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Linear Operators that Preserve Term Ranks of Matrices over Semirings

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Abstract. The term rank of a matrix A is the least number of lines (rows or columns) needed to include all the nonzero entries in A, and is a well-known upper bound for many standard and non-standard matrix ranks, and is one of the most important combinatorially. In this paper, we obtain a characterization of linear operators that preserve term ranks of matrices over antinegative semirings. That is, we show that a linear operator T on a matrix space over antinegative semirings preserves term rank if and only if T preserves any two term ranks k and l if and only if T strongly preserves any one term rank k.

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1. Introduction

A *semiring* [2] is a set $\mathbb{S}$ equipped with two binary operations $+$ and $\cdot$ such that $(\mathbb{S},+)$ is a commutative monoid with identity element 0 and $(\mathbb{S},\cdot)$ is a monoid with identity element 1. In addition, the operations $+$ and $\cdot$ are connected by distributivity and 0 annihilates $\mathbb{S}$.

A semiring $\mathbb{S}$ is called *antinegative* if 0 is the only element to have an additive inverse. The following are some examples of antinegative semirings which occur in combinatorics. Let $\mathbb{B}=\{0,1\}$. Then $(\mathbb{B},+,\cdot)$ is an antinegative semiring (the *binary Boolean semiring*) if arithmetic in $\mathbb{B}$ follows the usual rules except that $1+1=1$. If $\mathbb{F}$ is the real interval $[0,1]$, then $(\mathbb{F},+,\cdot)=(\mathbb{F},\max,\min)$ is an antinegative semiring (the *fuzzy semiring*). Any nonnegative subring of the real numbers, such as the nonnegative integers, is an antinegative semiring.

Hereafter, $\mathbb{S}$ will denote an arbitrary commutative and antinegative semiring. Let $\mathscr{M}_{m,n}(\mathbb{S})$ be the set of all $m\times n$ matrices with entries in a semiring $\mathbb{S}$. Algebraic operations on $\mathscr{M}_{m,n}(\mathbb{S})$ are defined as if the underlying scalars were in a field.

The *term rank*, $\tau(A)$, of a matrix A is the minimal number k such that all the nonzero entries of A are contained in h rows and $k-h$ columns. Term rank plays a central role

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in combinatorial matrix theory and has many applications in network and graph theory (see [4]).

From now on we will assume that $2 \le m \le n$ unless specified otherwise. It follows that $1 \le \tau(A) \le m$ for all nonzero $A \in \mathscr{M}_{m,n}(\mathbb{S})$.

Let $T : \mathscr{M}_{m,n}(\mathbb{S}) \to \mathscr{M}_{m,n}(\mathbb{S})$ be a linear operator. If f is a function defined on $\mathscr{M}_{m,n}(\mathbb{S})$, then T *preserves* the function f if $f(T(A)) = f(A)$ for all $A \in \mathscr{M}_{m,n}(\mathbb{S})$. If $\mathscr{X}$ is a subset of $\mathscr{M}_{m,n}(\mathbb{S})$, then T *preserves* the set $\mathscr{X}$ if $A \in \mathscr{X}$ implies $T(A) \in \mathscr{X}$. Further, T *strongly preserves* the set $\mathscr{X}$ if $A \in \mathscr{X}$ if and only if $T(A) \in \mathscr{X}$. There are many papers on linear operators that preserve matrix functions over $\mathbb{S}$ [1–3,5,6]. But there are few papers on term-rank preservers of matrices over semirings. Beasley and Pullman [2] have characterized linear operators on $\mathscr{M}_{m,n}(\mathbb{S})$ that preserve term rank, and the following are main results of their work: for a linear operator $T : \mathscr{M}_{m,n}(\mathbb{S}) \to \mathscr{M}_{m,n}(\mathbb{S})$,

(1.1) T preserves term rank if and only if T preserves term ranks 1 and 2;

(1.2) T preserves term rank if and only if T strongly preserves term rank 1 or m.

Kang and Song [5] also have characterized linear operators on $\mathscr{M}_{m,n}(\mathbb{B})$ that preserve term rank, and the following are main results of their work: for a linear operator T on $\mathscr{M}_{m,n}(\mathbb{B})$,

(1.3) T preserves term rank if and only if T preserves term ranks $1(\text{or } 2)$ and h,

for some $3 \le h \le m$.

In this paper, we extend their works to any two term rank preservers. A sectional summary is as follows: Some definitions and preliminaries are presented in Section 2. Section 3 generalizes $(1.1) \sim (1.3)$ by showing that T preserves the term rank of Boolean matrices if and only if T preserves term ranks k and l, where $1 \le k < l \le m \le n$. In Section 4, we show that T preserves the term rank of matrices over antinegative semiring $\mathbb{S}$ if and only if T preserves any two term ranks if and only if T strongly preserves any one term rank.

2. Preliminaries

The matrix I_n is the $n \times n$ identity matrix, $O_{m,n}$ is the $m \times n$ zero matrix, and $J_{m,n}$ is the $m \times n$ matrix all of whose entries are 1. We will suppress the subscripts on these matrices when the orders are evident from the context and we write I, O and J, respectively. Let $E_{i,j}$ be the $m \times n$ matrix whose (i,j)th entry is 1 and whose other entries are all 0, and we call $E_{i,j}$ a *cell*.

The following is obvious by the definition of term rank of matrices over antinegative semirings.

Lemma 2.1. *For matrices A and B in $\mathscr{M}_{m,n}(\mathbb{S})$, we have $\tau(A+B) \le \tau(A) + \tau(B)$ and $\tau(A) \le \tau(A+B)$.*

If A and B are matrices in $\mathscr{M}_{m,n}(\mathbb{S})$, we say that B *dominates* A (written $A \sqsubseteq B$ or $B \sqsupseteq A$) if $b_{i,j} = 0$ implies $a_{i,j} = 0$ for all i and j. This provides a reflexive and transitive relation on $\mathscr{M}_{m,n}(\mathbb{S})$.

The following is also obvious by the definition of term rank of matrices over antinegative semirings.

Lemma 2.2. *For matrices A and B in $\mathscr{M}_{m,n}(\mathbb{S})$, $A \sqsubseteq B$ implies that $\tau(A) \le \tau(B)$.*

As usual, for any matrix A and lists L_1 and L_2 of row and column indices respectively, $A(L_1 \mid L_2)$ denotes the submatrix formed by omitting the rows L_1 and columns L_2 from A and $A[L_1 \mid L_2]$ denotes the submatrix formed by choosing the rows L_1 and columns L_2 from A.

For matrices A and B in $\mathscr{M}_{m,n}(\mathbb{S})$, the matrix $A \circ B$ denotes the *Hadamard* or *Schur product*. That is, the (i,j)th entry of $A \circ B$ is $a_{i,j}b_{i,j}$. A nonzero $s \in \mathbb{S}$ is a *zero divisor* if $s's = 0$ for some nonzero $s' \in \mathbb{S}$.

If P and Q are permutation matrices and B is a matrix in $\mathscr{M}_{m,n}(\mathbb{S})$ none of whose entries is a zero divisor or zero, then an operator $T : \mathscr{M}_{m,n}(\mathbb{S}) \to \mathscr{M}_{m,n}(\mathbb{S})$ is called a (P,Q,B)-*operator* if $T(X) = P(X \circ B)Q$ for all X, or $m = n$ and $T(X) = P(X^t \circ B)Q$ for all X, where X^t denotes the transpose of X. If $B = J$ we say that T is a (P,Q)-*operator*.

3. A characterization of term rank preservers of Boolean matrices

For a linear operator $T : \mathscr{M}_{m,n}(\mathbb{B}) \to \mathscr{M}_{m,n}(\mathbb{B})$, we say that T

(1) *preserves term rank k* if $\tau(T(X)) = k$ whenever $\tau(X) = k$ for all $X \in \mathscr{M}_{m,n}(\mathbb{B})$;
(2) *strongly preserves term rank k* if $\tau(T(X)) = k$ if and only if $\tau(X) = k$ for all $X \in \mathscr{M}_{m,n}(\mathbb{B})$;
(3) *preserves term rank* if it preserves term rank k for every $k(\leq m)$.

In this section we provide characterizations of linear operators $T : \mathscr{M}_{m,n}(\mathbb{B}) \to \mathscr{M}_{m,n}(\mathbb{B})$ that preserve term ranks k and l, where $1 \leq k < l \leq m \leq n$.

Lemma 3.1. *Let $1 \leq k \leq m \leq n$. Let $T : \mathscr{M}_{m,n}(\mathbb{B}) \to \mathscr{M}_{m,n}(\mathbb{B})$ be a linear operator that preserves term rank k. If $A \in \mathscr{M}_{m,n}(\mathbb{B})$ and $\tau(A) \leq k$ then $\tau(T(A)) \leq k$.*

Proof. If $\tau(A) = k$, then $\tau(T(A)) = k$ since T preserves term rank k. Suppose that $\tau(A) = h < k$, and $\tau(T(A)) > k$ Then there exist a matrix B such that $\tau(A+B) = k$ and hence $\tau(T(A+B)) = k$, but by Lemma 2.1, $\tau(T(A+B)) = \tau(T(A)+T(B)) \geq \tau(T(A)) > k$, a contradiction. Thus $\tau(T(A)) \leq k$. ∎

Theorem 3.1. [5, Lemma 2.5] *Let $1 < k \leq m \leq n$. Let $T : \mathscr{M}_{m,n}(\mathbb{B}) \to \mathscr{M}_{m,n}(\mathbb{B})$ be a linear operator that preserves term rank 1 and term rank k. Then T is a (P,Q)-operator.*

Lemma 3.2. *Let $1 \leq k < l \leq m \leq n$. If $T : \mathscr{M}_{m,n}(\mathbb{B}) \to \mathscr{M}_{m,n}(\mathbb{B})$ is a linear operator that preserves term rank k and term rank l, then T is nonsingular.*

Proof. Suppose that $T(A) = O$ for some nonzero $A \in \mathscr{M}_{m,n}(\mathbb{B})$. Then, there is some cell $E_{i,j}$ such that $T(E_{i,j}) = O$. Without loss of generality, assume that $T(E_{1,1}) = O$. Since $\tau(E_{1,1} + E_{2,2} + \cdots + E_{l,l}) = l$ and T preserves term rank l, we have $\tau(T(E_{2,2} + E_{3,3} + \cdots + E_{l,l})) = \tau(T(E_{1,1} + E_{2,2} + \cdots + E_{l,l})) = l$. Let $X = T(E_{2,2} + \cdots + E_{l,l})$ then we can choose a set of cells $Y = \{F_1, F_2, \ldots, F_l\}$ such that $X \sqsupseteq F_i$ for all $i = 1, \ldots, l$, and $\tau(F_1 + F_2 + \cdots + F_l) = l$. Since $T(E_{2,2} + \cdots + E_{l,l}) = X$, there is some cell in $\{E_{2,2}, \ldots, E_{l,l}\}$ whose image under T dominates two cells in Y. Without loss of generality, we may assume that $T(E_{2,2}) \sqsupseteq F_1 + F_2$. Further, there is some cell in $\{E_{2,2}, \ldots, E_{l,l}\}$ whose image under T dominates F_3. Without loss of generality, we may assume that $T(E_{2,2} + E_{3,3}) \sqsupseteq F_1 + F_2 + F_3$. Continuing this process, there are k cells in $\{E_{2,2}, \ldots, E_{l,l}\}$ the sum of whose images under T dominate $F_1 + F_2 + \cdots F_{k+1}$. Without loss of generality, we may assume that $T(E_{2,2} + \cdots + E_{k+1,k+1}) \sqsupseteq F_1 + F_2 + \cdots + F_{k+1}$, a contradiction since $\tau(E_{2,2} + \cdots + E_{k+1,k+1}) = k$ and $\tau(F_1 + F_2 + \cdots + F_{k+1}) = k+1$. This contradiction establishes the lemma. ∎

Lemma 3.3. *Let $2 < k \leq m \leq n$ and $T : \mathscr{M}_{m,n}(\mathbb{B}) \to \mathscr{M}_{m,n}(\mathbb{B})$ be a linear operator that preserves term rank k. If T does not preserve term rank 1, then $\tau(T(J)) \leq k+2$.*

Proof. By Lemma 3.2, if T does not preserve term rank 1, then there is some rank 1 matrix whose image has term rank 2 or more. Without loss of generality, we may assume that $T(E_{1,1}+E_{1,2}) \sqsupseteq E_{1,1}+E_{2,2}$.

Suppose that $\tau(T(J)) \geq k+3$. Then, $\tau(T(J)[3,\ldots,m|3,\ldots,n]) \geq k-1$. Without loss of generality, we may assume that $T(J)[3,\ldots,m|3,\ldots,n] \sqsupseteq E_{3,3}+E_{4,4}+\cdots+E_{k+1,k+1}$. Thus, there are $k-1$ cells, $F_3,F_4,\ldots,F_{k+1}$ such that $T(F_3+F_4+\cdots+F_{k+1}) \sqsupseteq E_{3,3}+E_{4,4}+\cdots+E_{k+1,k+1}$. Then, $T(E_{1,1}+E_{1,2}+F_3+F_4+\cdots+F_{k+1}) \sqsupseteq I_{k+1}$. But, $\tau(E_{1,1}+E_{1,2}+F_3+F_4+\cdots+F_{k+1}) \leq k$ while $\tau(T(E_{1,1}+E_{1,2}+F_3+F_4+\cdots+F_{k+1})) \geq k+1$, a contradiction. Thus, $\tau(T(J)) \leq k+2$. ∎

Lemma 3.4. *Let $1 \leq k$, $k+3 \leq l \leq m \leq n$. Let $T : \mathscr{M}_{m,n}(\mathbb{B}) \to \mathscr{M}_{m,n}(\mathbb{B})$ be a linear operator that preserves term rank k and term rank l, then T preserves term rank 1.*

Proof. By Lemma 3.2, T is nonsingular. Let A be a term rank 1 matrix and suppose that $T(A)$ is not term rank 1. Then, A is dominated by a row or column and the image of the sum of two cells in that line has term rank at least two. Without loss of generality, we may assume that $T(E_{1,1}+E_{1,2}) \sqsupseteq E_{1,1}+E_{2,2}$. Now, by Lemma 3.3, if $B = T(C)$ is in the image of T, $\tau(B) \leq k+2 < l$. But if we take $B = T(I_l)$, then $T(I_l)$ must have term rank l, a contradiction.

That is, $\tau(T(A)) \leq 1$. Since T is nonsingular and A was an arbitrary term rank 1 matrix, T preserves term rank 1. ∎

Lemma 3.5. *Let $1 \leq k \leq m \leq n$. If $T : \mathscr{M}_{m,n}(\mathbb{B}) \to \mathscr{M}_{m,n}(\mathbb{B})$ is a linear operator that preserves term rank k and term rank $k+2$, then T strongly preserves term rank $k+1$.*

Proof. Let $A \in \mathscr{M}_{m,n}(\mathbb{B})$.

Case 1. Suppose that $\tau(A) = k+1$ and $\tau(T(A)) \geq k+2$. Let $A_1,A_2,\ldots,A_{k+1}$ be matrices of term rank 1 such that $A = A_1+A_2+\cdots+A_{k+1}$. Without loss of generality we may assume that $T(A) \sqsupseteq E_{1,1}+E_{2,2}+\cdots+E_{k+2,k+2}$ and, since the image of some A_i must have term rank at least 2, we may assume that $\tau(T(A_1+A_2+\cdots+A_i)) \geq i+1$, for every $i = 1,2,\ldots,k+1$. But then $\tau(A_1+A_2+\cdots+A_k) = k$ while $\tau(T(A_1+A_2+\cdots+A_k)) \geq k+1$, a contradiction. Thus if $\tau(A) = k+1$, $\tau(T(A)) \leq k+1$.

Case 2. Suppose that $\tau(A) = k+1$ and $\tau(T(A)) = s \leq k$. Without loss of generality, we may assume that $A = E_{1,1}+E_{2,2}+\cdots+E_{k+1,k+1}$ and $T(A) \sqsupseteq E_{1,1}+E_{2,2}+\cdots+E_{s,s}$. Then there are s members of $\{T(E_{1,1}),T(E_{2,2}),\ldots,T(E_{k+1,k+1})\}$ whose sum dominates $E_{1,1}+E_{2,2}+\cdots+E_{s,s}$. Say, without loss of generality, that $T(E_{1,1}+E_{2,2}+\cdots+E_{s,s}) \sqsupseteq E_{1,1}+E_{2,2}+\cdots+E_{s,s}$. Now, $\tau(A+E_{k+2,k+2}) = k+2$ so that $\tau(T(A+E_{k+2,k+2})) = k+2$. But since $\tau(T(A+E_{k+2,k+2})) = \tau((T(A)+T(E_{k+2,k+2})) \leq \tau(T(A))+\tau(T(E_{k+2,k+2}))$, it follows that $\tau(T(E_{k+2,k+2})) \geq k+2-s$ and there are s members of $\{T(E_{1,1}),T(E_{2,2}),\ldots,T(E_{k+1,k+1})\}$ whose sum together with $T(E_{k+2,k+2})$ has term rank $k+2$, say $\tau(T(E_{1,1}+E_{2,2}+\cdots+E_{s,s}+E_{k+2,k+2})) = k+2$. Since $s \leq k$, $\tau(E_{1,1}+E_{2,2}+\cdots+E_{s,s}+E_{k+2,k+2}) \leq k+1$ and $\tau(T(E_{1,1}+E_{2,2}+\cdots+E_{s,s}+E_{k+2,k+2})) = k+2$. By Case 1, we again arrive at a contradiction.

Therefore T strongly preserves term rank $k+1$. ∎

Lemma 3.6. *Let $1 \le k < m \le n$. If $\tau(E_{1,1}+\cdots+E_{k,k}+A) \ge k+1$ and $A[k+1,\ldots,m|k+1,\ldots,n] = O$, then there is some $i, 1 \le i \le k$, such that $\tau(E_{1,1}+\cdots+E_{i-1,i-1}+E_{i+1,i+1},\ldots E_{k,k}+A) \ge k+1$.*

Proof. Suppose that $B = E_{1,1}+\cdots+E_{k,k}+A$ and $\tau(B) \ge k+1$. Then there are $k+1$ cells $F_1, F_2, \ldots, F_{k+1}$ such that $B \sqsupseteq F_1+F_2+\cdots+F_{k+1}$ and $\tau(F_1+F_2+\cdots+F_{k+1}) = k+1$. If $F_1+F_2+\cdots+F_{k+1} \sqsupseteq I_k \oplus O$ then one cell F_j must be a cell $E_{p,q}$ where $p,q \ge k+1$, which contradicts the assumption $A[k+1,\ldots,m|k+1,\ldots,n] = O$. Thus $F_1+F_2+\cdots+F_{k+1}$ does not dominate $I_k \oplus O$. That is, there is some $i, 1 \le i \le k$, such that $\tau(E_{1,1}+\cdots+E_{i-1,i-1}+E_{i+1,i+1},\ldots E_{k,k}+A) \ge k+1$. ∎

Lemma 3.7. *Let $2 \le k+1 \le m \le n$. If $T : \mathscr{M}_{m,n}(\mathbb{B}) \to \mathscr{M}_{m,n}(\mathbb{B})$ is a linear operator that preserves term rank k and term rank $k+1$, then T preserves term rank 1.*

Proof. By Lemma 3.2 T is nonsingular.

If $k+1 = m$, then T strongly preserves term rank m and hence, by [2, Theorem 2], T preserves term rank, and hence, term rank 1. Thus we assume that $k+1 < m$.

Suppose that T does not preserve term rank 1. Then there is some matrix of term rank 1 whose image has term rank at least 2. Without loss of generality, we may assume that $T(E_{1,1}+E_{1,2}) \sqsupseteq E_{1,1}+E_{2,2}$. By Lemma 3.3 we have that $\tau(T(J)) \le k+2$. Since T preserves term rank $k+1$, $\tau(T(J)) \ge k+1$.

Thus, $\tau(T(J)) = k+i$ for either $i=1$ or $i=2$. Now, we may assume that for some r,s with $r+s = k+i$, $T(J)[r+1,\ldots,p|s+1,\ldots,q] = O$. Further, we may assume, without loss of generality, that there are $k+i$ cells $F_1, F_2, \ldots, F_{k+i}$ such that $T(F_l) \sqsupseteq E_{l,k+i-l+1}$ for $l = 1,\ldots,k+i$. Suppose the image of one of the cells in $F_1, F_2, \ldots, F_{k+i}$ dominates more than one cell in $\{E_{1,k+i}, E_{2,k+i-1}, \ldots, E_{k+1,i}\}$. Say, without loss of generality, that $T(F_1) \sqsupseteq E_{1,k+i}+E_{2,k+i-1}$, then, $T(F_1+F_3+\cdots+F_{k+1}) \sqsupseteq E_{1,k+i}+E_{2,k+i-1}+\cdots+E_{k+1,i}$, a contradiction since $\tau(F_1+F_3+\cdots+F_{k+1}) \le k$, and hence $\tau(T(F_1+F_3+\cdots+F_{k+1})) \le k$, and $\tau(E_{1,k+i}+E_{2,k+i-1}+\cdots+E_{k+1,i}) = k+1$. It follows that for each $j = 1,\ldots,k+1$, if $T(F_l) \sqsupseteq E_{j,k+i-j+1}$ then $l = j$ since $T(F_j) \sqsupseteq E_{j,k+i-j+1}$ is unique. Further, by permuting we may assume that $F_1+F_2+\cdots+F_k \sqsubseteq \begin{bmatrix} J_k & O_{k,n-k} \\ O_{m-k,k} & O_{m-k,n-k} \end{bmatrix}$.

Now, let $O \ne A \in \mathscr{M}_{m,n}(\mathbb{B})$ have term rank 1, and suppose that $A[1,2,\ldots,k|1,2,\ldots,n] = O$ and $A[k+1,\ldots m|1,\ldots,k] = O$. So that $A = \begin{bmatrix} O_k & O_{k,n-k} \\ O_{m-k,k} & A_1 \end{bmatrix}$. If $T(A)[k+1,\ldots,p|1,i] = O$, then, since $\tau(F_1+\cdots+F_k+A) = k+1$, $\tau(T(F_1+\cdots+F_k+A)) = k+1$. Applying Lemma 3.6, we have that there is some j such that $\tau(T(F_1+\cdots+F_{j-1}+F_{j+1}+\cdots+F_k+A)) = k+1$. But $\tau(F_1+\cdots+F_{j-1}+F_{j+1}+\cdots+F_k+A) = k$ while $\tau(T(F_1+\cdots+F_{j-1}+F_{j+1}+\cdots+F_k+A)) = k+1$, a contradiction. So we must have that $T(E_{k+1,1})[k+1,\ldots,p|1,i] \ne O$. If $T(E_{k+1,1})[k+1,\ldots,p|1,i] \ne O$ then $\tau(T(F_1+\cdots+F_k+E_{k+1,1})) = k+1$, a contradiction since $\tau(F_1+\cdots+F_k+E_{k+1,1} = k$. Suppose that the $(k,i+1)$ entry of $T(E_{k,k+1})$ is nonzero, then, $\tau(T(F_1+\cdots+F_{k-1}+E_{k,k+1}+E_{k+1,k+1})) = k+1$, a contradiction, since $\tau(F_1+\cdots+F_{k-1}+E_{k,k+1}+E_{k+1,k+1}) = k$.

Consider $T(F_1+\cdots+F_{k-1}+E_{k,k+1}+E_{k+1,k+2})$. This must have term rank $k+1$ and dominates $E_{1,k+i}+E_{2,k+i-1}+\cdots+E_{k-1,i+2}+E_{k+1,j}$ for some $j \in \{1,i\}$. Thus, by Lemma 3.6, there is some cell in $\{F_1,\ldots,F_{k-1}\}$, say F_j such that $\tau(T(F_1+\cdots+F_{j-1}+F_{j+1}+\cdots+F_{k-1}+E_{k,k+1}+E_{k+1,k+2})) = k+1$. But $\tau(F_1+\cdots+F_{j-1}+F_{j+1}+\cdots+F_{k-1}+E_{k,k+1}+E_{k+1,k+2}) = k$, a contradiction.

It follows that T must preserve term rank 1. ∎

Lemma 3.8. *Let* $1 \le k \le m \le n$. *If* $T : \mathscr{M}_{m,n}(\mathbb{B}) \to \mathscr{M}_{m,n}(\mathbb{B})$ *is a linear operator that strongly preserves term rank k, then T preserves term rank* 1.

Proof. Suppose that T does not preserve term rank $k-1$. The there is some matrix $A \in \mathscr{M}_{m,n}(\mathbb{B})$ such that $\tau(A) = k-1$ and $\tau(T(A)) = s \le k-2$ since $\tau(T(A)) < k$. Without loss of generality, we may assume that $A = E_{1,1} + E_{2,2} + \cdots + E_{k-1,k-1}$. Then $T(E_{k,k})$ must have term rank at least $k-s$ since using Lemma 2.1 $k = \tau(T(E_{1,1} + E_{2,2} + \cdots + E_{k-1,k-1} + E_{k,k})) = \tau(T(E_{1,1} + E_{2,2} + \cdots + E_{k-1,k-1}) + T(E_{k,k})) \le \tau(T(E_{1,1} + E_{2,2} + \cdots + E_{k-1,k-1})) + \tau(T(E_{k,k})) = s + \tau(T(E_{k,k}))$. Without loss of generality, we also assume that $T(E_{1,1} + E_{2,2} + \cdots + E_{k-1,k-1} + E_{k,k})) \sqsupseteq E_{1,1} + E_{2,2} + \cdots + E_{k-1,k-1} + E_{k,k}$ and $T(E_{k,k}) \sqsupseteq E_{s+1,s+1} + \cdots + E_{k,k}$. But then, there are s cells in $\{E_{1,1}, E_{2,2}, \ldots, E_{k,k}\}$ the image of whose sum dominates $E_{1,1} + E_{2,2} + \cdots + E_{s,s}$. Say $T(E_{i_1,i_1} + E_{i_2,i_2} + \cdots + E_{i_s,i_s}) \sqsupseteq E_{1,1} + E_{2,2} + \cdots + E_{s,s}$. Then, $T(E_{i_1,i_1} + E_{i_2,i_2} + \cdots + E_{i_s,i_s} + E_{k,k}) \sqsupseteq E_{1,1} + E_{2,2} + \cdots + E_{k,k}$, a contradiction since $\tau(E_{i_1,i_1} + E_{i_2,i_2} + \cdots + E_{i_s,i_s} + E_{k,k}) < k$ while $\tau(T(E_{i_1,i_1} + E_{i_2,i_2} + \cdots + E_{i_s,i_s} + E_{k,k})) = k$. Thus T preserves term rank $k-1$. By Lemma 3.7, T preserves term rank 1. ∎

Theorem 3.2. *Let* $1 \le k < l \le m \le n$. *If* $T : \mathscr{M}_{m,n}(\mathbb{B}) \to \mathscr{M}_{m,n}(\mathbb{B})$ *is a linear operator that preserves term rank k and term rank l, or if T strongly preserves term rank k, then T is a* (P,Q)*-operator.*

Proof. Application of Lemmas 3.4, 3.5, 3.7, and 3.8, give that T preserves term rank 1. By Lemma 3.1, T is a (P,Q)-operator. ∎

4. Term rank preservers of matrices over antinegative semirings.

Throughout this section, $\mathbb{S}$ will denote any commutative and antinegative semiring without zero divisors.

In this section we provide characterizations of linear operators $T : \mathscr{M}_{m,n}(\mathbb{S}) \to \mathscr{M}_{m,n}(\mathbb{S})$ that preserve term rank. Let $A \in \mathscr{M}_{m,n}(\mathbb{S})$ and define $\overline{A} \in \mathscr{M}_{m,n}(\mathbb{B})$ to be the matrix $[\overline{a}_{i,j}]$ where $\overline{a}_{i,j} = 1$ if and only if $a_{i,j} \ne 0$. $\overline{A}$ is called the *support* or *pattern* of A. Clearly $\tau(\overline{A}) = \tau(A)$. Let $T : \mathscr{M}_{m,n}(\mathbb{S}) \to \mathscr{M}_{m,n}(\mathbb{S})$ be a linear operator. Define $\overline{T} : \mathscr{M}_{m,n}(\mathbb{B}) \to \mathscr{M}_{m,n}(\mathbb{B})$ by $\overline{T}(E_{i,j}) = \overline{T(E_{i,j})}$, and extend linearly.

Lemma 4.1. *Let* $T : \mathscr{M}_{m,n}(\mathbb{S}) \to \mathscr{M}_{m,n}(\mathbb{S})$ *be a linear operator. Then T preserves term rank k if and only if* $\overline{T}$ *preserves term rank k, for any* $1 \le k \le m$.

Proof. Since the term rank of a matrix depends only upon the entries being zero or nonzero, we have $\tau(\overline{A}) = \tau(A)$. Further, $\overline{T}(\overline{A}) = \overline{T(A)}$. Thus, $\tau(T(A)) = \tau(\overline{T}(\overline{A}))$, and hence the results follows. ∎

Theorem 4.1. *Let* $T : \mathscr{M}_{m,n}(\mathbb{S}) \to \mathscr{M}_{m,n}(\mathbb{S})$ *be a linear operator. Then the following are equivalent*:

1. *T preserves term rank*;
2. *T preserves term rank k and term rank l, with* $1 \le k < l \le m \le n$;
3. *T strongly preserves term rank h, with* $1 \le h \le m \le n$;
4. *T is a* (P,Q,B)*-operator.*

Proof. It is obvious that 1 implies 2 and 3, and 4 implies 1, 2 and 3. In order to show that 2 (or 3) implies 4, assume that T preserves term rank k and term rank l, with $1 \le k < l \le m \le n$. By Lemma 4.1, $\overline{T}$ preserves term rank k and term rank l, with $1 \le k < l \le m \le n$.. Thus, by

Theorem 3.2, $\overline{T}$ is a (P,Q)-operator. It follows that for every cell $E_{i,j}$, there is some nonzero $b_{i,j} \in \mathbb{S}$ such that for $B = [b_{i,j}]$ either

(1) $T(E_{i,j}) = b_{i,j}(PE_{i,j}Q)$, and

$$\begin{aligned} T(X) &= T\left(\sum_{i=1}^{m}\sum_{j=1}^{n} x_{i,j}E_{i,j}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{i,j}T(E_{i,j}) = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{i,j}b_{i,j}(PE_{i,j}Q) \\ &= P(X \circ B)Q \end{aligned}$$

for every $X \in \mathscr{M}_{m,n}(\mathbb{S})$; or

(2) $m = n$ and $T(E_{i,j}) = b_{i,j}(PE^t_{i,j}Q)$, and

$$\begin{aligned} T(X) &= T\left(\sum_{i=1}^{m}\sum_{j=1}^{n} x_{i,j}E_{i,j}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{i,j}T(E_{i,j}) = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{i,j}b_{i,j}(PE^t_{i,j}Q) \\ &= P(X^t \circ B)Q \end{aligned}$$

for every $X \in \mathscr{M}_{m,n}(\mathbb{S})$.

Thus, T is a (P,Q,B)-operator.

In order to show that 3 implies 4, if we apply Lemma 4.1 and Theorem 3.2, the proof is parallel to the above. ∎

In conclusion, we have obtained new characterizations of the linear operators that preserve term rank, which generalized the previous results in [2] and [5]. Further research should include application of these results to characterize linear transformations between the set of $m \times n$ matrices and $p \times q$ matrices.

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