(5)$ and $|Y_{t'}| = 5^3(5^3 - 5^2)$. We use Lemma 2.2 to get, that each element of X_t is adjacent to all the elements of Y_{5-t}, where $t \in U(5) = \{1,2,3,4\}$. For instance, each element of X_1 is adjacent to all the elements of Y_4 and no element of X_1 is adjacent to any element of the sets Y_1, Y_2, or Y_3. Now, let us see how the elements of $S_{5^4,5^1}$ are adjacent to each other. According to Lemma 2.2, we divide the elements of the set $S_{5^4,5^1}$ into $5^2-5=20$ subsets. These subsets are $W_t=\{r5^4+5(s5^2+\overline{rt})x:\ r\in U(5^3),\ 0\leq s\leq 5^4-1\}$ where $t\in U(5^2)$. Using Lemma 2.2, every element of W_t is adjacent to all the elements of W_{5^2-t} where $t\in U(5^2)$. And no element of W_t is adjacent to any element of $S_{5^4,5^1}-W_{5^2-t}$. Hence, the induced subgraph on each pair W_t, W_{5^2-t} is a complete bipartite graph that is isomorphic to $K_{5^4(5^3-5^2),5^4(5^3-5^2)}$.

This way we are able to construct the zero divisor graph $\Gamma(\mathbb{Z}_{5^7}(x))$.

3. The independence number of $\Gamma(\mathbb{Z}_{p^n}(x))$

This section is devoted to compute the independence number of the zero divisor graph $\Gamma(\mathbb{Z}_{p^n}(x))$. In finding the independence number, we will study the structure of $\Gamma(\mathbb{Z}_{p^n}(x))$ thoroughly. Suppose that I is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{p^n}(x))$. We present a series of lemmas that determine the elements of the set I.

In the following lemma, we show that I contains exactly one element of the set $\bigcup_{i=0}^{n-1}E_{p^i}$.

Lemma 3.1. *Suppose that I is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{p^n}(x))$. Then I contains exactly one element of the set $\bigcup_{i=0}^{n-1}E_{p^i}$. Moreover, I can be chosen so that this element sits in E_{p^0}.*

Proof. Suppose that I is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{p^n}(x))$. Since $\bigcup_{i=0}^{n-1}E_{p^i}$ is a complete subgraph of $\Gamma(\mathbb{Z}_{p^n}(x))$, the set I contains at most one element of $\bigcup_{i=0}^{n-1}E_{p^i}$. Suppose I contains an element w in $\bigcup_{i=1}^{n-1}E_{p^i}$ and suppose $w_0\in E_{p^0}$. The element w_0 is not adjacent to any element in $V(\Gamma(\mathbb{Z}_{p^n}(x)))-\bigcup_{i=0}^{n-1}E_{p^i}=\bigcup_{i=1}^{n-1}S_{p^i\cup\bigcup_{i=1}^{n-1}\bigcup_{j=0}^{n-1}S_{p^i,p^j}}$. Hence $I_0=(I\cup\{w_0\})-\{w\}$ is an independent set with maximum cardinality. If the set I contains no elements of $\bigcup_{i=1}^{n-1}E_{p^i}$, then I must contain an element of E_{p^0}. This is because I has a maximum cardinality and none of the elements of E_{p^0} are adjacent to any element of $\bigcup_{i=1}^{n-1}S_{p^i}\cup\bigcup_{i=1}^{n-1}\bigcup_{j=0}^{n-1}S_{p^i,p^j}$. So either I contains an element of E_{p^0} or we can replace I by a independent set with maximum cardinality that contains an element of E_{p^0}. ∎

In the following, we always assume that I is taken as in Lemma 3.1. In the following two lemmas, we show that I contains no elements of the set $\bigcup_{i=\lfloor n/2\rfloor+1}^{n-1}S_{p^i}\cup\bigcup_{i=\lfloor n/2\rfloor+1}^{n-1}\bigcup_{j=\lfloor n/2\rfloor+1}^{n-1}S_{p^i,p^j}$.

Lemma 3.2. *Suppose that I is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{p^n}(x))$, where $n\geq 3$. Then I can not contain any element of the set $\bigcup_{i=\lfloor n/2\rfloor+1}^{n-1}S_{p^i}$.*

Proof. Since $\bigcup_{i=\lfloor n/2\rfloor+1}^{n-1}S_{p^i}$ is a complete subgraph of $\Gamma(\mathbb{Z}_{p^n}(x))$, then I contains at most one element of $\bigcup_{i=\lfloor n/2\rfloor+1}^{n-1}S_{p^i}$. Suppose that $x_0\in S_{p^{i_0}}\cap I$, where $i_0\geq\lfloor n/2\rfloor+1$. Since all the elements of $\bigcup_{i=n-i_0}^{n-1}S_{p^i}\cup\bigcup_{i=n-i_0}^{n-1}\bigcup_{j=n-i_0}^{n-1}S_{p^i,p^j}\cup\bigcup_{i=n-i_0}^{n-1}E_{p^i}$ are adjacent to x_0, then $I\cap\left(\bigcup_{i=n-i_0}^{n-1}S_{p^i}\cup\bigcup_{i=n-i_0}^{n-1}\bigcup_{j=n-i_0}^{n-1}S_{p^i,p^j}\cup\bigcup_{i=n-i_0}^{n-1}E_{p^i}\right)=\emptyset$. Let $I_0=(I\cup S_{p^{n-i_0}})-\{x_0\}$. Note that the induced graph on the vertices $S_{p^{n-i_0}}$ in $\Gamma(\mathbb{Z}_{p^n}(x))$ is the null graph. Also, the elements of $S_{p^{n-i_0}}$ are not adjacent to any element of $I-\{x_0\}$ because $I\cap\left(\bigcup_{i=n-i_0}^{n-1}S_{p^i}\cup\bigcup_{i=n-i_0}^{n-1}\bigcup_{j=n-i_0}^{n-1}S_{p^i,p^j}\cup\bigcup_{i=n-i_0}^{n-1}E_{p^i}\right)=\emptyset$. Hence I_0 is an independent set. But $|I_0|=|I|-1+p^{i_0}-p^{i_0-1}>|I|$. So, I contains no elements of $\bigcup_{i=\lfloor n/2\rfloor+1}^{n-1}S_{p^i}$. ∎

Lemma 3.3. *Suppose that I is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{p^n}(x))$, where $n \geq 3$. Then I contains no elements of the set $\bigcup_{i=\lfloor n/2\rfloor+1}^{n-1}\bigcup_{j=\lfloor n/2\rfloor+1}^{n-1} S_{p^i,p^j}$.*

Proof. The proof is similar to that of the last lemma and so we skip this proof. ∎

In the following lemma, we show that $\bigcup_{i=1}^{\lfloor (n-1)/2\rfloor} S_{p^i}$ is a subset of I.

Lemma 3.4. *Suppose that I is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{p^n}(x))$, where $n \geq 3$. Then $\bigcup_{i=1}^{\lfloor (n-1)/2\rfloor} S_{p^i}$ is a subset of I.*

Proof. The induced graph on the vertices $\bigcup_{i=1}^{\lfloor (n-1)/2\rfloor} S_{p^i}$ in $\Gamma(\mathbb{Z}_{p^n}(x))$ is the null graph and the adjacency set of these vertices is a subset of the vertices $\bigcup_{i=\lfloor n/2\rfloor+1}^{n-1} S_{p^i} \cup \bigcup_{i=\lfloor n/2\rfloor+1}^{n-1}\bigcup_{j=\lfloor n/2\rfloor+1}^{n-1} S_{p^i,p^j} \cup \bigcup_{i=\lfloor n/2\rfloor+1}^{n-1} E_{p^i}$. Using Lemmas 3.1, 3.2 and 3.3, I contains no elements of $\bigcup_{i=\lfloor n/2\rfloor+1}^{n-1} S_{p^i} \cup \bigcup_{i=\lfloor n/2\rfloor+1}^{n-1}\bigcup_{j=\lfloor n/2\rfloor+1}^{n-1} S_{p^i,p^j} \cup \bigcup_{i=\lfloor n/2\rfloor+1}^{n-1} E_{p^i}$. Hence an independent set with maximum cardinality contains all the elements of $\bigcup_{i=1}^{\lfloor (n-1)/2\rfloor} S_{p^i}$. ∎

In the following lemma, we show that $\bigcup_{i=1}^{\lfloor (n-1)/2\rfloor}\bigcup_{j=0}^{n-1} S_{p^i,p^j}$ is a subset of I and $I \cap \left(\bigcup_{i=\lfloor n/2\rfloor+1}^{n-1}\bigcup_{j=i-\lfloor (n-1)/2\rfloor}^{n-1} S_{p^i,p^j}\right) = \emptyset$

Lemma 3.5. *Suppose that I is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{p^n}(x))$, where $n \geq 3$. Then $I \cap \left(\bigcup_{i=\lfloor n/2\rfloor+1}^{n-1}\bigcup_{j=i-\lfloor (n-1)/2\rfloor}^{n-1} S_{p^i,p^j}\right) = \emptyset$ and $\bigcup_{i=1}^{\lfloor (n-1)/2\rfloor}\bigcup_{j=0}^{n-1} S_{p^i,p^j}$ is a subset of I.*

Proof. Take S_{p^i,p^j} with $i \leq \lfloor (n-1)/2\rfloor$. Using Lemma 3.2 and Lemma 3.4, there are no elements in $\bigcup_{i=1}^{n-1} S_{p^i} \cap I$ adjacent to S_{p^i,p^j}. For S_{p^k,p^m} with $i+k \geq n$, $i+m$, $k+j \geq n$, we get $k \geq \lfloor n/2\rfloor + 1$ and $m \geq \lfloor n/2\rfloor + 1$. Hence, using Lemma 3.3, none of the elements of S_{p^k,p^m} are in I, where $i+k \geq n$, $i+m$, $k+j \geq n$. So, in this case the elements of S_{p^i,p^j} are adjacent to some elements that are outside I.

Using Lemma 2.2, every element of S_{p^i,p^j} is adjacent to some of the elements of $S_{p^{n-i},p^{n-2i+j}}$ provided that $n-2i+j \leq n-1$. In case $n-2i+j \geq n$, the induced subgraph on S_{p^i,p^j} is a set of null vertices whose elements are adjacent to some elements that are outside I and hence S_{p^i,p^j} is a subset of I. For the case where $n-2i+j \leq n-1$, every element of S_{p^i,p^j} is adjacent to some of the elements of $S_{p^{n-i},p^{n-2i+j}}$, $S_{p^{n-i+1},p^{n-2i+j+1}}$, …, $S_{p^{n-i+s},p^{n-2i+j+s}}$ where $n-i+s = n-1$, $n-2i+j+s \leq n-1$, or $n-i+s \leq n-1$, $n-2i+j+s = n-1$. Hence, if $n-2i+j \leq n-1$, then every element of S_{p^i,p^j} is adjacent to some elements of $A = \bigcup_{r=0}^{s} S_{p^{n-i+r},p^{n-2i+j+r}}$. Note that if $(n-i)+(n-2i+j) \geq n$, then the induced subgraph on A is complete. Since the induced graph on the vertices S_{p^i,p^j} in $\Gamma(\mathbb{Z}_{p^n}(x))$ is the null graph, we get $I \cap A = \emptyset$ and $S_{p^i,p^j} \subseteq I$. Also, if $j \geq i$, then $(n-i)+(n-2i+j) \geq n$ because $i \leq \lfloor (n-1)/2\rfloor$. Hence the induced subgraph on A is complete. Since S_{p^i,p^j} is a set of null vertices, we get $I \cap A = \emptyset$ and $S_{p^i,p^j} \subseteq I$. So, we can assume that $j < i$ and $(n-i)+(n-2i+j) \leq n-1$. Our goal is to show that $I \cap A = \emptyset$. Assume on the contrary that $I \cap A \neq \emptyset$. For $0 \leq r \leq i-j-1$, Lemma 2.2 gives $S_{p^i,p^j} = \bigcup_{t\in U(p^{i-j-r})} X_t^r$ where $X_t^r = \{a_r p^i + p^j(b_r p^{i-j-r} + \overline{a_r t})x :\ a_r \in U(p^{n-i}),\ 0 \leq b_r \leq p^{n-i+r} - 1\}$, and $S_{p^{n-i+r},p^{n-2i+j+r}} = \bigcup_{t'\in U(p^{i-j-r})} Y_{t'}^r$ where $Y_{t'}^r = \{a'_r p^{n-i+r} + p^{n-2i+j+r}(b'_r p^{i-j-r} + \overline{a'_r t'})x :\ a'_r \in U(p^{i-r}),\ 0 \leq b'_r \leq p^i - 1\}$. For $0 \leq r \leq i-j-1$, define $T^r = \{t' : I \cap Y_{t'}^r \neq \emptyset\}$, $T = \text{maximum}\{|T^r| : 0 \leq r \leq i-j-1\}$, $r_0 = \text{maximum}\{r : |T^r| = T \text{ and } 0 \leq r \leq i-j-1\}$, and $\overline{T^{r_0}} = \{p^{i-j-r_0} - t' : t' \in T^{r_0}\}$. Since $|Y^r_{t'}| = (p^{i-r} - p^{i-r-1})p^i$, then I contains at most $T\sum_{r=0}^{i-j-1}(p^{i-r} - p^{i-r-1})p^i$ elements of $\bigcup_{r=0}^{i-j-1} S_{p^{n-i+r},p^{n-2i+j+r}}$ and since the induced subgraph on $\bigcup_{r=i-j}^{s} S_{p^{n-i+r},p^{n-2i+j+r}}$ is complete, then I contains at most one element of $\bigcup_{r=i-j}^{s} S_{p^{n-i+r},p^{n-2i+j+r}}$. Hence I contains at most $T\sum_{r=0}^{i-j-1}(p^{i-r} - p^{i-r-1})p^i + 1$ elements of A. Take $I_0 = (I - A) \cup \bigcup_{t\in\overline{T^{r_0}}} X^{r_0}_t$. Note that, the induced subgraph on $\bigcup_{t\in\overline{T^{r_0}}} X^{r_0}_t$ is a set of null vertices and the elements of $\bigcup_{t\in\overline{T^{r_0}}} X^{r_0}_t$ are either adjacent to some elements that are outside I or to some elements of A. Hence I_0 is an independent set. Since I contains at least one element from each $Y^{r_0}_{t'}$ where $t' \in T^{r_0}$ and this element is adjacent to all the elements of $X^{r_0}_{p^{i-j-r_0}-t'}$, then $I \cap \bigcup_{t\in\overline{T^{r_0}}} X^{r_0}_t = \emptyset$. So, $|I_0| \ge |I| - (T\sum_{r=0}^{i-j-1}(p^{i-r} - p^{i-r-1})p^i + 1) + T|X^r_t| = |I| - \big(T(p^i - p^{i-1})p^i((p^{i-j}-1)p)/(p^{i-j}(p-1)) + 1\big) + T(p^{n-i} - p^{n-i-1})p^{n-i+r_0} > |I| - (T(p^i - p^{i-1})p^{i+1}) + T(p^{n-i} - p^{n-i-1})p^{n-i+r_0} > |I|$. The last greater than follows because $i \le \lfloor (n-1)/2 \rfloor < n-i$. This contradicts the fact that I is an independent set with maximum cardinality. Hence $I \cap A = \emptyset$. As above by taking $S_{p^i,p^j} = S_{p^{\lfloor (n-1)/2\rfloor},p^j}$, $0 \le j \le n-1$, we get $I \cap \big(\bigcup_{i=\lfloor n/2\rfloor+1}^{n-1} \bigcup_{j=i-\lfloor (n-1)/2\rfloor}^{n-1} S_{p^i,p^j}\big) = \emptyset$ and $\bigcup_{i=1}^{\lfloor (n-1)/2\rfloor} \bigcup_{j=0}^{n-1} S_{p^i,p^j} \subseteq I$. ∎

We summarize what we have about an independent set I with maximum cardinality in $\Gamma(\mathbb{Z}_{p^n}(x))$, where $n \ge 3$. Using Lemmas 3.1 to 3.5 to get $\bigcup_{i=1}^{\lfloor (n-1)/2\rfloor} S_{p^i} \cup \bigcup_{i=1}^{\lfloor (n-1)/2\rfloor} \bigcup_{j=0}^{n-1} S_{p^i,p^j} \cup \{w_0\} \subseteq I$ and $I \cap \big(\bigcup_{i=\lfloor n/2\rfloor+1}^{n-1} S_{p^i} \cup \bigcup_{i=\lfloor n/2\rfloor+1}^{n-1} \bigcup_{j=i-\lfloor (n-1)/2\rfloor}^{n-1} S_{p^i,p^j} \cup \bigcup_{i=0}^{n-1}(E_{p^i} - \{w_0\})\big) = \emptyset$, where w_0 is an element of E_{p^0}. Now, $|\bigcup_{i=1}^{\lfloor (n-1)/2\rfloor} S_{p^i} \cup \bigcup_{i=1}^{\lfloor (n-1)/2\rfloor} \bigcup_{j=0}^{n-1} S_{p^i,p^j} \cup \{w_0\}| = \sum_{i=1}^{\lfloor (n-1)/2\rfloor}(p^{n-i} - p^{n-i-1}) + \sum_{i=1}^{\lfloor (n-1)/2\rfloor}\sum_{j=0}^{n-1}(p^{n-i} - p^{n-i-1})(p^{n-j} - p^{n-j-1}) + 1 = p^{2n-\lfloor (n-1)/2\rfloor-1}(p^{\lfloor (n-1)/2\rfloor} - 1) + 1$. We state this in the following corollary.

Corollary 3.1. *Suppose that I is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{p^n}(x))$, where $n \ge 3$. Then I contains exactly $p^{2n-\lfloor (n-1)/2\rfloor-1}(p^{\lfloor (n-1)/2\rfloor} - 1) + 1$ elements from the set $\bigcup_{i=1}^{\lfloor (n-1)/2\rfloor} S_{p^i} \cup \bigcup_{i=1}^{\lfloor (n-1)/2\rfloor} \bigcup_{j=0}^{n-1} S_{p^i,p^j} \cup \bigcup_{i=\lfloor n/2\rfloor+1}^{n-1} S_{p^i} \cup \bigcup_{i=\lfloor n/2\rfloor+1}^{n-1} \bigcup_{j=i-\lfloor (n-1)/2\rfloor}^{n-1} S_{p^i,p^j} \cup \bigcup_{i=0}^{n-1} E_{p^i}$.*

To determine I completely we have to determine which elements of $\bigcup_{i=\lfloor n/2\rfloor+1}^{n-1} \bigcup_{j=0}^{i-\lfloor (n-1)/2\rfloor-1} S_{p^i,p^j}$ are in I when n is an odd integer and which elements of $S_{p^{\lfloor n/2\rfloor}} \cup \bigcup_{i=\lfloor n/2\rfloor+1}^{n-1} \bigcup_{j=0}^{i-\lfloor (n-1)/2\rfloor-1} S_{p^i,p^j} \cup \bigcup_{j=0}^{n-1} S_{p^{\lfloor n/2\rfloor},p^j}$ are in I when n is an even integer. Observe that $\bigcup_{i=\lfloor n/2\rfloor+1}^{n-1} \bigcup_{j=0}^{i-\lfloor (n-1)/2\rfloor-1} S_{p^i,p^j} = \bigcup_{i=0}^{\lfloor (n-1)/2\rfloor-1} \bigcup_{j=0}^{\lfloor (n-1)/2\rfloor-i-1} S_{p^{\lfloor n/2\rfloor+1+i+j},p^j}$. First, we consider the case where $n = 4n_1 + 3$ with n_1 is a positive integer. So, for any i with $0 \le i \le \lfloor (n-1)/2\rfloor - 1 = \lfloor (4n_1 + 3 - 1)/2\rfloor - 1 = 2n_1$, we want to determine which elements of $\bigcup_{j=0}^{\lfloor (n-1)/2\rfloor-i-1} S_{p^{\lfloor \frac{n}{2}\rfloor+1+i+j},p^j} = \bigcup_{j=0}^{2n_1+1-i-1} S_{p^{2n_1+1+1+i+j},p^j}$ are in I. First, take $i = 0$, so we want to determine which elements of $\bigcup_{j=0}^{2n_1} S_{p^{2n_1+2+j},p^j}$ are in I. Observe that $\bigcup_{j=n_1+1}^{2n_1} S_{p^{2n_1+2+j},p^j}$ is a complete subgraph of $\bigcup_{j=0}^{2n_1} S_{p^{2n_1+2+j},p^j}$ and hence I can contain at most one element from it. We want to see which elements of $\bigcup_{j=0}^{n_1} S_{p^{2n_1+2+j},p^j}$ are in I. To do that, we divide $S_{p^{2n_1+2+0},p^0}$, $S_{p^{2n_1+2+1},p^1}, \ldots, S_{p^{2n_1+2+j},p^j}, \ldots, S_{p^{2n_1+2+n_1},p^{n_1}}$ into $X^0_{t_0} = \{r_0p^{2n_1+2} + p^0(s_0p^{2n_1+1} + \overline{r_0t_0})x :$ $r_0 \in U(p^{2n_1+1})$ and $0 \le s_0 \le p^{2n_1+2} - 1\}$ where $t_0 \in U(p^{2n_1+1})$, $X^1_{t_1} = \{r_1p^{2n_1+3} + p^1(s_1p^{2n_1}$

$+\overline{r_1t_1})x\colon\ r_1\in U(p^{2n_1})$ and $0\le s_1\le p^{2n_1+2}-1\}$ where $t_1\in U(p^{2n_1})$, $\dots X^j_{t_j}=\{r_jp^{2n_1+2+j}+p^j(s_jp^{2n_1+1-j}+\overline{r_jt_j})x\colon\ r_j\in U(p^{2n_1+1-j})$ and $0\le s_j\le p^{2n_1+2}-1\}$ where $t_j\in U(p^{2n_1+1-j})$, $\dots X^{n_1}_{t_{n_1}}=\{r_{n_1}p^{2n_1+2+n_1}+p^{n_1}(s_{n_1}p^{2n_1+1-n_1}+\overline{r_{n_1}t_{n_1}})x\colon\ r_{n_1}\in U(p^{2n_1+1-n_1})$ and $0\le s_{n_1}\le p^{2n_1+2}-1\}$ where $t_{n_1}\in U(p^{2n_1+1-n_1})$ respectively.

The bar on r_jt_j means that we are taking r_jt_j mod p^{2n_1+1-j}. Observe that when t_j runs over all the elements of $U(p^{2n_1+1-j})$ and $r_j\in U(p^{2n_1+1-j})$, then $\overline{r_jt_j}$ runs over all the elements of $U(p^{2n_1+1-j})$. So, for any j with $0\le j\le n_1$, we have $S_{p^{2n_1+2+j},p^j}=\bigcup_{t_j\in U(p^{2n_1+1-j})}X^j_t$ i.e. $S_{p^{2n_1+2+j},p^j}$ has been divided into $|U(p^{2n_1+1-j})|$ subsets and these subsets are $X^j_{t_j}$ where $t_j\in U(p^{2n_1+1-j})$. We want to see how the elements of these subsets are adjacent. Suppose that $y_1\in X^j_{t_j}$ and $y_2\in X^{j'}_{s_{j'}}$, where $0\le j,j'\le n_1$, $t_j\in U(p^{2n_1+1-j})$, and $s_{j'}\in U(p^{2n_1+1-j'})$. As we have explained for Lemma 2.2, we get y_1 and y_2 are adjacent if and only if $p^{2n_1+1-j-j'}$ divides $t_j+s_{j'}$. So, each element of X^j_t is adjacent to all the elements of $X^{j'}_{s_{j'}}$ if and only if $p^{2n_1+1-j-j'}$ divides $t_j+s_{j'}$. Note that the exponent $2n_1+1-j-j'$ is positive and this is because j and j' are less than or equal to n_1. By taking $j=j'$, we get each element of $X^j_{t_j}$ is adjacent to every element of $X^j_{s_j}$ if and only if $t_j+s_j\equiv 0 \pmod{p^{2n_1+1-2j}}$. Note that both t_j and s_j are elements of $U(p^{2n_1+1-j})$. Hence, if $l\in U(p^{2n_1+1-2j})$, then each element of X^j_l is adjacent to every element of $X^j_{(p^{2n_1+1-2j}-l)+cp^{2n_1+1-2j}}$ where $0\le c\le p^j-1$. So, if $l\in U(p^{2n_1+1-2j})$, then every element of $\bigcup_{b=0}^{p^j-1}X^j_{l+bp^{2n_1+1-2j}}$ is adjacent to every element of $\bigcup_{c=0}^{p^j-1}X^j_{(p^{2n_1+1-2j}-l)+cp^{2n_1+1-2j}}$. Whereas, no element of $\bigcup_{b=0}^{p^j-1}X^j_{l+bp^{2n_1+1-2j}}$ is adjacent to any element of $S_{p^{2n_1+2+j},p^j}-\bigcup_{c=0}^{p^j-1}X^j_{(p^{2n_1+1-2j}-l)+cp^{2n_1+1-2j}}$. So, if $l\in U(p^{2n_1+1-2j})$, then the induced subgraph on each pair $\bigcup_{b=0}^{p^j-1}X^j_{l+bp^{2n_1+1-2j}}$, $\bigcup_{c=0}^{p^j-1}X^j_{(p^{2n_1+1-2j}-l)+cp^{2n_1+1-2j}}$ is isomorphic to the complete bipartite graph $K_{(p^{2n_1+1-j}-p^{2n_1-j})p^{2n_1+2}p^j,(p^{2n_1+1-j}-p^{2n_1-j})p^{2n_1+2}p^j}$ $=K_{p^{4n_1+2}(p-1),p^{4n_1+2}(p-1)}$. So, the induced subgraph on $S_{p^{2n_1+2+j},p^j}$ is isomorphic to the disjoint union of $|U(p^{2n_1+1-2j})|/2=(p^{2n_1+1-2j}-p^{2n_1-2j})/2$ copies of the complete bipartite graph $K_{p^{4n_1+2}(p-1),p^{4n_1+2}(p-1)}$. Recall that I is an independent set with maximum cardinality of $\Gamma(\mathbb{Z}_{p^n}(x))$. So, for $0\le j\le n_1$, I can contain at most $p^{4n_1+2}(p-1)(p^{2n_1+1-2j}-p^{2n_1-2j})/2$ elements of $S_{p^{2n_1+2+j},p^j}$. Thus, I can contain *at most* $\sum_{j=0}^{n_1}p^{4n_1+2}(p-1)(p^{2n_1+1-2j}-p^{2n_1-2j})/2=p^{6n_1+2}\,(p-1)^2/2\sum_{j=0}^{n_1}\left(1/p^2\right)^j=p^{4n_1+2}(p-1)(p^{2n_1+2}-1)/(2(p+1))$ elements of $\bigcup_{j=0}^{n_1}S_{p^{2n_1+2+j},p^j}$.

We summarize this in the following lemma.

Lemma 3.6. *If $0\le j\le n_1$, then the induced subgraph on $S_{p^{2n_1+2+j},p^j}$ is isomorphic to the disjoint union of $(p^{2n_1+1-2j}-p^{2n_1-2j})/2$ copies of the complete bipartite graph $K_{p^{4n_1+2}(p-1),p^{4n_1+2}(p-1)}$. For $n_1+1\le j\le 2n_1$, the induced subgraph on $S_{p^{2n_1+2+j},p^j}$ is complete. Moreover, I contains at most $p^{4n_1+2}(p-1)(p^{2n_1+2}-1)/(2(p+1))$ elements of $\bigcup_{j=0}^{n_1}S_{p^{2n_1+2+j},p^j}$.*

If $0\le j\le n_1$, then define $X^j=\{X^j_t : t\in\{1,2,\dots,(p-1)/2\} \bmod p\}$, $Y^j=\{X^j_t : t\in\{(p+1)/2,(p+3)/2,\dots,p-1\} \bmod p\}$, $X_0=\bigcup_{j=0}^{n_1}X^j$, and $Y_0=\bigcup_{j=0}^{n_1}Y^j$. We have the following lemma about X_0 and Y_0.

Lemma 3.7. *X_0 and Y_0 form a partition of $\bigcup_{j=0}^{n_1} S_{p^{2n_1+2+j},p^j}$. Moreover, the induced subgraph on each one of X_0 and Y_0 is a set of null vertices and $|X_0| = |Y_0| = |\bigcup_{j=0}^{n_1} S_{p^{2n_1+2+j},p^j}|/2$ $= p^{4n_1+2}(p-1)(p^{2n_1+2}-1)/(2(p+1))$.*

Proof. The proof follows immediately from the definition of X_0, Y_0 and the X_t^j's. ∎

We want to show that I contains exactly $p^{4n_1+2}(p-1)(p^{2n_1+2}-1)/(2(p+1))$ elements of $\bigcup_{j=0}^{2n_1} S_{p^{2n_1+2+j},p^j}$. Suppose that $C = \bigcup_{j=0}^{n_1} S_{p^{2n_1+2+j},p^j}$ and B is a subset of $\Gamma(\mathbb{Z}_{p^n}(x))$ with $B \cap (\bigcup_{i=\lfloor n/2\rfloor+1}^{n-1} \bigcup_{j=i-\lfloor (n-1)/2\rfloor}^{n-1} S_{p^i,p^j} \cup \bigcup_{i=\lfloor n/2\rfloor+1}^{n-1} S_{p^i} \cup \bigcup_{i=1}^{n-1} E_{p^i}) = \emptyset$, $B \cap C = \emptyset$, and any element of B is adjacent to some element of C. Our goal is to show that I does not contain any elements of B. It is straight forward to check that B does not contain any elements of $\bigcup_{i=1}^{\lfloor (n-1)/2\rfloor} \bigcup_{j=0}^{n-1} S_{p^k,p^m} \cup \bigcup_{i=1}^{\lfloor (n-1)/2\rfloor} S_{p^i} \cup E_{p^0}$ and $\bigcup_{j=n_1+1}^{2n_1} S_{p^{2n_1+2+j},p^j}$ is a subset of C. Hence B is a subset of $\bigcup_{i=0}^{2n_1} \bigcup_{j=0}^{2n_1-i} S_{p^{2n_1+2+i+j},p^j} - C$. So, it is straightforward to check that the induced subgraph on B is complete and any element of B is adjacent to all the elements of $S_{p^{2n_1+2+n_1}}$. We state this in the following lemma.

Lemma 3.8. *For the set B that is defined above, the induced subgraph on B is complete and any element of B is adjacent to all the elements of $S_{p^{2n_1+2+n_1}}$.*

In the following lemma, we show that I does not contain any element of B.

Lemma 3.9. *Suppose that I is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{p^n}(x))$, where $n = 4n_1+3$ and n_1 is a positive integer. Then $I \cap B = \emptyset$.*

Proof. Since B is complete, I can contain at most one element of B. Suppose that $y \in I \cap B$. Since any element of B is adjacent to all the elements of $S_{p^{2n_1+2+n_1},p^{n_1}}$, then I contains no elements of $S_{p^{2n_1+2+n_1},p^{n_1}}$. So, using Lemma 3.6, I can contain at most $[p^{4n_1+2}(p-1)(p^{2n_1+2}-1)/(2(p+1))] - |S_{p^{2n_1+2+n_1},p^{n_1}}| + 1$ elements from the set $\bigcup_{j=0}^{n_1} S_{p^{2n_1+2+j},p^j} \cup B$. Define $I_0 = (I - \bigcup_{j=0}^{n_1} S_{p^{2n_1+2+j},p^j} \cup B) \cup X_0$, where X_0 is defined in Lemma 3.7. Using Lemma 3.7 and Lemma 3.8, we get the induced subgraph on I_0 is a set of null vertices. So, I_0 is an independent set with $|I_0| > |I|$. This contradicts the fact that I is an independent set with maximum cardinality. Hence I does not contain any element of B. ∎

Lemma 3.10. *Suppose that I is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{p^n}(x))$, where $n = 4n_1+3$ and n_1 is a positive integer. Then I contains exactly $\sum_{j=0}^{n_1} |S_{p^{2n_1+2+j},p^j}|/2 = p^{4n_1+2}(p-1)(p^{2n_1+2}-1)/(2(p+1))$ elements of $\bigcup_{j=0}^{2n_1} S_{p^{2n_1+2+j},p^j}$.*

Proof. The result is an immediate consequence of Lemmas 3.6, 3.7, and Lemma 3.9. ∎

Note that, I might contain the elements of X_0 from $\bigcup_{j=0}^{2n_1} S_{p^{2n_1+2+j},p^j}$, where X_0 is defined in Lemma 3.7. For $i = 0$, we determined how many elements I contains from $\bigcup_{j=0}^{2n_1} S_{p^{2n_1+2+i+j},p^j}$. This was done in Lemma 3.10. Similar to the case where $i = 0$, one can show that I contains exactly $\sum_{j=0}^{n_1-1} |S_{p^{2n_1+2+1+j},p^j}|/2$ elements from $\bigcup_{j=0}^{n_1-1} S_{p^{2n_1+2+1+j},p^j}$ and one element from $S_{p^{2n_1+2+1+n_1},p^{n_1}}$. Observe that the induced subgraph on $S_{p^{2n_1+2+1+n_1},p^{n_1}}$ is complete and hence I contains exactly one element of it. Continuing this way for $i = 2, 3, \ldots, 2n_1$ and we get that I contains exactly $\sum_{i=0}^{2n_1} \sum_{j=0}^{\lfloor (2n_1-i)/2\rfloor} |S_{p^{2n_1+2+i+j},p^j}|/2 + n_1$ elements from $\bigcup_{i=0}^{2n_1} \bigcup_{j=0}^{2n_1-i} S_{p^{2n_1+2+i+j},p^j}$. This will determine the independence number of

$\Gamma(\mathbb{Z}_{p^n}(x))$, where $n = 4n_1 + 3$ and n_1 is a positive integer. We state this in the following theorem.

Theorem 3.1. *Suppose that $n = 4n_1 + 3$ where n_1 is a positive integer. Then the independence number of $\Gamma(\mathbb{Z}_{p^n}(x))$ is equal to $(1-p)p^{4n_1+3}/(2(1+p))[((n_1+1)/p+n_1)-(1-p^{2n_1+1})p/(1-p)]+p^{6n_1+4}(p^{2n_1+1}-1)+n_1+1$.*

Proof. If I is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{p^n}(x))$, then I contains $\sum_{i=0}^{2n_1}\sum_{j=0}^{\lfloor(2n_1-i)/2\rfloor} 1/2|S_{p^{2n_1+2+i+j},p^j}|+n_1$ elements from $\bigcup_{i=0}^{2n_1}\bigcup_{j=0}^{2n_1-i} S_{p^{2n_1+2+i+j},p^j}$. We have

$$\begin{aligned}
\sum_{i=0}^{2n_1}\sum_{j=0}^{\lfloor\frac{2n_1-i}{2}\rfloor}\left|S_{p^{2n_1+2+i+j},p^j}\right| &= \sum_{i=0}^{2n_1}\sum_{j=0}^{\lfloor\frac{2n_1-i}{2}\rfloor} p^{6n_1+2}\left(p^{-i-j+1}-p^{-i-j}\right)\left(p^{-j+1}-p^{-j}\right)\\
&= p^{6n_1+2}(p-1)^2\sum_{i=0}^{2n_1}\sum_{j=0}^{\lfloor\frac{2n_1-i}{2}\rfloor}\left(\frac{1}{p}\right)^i\left(\frac{1}{p}\right)^{2j}\\
&= p^{6n_1+2}(p-1)^2\sum_{i=0}^{2n_1}\left(\frac{1}{p}\right)^i\frac{\left(\left(\frac{1}{p^2}\right)^{\lfloor\frac{2n_1-i}{2}\rfloor+1}-1\right)}{\left(\frac{1}{p^2}-1\right)}\\
&= \frac{(1-p)p^{6n_1+4}}{1+p}\left[\left(\frac{n_1+1}{p^{2n_1+2}}\right)+\left(\frac{n_1}{p^{2n_1+1}}\right)-\sum_{i=0}^{2n_1}\left(\frac{1}{p}\right)^i\right]\\
&= \frac{(1-p)p^{4n_1+3}}{1+p}\left[\left(\frac{n_1+1}{p}+n_1\right)-\frac{(1-p^{2n_1+1})p}{1-p}\right]
\end{aligned}$$

Using Corollary 3.1, I contains $p^{6n_1+4}(p^{2n_1+1}-1)+1$ elements from $\Gamma(\mathbb{Z}_{p^n}(x))-\bigcup_{i=0}^{2n_1}\bigcup_{j=0}^{2n_1-i} S_{p^{2n_1+2+i+j},p^j}$. Now, the result follows immediately. ∎

We give the following example to explain Theorem 3.1.

Example 3.1. Consider the zero divisor graph $\Gamma(\mathbb{Z}_{5^7}(x))$. Partition the sets $S_{5^4,5^0}$ and $S_{5^5,5^1}$ into $X_t^0 = \{r_0 5^4 + (s_0 5^3 + \overline{r_0 t})x : r_0 \in U(5^3)$ and $0 \le s_0 \le 5^4-1\}$ where $t \in U(5^3)$ and $X_{t'}^1 = \{r_1 5^5 + 5^1(s_1 5^2 + \overline{r_1 t'})x : r_1 \in U(5^2)$ and $0 \le s_1 \le 5^4-1\}$ where $t' \in U(5^2)$. Let $X^0 = \{X_t^0 : t \in \{1,2\} \bmod 5\}$ and $X^1 = \{X_{t'}^1 : t' \in \{1,2\} \bmod 5\}$. Also, we partition the sets $S_{5^5,5^0}$ and $S_{5^6,5^1}$ into $Y_t^0 = \{r_0 5^5 + (s_0 5^2 + \overline{r_0 t})x : r_0 \in U(5^2)$ and $0 \le s_0 \le 5^5-1\}$ where $t \in U(5^2)$ and $Y_{t'}^1 = \{r_1 5^6 + 5^1(s_1 5^1 + \overline{r_1 t'})x : r_1 \in U(5^1)$ and $0 \le s_1 \le 5^5-1\}$ where $t' \in U(5^1)$. Take $Y^0 = \{Y_t^0 : t \in \{1,2\} \bmod 5\}$ and $Y^1 = \{Y_{t'}^1 : t' \in \{1,2\} \bmod 5\}$. Lastly, we partition $S_{5^6,5^0}$ into $Z_t^0 = \{r_0 5^6 + (s_0 5^1 + \overline{r_0 t})x : r_0 \in U(5^1)$ and $0 \le s_0 \le 5^6-1\}$ where $t \in U(5^1)$ and take $Z^0 = \{Z_t^0 : t \in \{1,2\} \bmod 5\}$. If $I = \bigcup_{i=1}^{3}\bigcup_{j=0}^{6} S_{p^i,p^j} \cup \bigcup_{i=1}^{3} S_{p^i} \cup X^0 \cup X^1 \cup Y^0 \cup Z^0 \cup \{5^6 + 5(5+1)x\} \cup \{w_0\}$, where $5^6 + 5(5+1)x$ is an element of Y^1 and $w_0 \in E_{p^0}$, then I is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{5^7}(x))$. Hence the independence number of $\Gamma(\mathbb{Z}_{5^7}(x))$ is equal to $|I| = 5^{10}(5^3-1)+5^6(5-1)^2(5^2+5+2)/2+2$.

Similarly, one can find the independence number of $\Gamma(\mathbb{Z}_{p^n}(x))$ where $n = 4n_1+1$ and n_1 is a positive integer. We state the following.

Theorem 3.2. *Suppose that $n = 4n_1 + 1$ where n_1 is a positive integer. Then the independence number of $\Gamma(\mathbb{Z}_{p^n}(x))$ is equal to $\sum_{i=0}^{2n_1-1}\sum_{j=0}^{\lfloor(2n_1-i-1)/2\rfloor}|S_{p^{2n_1+1+i+j},p^j}|/2+n_1+$*

$|\bigcup_{i=1}^{\lfloor(n-1)/2\rfloor} S_{p^i} \cup \bigcup_{i=1}^{\lfloor(n-1)/2\rfloor} \bigcup_{j=0}^{n-1} S_{p^i,p^j} \cup \{w_0\}| = p^{4n_1+1}(p-1)/(2(p+1))[(p^{2n_1}-1)p/(p-1)-(n_1+n_1/p)]+n_1+p^{6n_1+1}(p^{2n_1}-1)+1$.

We want to determine the independence number of $\Gamma(\mathbb{Z}_{p^n}(x))$ when n is an even integer. If I is an independence set with maximum cardinality in $\Gamma(\mathbb{Z}_{p^n}(x))$ and n is an even integer, then the use of Lemma 3.1, Lemma 3.2, Lemma 3.3, Lemma 3.4, and Lemma 3.5 gives $\bigcup_{i=1}^{\lfloor(n-1)/2\rfloor} S_{p^i} \cup \bigcup_{i=1}^{\lfloor(n-1)/2\rfloor} \bigcup_{j=0}^{n-1} S_{p^i,p^j} \cup \{w_0\} \subseteq I$ and $I \cap \left(\bigcup_{i=\lfloor n/2\rfloor+1}^{n-1} S_{p^i} \cup \bigcup_{i=0}^{n-1} E_{p^i} - \{w_0\} \cup \bigcup_{i=\lfloor n/2\rfloor+1}^{n-1} \bigcup_{j=i-\lfloor(n-1)/2\rfloor}^{n-1} S_{p^i,p^j}\right) = \emptyset$, where w_0 is an element of E_{p^0}. To determine I completely, we have to determine which elements of $S_{p^{\lfloor n/2\rfloor}} \cup \bigcup_{i=\lfloor n/2\rfloor+1}^{n-1} \bigcup_{j=0}^{i-\lfloor(n-1)/2\rfloor-1} S_{p^i,p^j} \cup \bigcup_{j=0}^{n-1} S_{p^{\lfloor n/2\rfloor},p^j}$ are in I. For $1 \leq k \leq \lfloor n/2\rfloor - 1$, the use of Lemma 2.2 gives that the induced subgraph on $S_{p^{\lfloor n/2\rfloor},p^k}$ is isomorphic to $(p^{n-\lfloor n/2\rfloor-k} - p^{n-\lfloor n/2\rfloor-k-1})/2$ disjoint copies of the complete bipartite graph $K_{(p-1)p^{n-1},(p-1)p^{n-1}}$. Observe that, for $1 \leq k \leq \lfloor n/2\rfloor - 1$, no element of $S_{p^{\lfloor n/2\rfloor},p^k}$ is adjacent to any of the elements of $S_{p^{\lfloor n/2\rfloor}} \cup \bigcup_{i=\lfloor n/2\rfloor+1}^{n-1} \bigcup_{j=0}^{i-\lfloor(n-1)/2\rfloor-1} S_{p^i,p^j} \cup \bigcup_{j\neq k} S_{p^{\lfloor n/2\rfloor},p^j}$ or $\bigcup_{i=1}^{\lfloor(n-1)/2\rfloor} S_{p^i} \cup \bigcup_{i=1}^{\lfloor(n-1)/2\rfloor} \bigcup_{j=0}^{n-1} S_{p^i,p^j} \cup \{w_0\}$. Hence, for $1 \leq k \leq \lfloor n/2\rfloor - 1$, I contains $(p^{n-\lfloor n/2\rfloor-k} - p^{n-\lfloor n/2\rfloor-k-1})(p-1)p^{n-1}/2$ elements of $S_{p^{\lfloor n/2\rfloor},p^k}$. So, I contains $\sum_{k=1}^{\lfloor(n-1)/2\rfloor}(p^{n-\lfloor n/2\rfloor-k} - p^{n-\lfloor n/2\rfloor-k-1})(p-1)p^{n-1}/2 = p^{n-1}(p-1)(p^{\lfloor(n-1)/2\rfloor}-1)/2$ elements of $\bigcup_{k=1}^{\lfloor(n-1)/2\rfloor} S_{p^{\lfloor n/2\rfloor},p^k}$. We state this result in the following lemma.

Lemma 3.11. *Suppose that I is an independent set with maximum cardinality in* $\Gamma(\mathbb{Z}_{p^n}(x))$, *where n is a positive even integer. Then I contains exactly* $p^{n-1}(p-1)(p^{\lfloor(n-1)/2\rfloor}-1)/2$ *elements of* $\bigcup_{k=1}^{\lfloor(n-1)/2\rfloor} S_{p^{\lfloor n/2\rfloor},p^k}$.

Now, we determine which elements of $\bigcup_{i=\lfloor n/2\rfloor+1}^{n-1} \bigcup_{j=0}^{i-\lfloor(n-1)/2\rfloor-1} S_{p^i,p^j} \cup S_{p^{\lfloor n/2\rfloor},p^0}$ are in I. Observe that $\bigcup_{i=\lfloor n/2\rfloor+1}^{n-1} \bigcup_{j=0}^{i-\lfloor(n-1)/2\rfloor-1} S_{p^i,p^j} \cup S_{p^{\lfloor n/2\rfloor},p^0} = \bigcup_{i=0}^{\lfloor(n-1)/2\rfloor} \bigcup_{j=0}^{\lfloor(n-1)/2\rfloor-i} S_{p^{\lfloor n/2\rfloor+i+j},p^j}$. Similar to the case where n is odd, we determine the number of elements of $\bigcup_{i=0}^{\lfloor(n-1)/2\rfloor} \bigcup_{j=0}^{\lfloor n-1/2\rfloor-i} S_{p^{\lfloor n/2\rfloor+i+j},p^j}$ that are in I. We state that in the following lemma.

Lemma 3.12. *Suppose that I is an independent set with maximum cardinality in* $\Gamma(\mathbb{Z}_{p^n}(x))$. *If* $n = 4n_1+2$, *where* n_1 *is a positive integer, then I contains exactly* $(p-1)p^{4n_1+2}/(2(p+1))[(-(n_1+1)/p-n_1)+(p^{2n_1+1}-1)p/(p-1)]+n_1$ *elements of* $\bigcup_{i=0}^{\lfloor(n-1)/2\rfloor} \bigcup_{j=0}^{\lfloor(n-1)/2\rfloor-i} S_{p^{\lfloor n/2\rfloor+i+j},p^j}$. *If* $n = 4n_1$, *where* n_1 *is a positive integer, then I contains exactly* $(p-1)p^{4n_1+2}/(2(p+1))[(-n_1/p-n_1)+(p^{2n_1}-1)p/(p-1)]+n_1$ *elements of* $\bigcup_{i=0}^{\lfloor(n-1)/2\rfloor} \bigcup_{j=0}^{\lfloor(n-1)/2\rfloor-i} S_{p^{\lfloor n/2\rfloor+i+j},p^j}$.

We state the independence number of $\Gamma(\mathbb{Z}_{p^n}(x))$, where n is a positive even integer in the following two theorems.

Theorem 3.3. *Suppose that* $n = 4n_1+2$ *where* n_1 *is a positive integer. Then the independence number of* $\Gamma(\mathbb{Z}_{p^n}(x))$ *is equal to* $(p-1)p^{4n_1+2}/(2(p+1))[(-(n_1+1)/p-n_1)+(p^{2n_1+1}-1)p/(p-1)]+n_1+p^{6n_1+3}(p^{2n_1}-1)+1+p^{4n_1+1}(p^{2n_1}-1)(p-1)/2+1$.

Proof. Suppose that I is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{p^n}(x))$. Using Lemma 3.11 and Lemma 3.12, I contains $(p-1)p^{4n_1+2}/(2(p+1))$ $[(-(n_1+1)/p-$

$n_1) + (p^{2n_1+1} - 1)p/(p-1)] + n_1 + p^{4n_1+1}(p^{2n_1} - 1)(p-1)/2$ elements from $\bigcup_{i=0}^{\lfloor (n-1)/2 \rfloor} \bigcup_{j=0}^{\lfloor (n-1)/2 \rfloor - i} S_{p^{\lfloor n/2 \rfloor + i + j}, p^j} \cup \bigcup_{k=1}^{\lfloor (n-1)/2 \rfloor} S_{p^{\lfloor n/2 \rfloor}, p^k}$. Using Corollary 3.1, I contains exactly $p^{6n_1+3}(p^{2n_1} - 1) + 1$ elements from $\Gamma(\mathbb{Z}_{p^n}(x)) - \bigcup_{i=0}^{\lfloor (n-1)/2 \rfloor} \bigcup_{j=0}^{\lfloor (n-1)/2 \rfloor - i} S_{p^{\lfloor n/2 \rfloor + i + j}, p^j} \cup \bigcup_{k=1}^{n-1} S_{p^{\lfloor n/2 \rfloor}, p^k} \cup S_{p^{\lfloor n/2 \rfloor}}$. Observe that I contains exactly one element of $\bigcup_{k=\lfloor n/2 \rfloor}^{n-1} S_{p^{\lfloor n/2 \rfloor}, p^k} \cup S_{p^{\lfloor n/2 \rfloor}}$. Hence the result follows immediately. ∎

Theorem 3.4. *Suppose that $n = 4n_1$ where n_1 is a positive integer. Then the independence number of $\Gamma(\mathbb{Z}_{p^n}(x))$ is equal to $(p-1)p^{4n_1+2}/(2(p+1))[(-n_1/p - n_1) + (p^{2n_1} - 1)p/(p-1)] + n_1 + p^{6n_1}(p^{2n_1-1} - 1) + 1 + p^{4n_1-1}(p^{2n_1-1} - 1)(p-1)/2 + 1$.*

Proof. The proof is similar to that one of last theorem. ∎

Our theorems do not cover the cases where $n = 1, 2, 3$. We cover these cases in the following theorem.

Theorem 3.5. *The independence number of $\Gamma(\mathbb{Z}_{p^n}(x))$ where $n = 1, 2, 3$ is 1, $(p^2 - p)(p-1)/2 + 1$, $(p^2 - p)p^3 + (p^3 - p^2)(p-1)/2 + 1$ respectively.*

Proof. The independence number of $\Gamma(\mathbb{Z}_{p^1}(x))$ is equal to 1 and this is because the graph $\Gamma(\mathbb{Z}_{p^1}(x))$ is complete. The set $I = \{ap + (bp + c)x : a \in \{1, 2, \ldots, p-1\}, b \in \{0, 1, \ldots, p-1\}$ and $c \in \{1, \ldots, (p-1)/2\}\} \cup \{w_0\}$, where $w_0 \in E_{p^0}$, is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{p^2}(x))$ and hence the independence number of $\Gamma(\mathbb{Z}_{p^2}(x))$ is equal to $|I| = (p^2 - p)(p-1)/2 + 1$. The set $I = \bigcup_{j=0}^{2} S_{p^1, p^j} \cup S_{p^1} \cup \{ap^2 + (bp + c)x : a \in \{1, 2, \ldots, p-1\}, b \in \{0, 1, \ldots, p^2 - 1\}$ and $c \in \{1, \ldots, (p-1)/2\}\} \cup \{w_0\}$, where $w_0 \in E_{p^0}$, is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{p^3}(x))$. Hence the independence number of $\Gamma(\mathbb{Z}_{p^3}(x))$ is equal to $|I| = (p^2 - p)p^3 + (p^3 - p^2)(p-1)/2 + 1$. ∎

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