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Generalized Solution to the Volterra Equations with Piecewise Continuous Kernels

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Abstract. Sufficient conditions for existence and uniqueness of the solution of the Volterra integral equations of the first kind with piecewise continuous kernels are derived in framework of Sobolev-Schwartz distribution theory. The asymptotic approximation of the parametric family of generalized solutions is constructed. The method for the solution's regular part refinement is proposed using the successive approximations method.

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1. Introduction

Let us define the triangular region $D = \{s,t; 0 < s < t < T\}$ and introduce the functions $s = \alpha_i(t), i = \overline{1,n}$, which are continuous and have continuous derivatives for $t \in (0,T)$. We suppose $\alpha_i(0) = 0$, $0 < \alpha_1(t) < \cdots < \alpha_{n-1}(t) < t$ for $t \in (0,T)$, $0 < \alpha_1'(0) < \cdots < \alpha_{n-1}'(0) < 1$, and functions $s = \alpha_i(t)$, $i = \overline{0,n}$, $\alpha_0(t) = 0$, $\alpha_n(t) = t$, split the region D into the following disjoint sectors $D_1 = \{s,t : 0 \leq s < \alpha_1(t)\}$, $D_i = \{s,t : \alpha_{i-1}(t) < s < \alpha_i(t), i = \overline{2,n}\}$, $\overline{D} = \bigcup\limits_1^n D_i$. Let us introduce the continuous functions $K_i(t,s)$ defined for $t,s \in D_i$, and differentiable wrt t, $i = \overline{1,n}$.

Let us consider the integral operator

$$\int\limits_0^t K(t,s)u(s)ds \overset{\text{def}}{=} \sum_{i=1}^n \int\limits_{\alpha_{i-1}(t)}^{\alpha_i(t)} K_i(t,s)u(s)ds \tag{1.1}$$

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with piecewise continuous kernels

$$K(t,s) = \begin{cases} K_1(t,s), \ t,s \in D_1, \\ \dots \quad \dots\dots \\ K_n(t,s), \ t,s \in D_n. \end{cases} \tag{1.2}$$

In this paper we deal with the following Volterra integral equation

$$\int\limits_0^t K(t,s)u(s)\,ds = f(t), 0 < t < T \leq \infty, \tag{1.3}$$

where function $f(t)$ has a continuous derivative for $t \in (0,T)$, $f(0) \neq 0$. Equation (1.3) we call the Volterra integral equation (VIE) with piecewise continuous kernel. Our objective is to construct the solution of VIE (1.3) in the space of Sobolev-Schwartz distributions [19]. Obviously, VIE (1.3) does not have classic solutions since $f(0) \neq 0$.

The differentiation of VIE (1.3) leads to integral-functional equation and its solution is not unique in the general case [6]. That is why study of VIE (1.3) cannot be performed using only the classic methods in the Volterra theory [1,2,5,7]. In this paper we continue our results on VIE studies [9–13,15]. We consider the equation (1.3) using the elementary results of the theory of integral and difference equations, functional analysis [18], Sobolev-Schwartz distributions and theory of functional equations with perturbed argument of neutral type [9].

This paper is organized as follows.

Section 2 outlines the construction of the singular component of the solution and the integral-functional equation for the regular component of the solution is derived. In Section 3 we obtain the sufficient conditions for existence and uniqueness of solution of VIE (1.3) in the following form $u(t) = a\delta(t) + x(t)$, were $\delta(t)$ is Dirac delta function, $x(t)$ is regular continuous function. Such solutions satisfy to the equation (1.3) in the sense of Sobolev-Schwartz distributions [19]. To the best of our knowledge, similar studies on VIE (1.3) have not yet been reported in literature. In Section 3 we construct the regular part of the solution using the "step method" [3] from the theory of functional equations and successive approximations method. In Sections 4 and 5 we address the most interesting case when VIE (1.3) has family of solutions depending on free parameters. The method for construction of asymptotic approximations of parametric solutions is proposed and iterative refinement method is constructed. It is to be noted that known method of A.O. Gelfond (readers may refer to [4, p. 338] of solution of difference equations is employed).

2. Definition of the singular component of the solution

Let us extend $f(t)$ on negative semiaxis with zero and differentiation of VIE (1.3) yields the following equivalent equation

$$\begin{aligned} F(u) \overset{\text{def}}{=} K_n(t,t)u(t) + \sum_{i=1}^{n-1} \alpha_i'(t) \Big\{ K_i(t,\alpha_i(t)) - K_{i+1}(t,\alpha_i(t)) \Big\} u(\alpha_i(t)) \\ + \sum_{i=1}^{n} \int\limits_{\alpha_{i-1}(t)}^{\alpha_i(t)} K_i^{(1)}(t,s)u(s)\,ds = f^{(1)}(t) + f(0)\delta(t), \end{aligned} \tag{2.1}$$

where $\alpha_0 = 0$, $\alpha_n(t) = t$. Let us assume $K_1(0,0) \neq 0$, $K_n(t,t) \neq 0$, for $t \in [0,T]$. Let us introduce the following functional operator

$$Au \overset{\text{def}}{=} \sum_{i=1}^{n-1} K_n^{-1}(t,t)\alpha_i'(t)\{K_i(t,\alpha_i(t)) - K_{i+1}(t,\alpha_i(t))\}u(\alpha_i(t))$$

and integral operator $Ku \overset{\text{def}}{=} \sum_{i=1}^{n} \int\limits_{\alpha_{i-1}(t)}^{\alpha_i(t)} K_n^{-1}(t,t)K_i^{(1)}(t,s)u(s)ds$.

Then equation (2.1) can be reduced to the following equation

$$u(t) + Au + Ku = K_n^{-1}(t,t)f^{(1)}(t) + K_n^{-1}(0,0)f(0)\delta(t). \tag{2.2}$$

Let us search for a solution to VIE (3.2) of the form $u(t) = a\delta(t) + x(t)$, where a is constant, $x(t) \in C_{(0,T)}$.

It is easy to verify the following identities:

$$\int\limits_0^{\alpha_1(t)} \frac{\partial K_1(t,s)}{\partial t}\delta(s)ds = \frac{\partial K_1(t,0)}{\partial t},$$

$$\int\limits_{\alpha_{i-1}(t)}^{\alpha_i(t)} \frac{\partial K_i(t,s)}{\partial t}\delta(s)ds = 0$$

for $i = \overline{2,n}$. Indeed, the first identity holds because $\alpha_1(t) > 0$,

$$\frac{\partial K_1(t,s)}{\partial t}\delta(s) = \frac{\partial K_1(t,0)}{\partial t}\delta(s),$$

$\int_0^{\alpha_1(t)} \delta(s)ds = \theta(\alpha_i(t)) = 1$ for $t > 0$, were θ is Heaviside function. The second identity becomes trivial if we notice that for $i = \overline{2,n}$ $supp\ \delta(s) \cap D_i = 0$, $\int\limits_{\alpha_{i-1}(t)}^{\alpha_i(t)} \delta(s)ds = \theta(\alpha_i(t)) - \theta(\alpha_{i-1}(t)) = 0$, since $0 < \alpha_1(t) < \alpha_2(t) < \cdots < \alpha_n(t) = t$. Let us also recall the identity $\delta(\alpha_i(t)) = \frac{\delta(t)}{|\alpha_i'(0)|}$ (see, e.g., [19, p. 34]). Let us take into account the outlined identities and substitution $u = a\delta(t) + x(t)$ leads the equation (2.2) to the following equation $K_n^{-1}(0,0)K_1(0,0)a\delta(t) + K_n^{-1}(t,t)\frac{\partial K_1(t,0)}{\partial t}a + x(t) + Ax + Kx = K_n^{-1}(t,t)f^{(1)}(t) + K_n^{-1}(0,0)$ $f(0)\delta(t)$. Equating the last equation coefficients of $\delta(t)$ results $a = \frac{f(0)}{K_1(0,0)}$. It is remained to determine the regular part from the equation

$$x(t) + Ax + Kx = \overline{f}(t), \tag{2.3}$$

where $\overline{f}(t) = K_n^{-1}(t,t)\left\{f^{(1)}(t) - \frac{\partial K_1(t,0)}{\partial t}\frac{f(0)}{K_1(0,0)}\right\}$. It is to be noted that due to the operator equality

$$K_n(t,t)(I + A + K)x = F(x)$$

the equation (2.3) can be written as follows

$$F(x) = f'(t) - \frac{\partial K_1(t,0)}{\partial t}\frac{f(0)}{K_1(0,0)}. \tag{2.4}$$

3. Sufficient conditions for the existence of a unique generalized solution

Since $K_1(0,0) \neq 0$ then homogeneous equation (2.1) has only trivial solution of singular functions

$$u_{sing} \stackrel{\text{def}}{=} \sum_0^m c_i \delta^{(i)}(t)$$

with support at the origin. Therefore, the existence and uniqueness of generalized solutions of the equation (2.1)

$$u(t) = u_{sing} + x(t),$$

$x(t) \in \mathbb{C}_{(0,T)}$ is equivalent to proving the existence of a unique solution of equation (2.3) in $\mathbb{C}_{(0,T)}$. Let us introduce the function

$$|A(t)| \stackrel{\text{def}}{=} \sum_{i=1}^{n-1} \left| \alpha_i^{(1)}(t) K_n^{-1}(t,t) \right| |K_i(t,\alpha_i(t)) - K_{i+1}(t,\alpha_i(t))|. \qquad (*)$$

Let the following condition be fulfilled

(A) $|A(0)| < 1, \sup\limits_{0<s<t<T} |K_n^{-1}(t,t)K(t,s)| \leq c < \infty.$

Condition (A) is fulfilled if $\alpha_i^{(1)}(0)$ are sufficiently small. Here and below the kernel $K(t,s)$ in $\bigcup\limits_1^n D_i$ is defined as (1.2). It's derivative wrt t for $t,s \in \bigcup\limits_1^n D_i$ is defined as follows:

$$K^{(1)}(t,s) = \begin{cases} K_1^{(1)}(t,s), \; t,s \in D_1, \\ \dots \quad \dots\dots \\ K_n^{(1)}(t,s), \; t,s \in D_n. \end{cases}$$

Theorem 3.1. *(Sufficient conditions for existence and uniqueness of generalized solutions). Let condition (A) be fulfilled,* $K_i(t,s)$ *in* (1.2) *are continuous functions, and have continuous derivatives wrt t, function* $f(t)$ *has continuous derivative,* $f(0) \neq 0$. *Let* $K_1(0,0) \neq 0$. *Then equation* (1.3) *has the unique solution*

$$u(t) = \frac{f(0)}{K_1(0,0)} \delta(t) + x(t),$$

where $x(t) \in \mathbb{C}_{(0,T)}$. *At the same time we can find* $x(t)$ *using the step method combined with successive approximations.*

Proof. Since the singular part of the solution is defined let us consider the equation (2.3) satisfied by the regular component $x(t)$.

Let us fix $q < 1$ and select $h_1 > 0$ such as $\sup\limits_{0 \leq t \leq h_1} |A(t)| = q < 1$. Due to condition (A) such a variable $h_1 > 0$ exists. Let $0 < h < \min\{h_1, \frac{1-q}{c}\}$, where variable c is defined in condition (A). Let us divide the interval $[0,T]$ into subintervals

$$(3.1) \qquad [0,h], [h, h+\varepsilon h], [h+\varepsilon h, h+2\varepsilon h], \dots.$$

We denote by $x_0(t)$ the restriction of the solution $x(t)$ into $[0,h]$, and by $x_m(t)$ we denote it's restriction into subintervals

$$I_m = [(1+(m-1)\varepsilon)h, (1+m\varepsilon)h], m = 1,2,\dots.$$

Let us select ε from $(0,1]$ such as for $t \in I_m$ “perturbed” arguments $\alpha_i(t) \in \bigcup\limits_{k=1}^{m-1} I_k$, $i = \overline{1,n-1}$. If $0 < \alpha_i^{(1)}(t) < \frac{1}{1+\varepsilon}$ for $t \in [0,T)$, $i = \overline{1,n-1}$, then the above inclusion holds in the interval $[0,T)$. This inclusion makes it possible to apply the well-known in the theory of functional differential equations the method of steps. The readers may refer to [3, p. 199].

Let us construct the sequence $\{x_0^n(t)\}$:

$$x_0^n(t) = -Ax_0^{n-1} - Kx_0^{n-1} + \overline{f}(t),$$

$$x_0^0(t) = \overline{f}(t), t \in [0,h].$$

to define $x_0(t) \in \mathbb{C}_{[0,h]}$

Due to the selection of h we have an estimate $||A+K||_{\mathscr{L}(\mathbb{C}_{(0,h)} \to \mathbb{C}_{(0,h)})} < 1$.

Therefore for $t \in [0,h]$ exists a unique solution $x_0(t)$ of equation (2.3). The sequence $x_0^n(t)$ uniformly converges to the solution. We continue the process of constructing the desired solution for $t \geq h$, i.e. on the intervals I_n, $n = 1,2,\ldots$. For the sake of clarity let $\varepsilon = 1$ in (3.1).

Once we get the element $x_0(t) \in \mathbb{C}_{[0,h]}$ computed we will look for element $x_1(t)$ in the space $\mathbb{C}_{(h,2h)}$. We will find $x_1(t)$ from the Volterra integral equation of the 2nd kind

$$x(t) + \int\limits_h^t K_n^{-1}(t,t)K_t'(t,s)x(s)\,ds = \overline{f}(t) - Ax_0 - \int\limits_0^h K_n^{-1}(t,t)K_t'(t,s)x_0(s)\,ds$$

using the successive approximations, with already defined right hand side.

Let us introduce the continuous function

$$\overline{x}_1(t) = \begin{cases} x_0(t), & 0 \leq t \leq h, \\ x_1(t), & h \leq t \leq 2h, \end{cases} \tag{3.2}$$

which is the reduction of continuous solution $x(t)$ on to $[0,2h]$. Then we can find element $x_2(t) \in \mathbb{C}_{(2h,3h)}$ using the successive approximations from the Volterra integral equation of the 2nd kind

$$x(t) + \int\limits_{2h}^t K_n^{-1}(t,t)K_t'(t,s)x(s)\,ds = \overline{f}(t) - A\overline{x}_1 - \int\limits_0^{2h} K_n^{-1}(t,t)K_t'(t,s)\overline{x}_1(s)\,ds.$$

The desired solution $x(t) \in \mathbb{C}_{(0,T)}$ of VIE (1.3) can be finally constructed by continuation of this process for N steps, $N \geq \frac{T}{h}$. This completes the proof of the theorem. ▮

4. Construction of an asymptotic approximation $\hat{x}(t)$ of the regular part of parametric family of the desired solution

Let us consider the equation (2.4) which is satisfied by the regular part of generalized solution. Let the following condition be fulfilled

(B) Exist polynomials $\mathscr{P}_i = \sum_{\nu+\mu=1}^N K_{i\nu\mu} t^\nu s^\mu$, $i = \overline{1,n}$, $f^N(t) = \sum_{\nu=1}^N f_\nu t^\nu$, $\alpha_i^N(t) = \sum_{\nu=1}^N \alpha_{i\nu} t^\nu$, $i = \overline{1,n-1}$, where $0 < \alpha_{11} < \alpha_{21} < \cdots < \alpha_{n-1,1} < 1$, such as for $t \to +0$, $s \to +0$ we have the following estimates $|K_i(t,s) - \mathscr{P}_i(t,s)| = \mathscr{O}((t+s)^{N+1})$, $i = \overline{1,n}$, $|f(t) - f^N(t)| = \mathscr{O}(t^{N+1})$, $|\alpha_i(t) - \alpha_i^N(t)| = \mathscr{O}(t^{N+1})$, $i = \overline{1,n-1}$.

Expansion in powers of t,s which are presented in condition (B) we call as "Taylor polynomials" of the corresponding functions. Let us introduce the function

$$B(j) = K_n(0,0) + \sum_{i=1}^{n-1} (\alpha_i'(0))^{1+j}(K_i(0,0) - K_{i+1}(0,0)),$$

which depends on argument j, $j \in \mathbb{N} \cup 0$. Function $B(j)$ which corresponds to the main "functional" part of the equation (2.4) is called as *characteristic function* of equation (2.4). Let us consider the construction of asymptotic solution of equation (2.4).

In contrast to Section 3, in Sections 4 and 5 it is not supposed that homogeneous equation for equation (1.3) has only trivial solution. Therefore the solution of integral-functional equation (2.4) can be non unique. Let us follow paper [8] and search for the asymptotic approximation of a particular solution of the inhomogeneous equation (2.4) as following polynomial

$$\hat{x}(t) = \sum_{j=0}^{N} x_j(\ln t)t^j. \tag{4.1}$$

Let us demonstrate that coefficients x_j depend on $\ln t$ and free parameters in general irregular case. This is consistent with the possibility of the existence of nontrivial solutions of the homogeneous equation.

For computation of the coefficients x_j we consider regular and irregular cases.

Definition 4.1. *Point j^* is called regular point of characteristic function $B(j)$, if $B(j^*) \neq 0$ and irregular point otherwise.*

4.1. The regular case: characteristic function $B(j) \neq 0$ for $j \in (0,1,\dots,N)$, where N is sufficiently large

In this case, the coefficients x_j will be constant, i.e. independent on $\ln t$. Indeed, lets substitute expansion (4.1) into equation (2.4). Using the method of undetermined coefficients and taking into account conditions (B), lead to the recursive sequence of the systems of linear algebraic equations wrt x_j :

$$B(0)x_0 = f'(0) - \frac{f(0)}{K_1(0,0)} - \frac{f(0)}{K_1(0,0)} \frac{\partial K_1(t,0)}{\partial t}\bigg|_{t=0}, \tag{4.2}$$

$$B(j)x_j = M_j(x_0,\dots,x_{j-1}),\ j = 1,\dots,N. \tag{4.3}$$

M_j are expressed in terms of solutions $x_0,\dots,x_{j-1}$ of previous equations and coefficients of the Taylor polynomials from the condition (B).

Since in the regular case $B(j) \neq 0$ the coefficients $x_0,\dots,x_N$ can be uniquely determined and the asymptotic expansion (4.1) can be constructed by this means.

4.2. Irregular case: characteristic function $B(j)$ in $(0,1,\dots,N)$ has zeros

Let us demonstrate that in irregular case the coefficients x_j are polynomials in powers of $\ln t$ and depends upon arbitrary constants. The order of polynomials and the number of arbitrary constants are related to the multiplicities of integer solutions of the equation $B(j) = 0$.

Indeed, since the coefficient x_0 in the irregular case can depend on $\ln t$, then based on the method of undetermined coefficients x_0 can be found as the solution of the difference equation

(4.4)
$$K_n(0,0)x_0(z)+\sum_{i=1}^{n-1}\alpha_i'(0)(K_i(0,0)-K_{i+1}(0,0))x_0(z+a_i)=f'(0)-\frac{f(0)}{K_1(0,0)}\frac{K_1(t,0)}{\partial t}\bigg|_{t=0},$$

where $a_i=\ln\alpha'(0), z=\ln t$. There are three possible cases here:
1st case. In this case the coefficient x_0 does not depend on z and can be determined uniquely from the equation (4.2).
2nd case. $(B(0)=0)$.
Let $j=0$ be simple zero of the function $B(j)$, i.e. $B(0)=0, B'(0)\neq 0$. Then the coefficient $x_0(z)$ we can find from the difference equation (4.4) as linear function

$$x_0(z)=x_{01}z+x_{02}. \tag{4.5}$$

Lets substitute (4.5) into (4.4). Thus for determination of the coefficients x_{01},x_{02} we obtain two equations as follows:

$$B(0)x_{01}=0, \tag{4.6}$$

$$B(0)x_{02}+B^{(1)}(0)x_{01}=f'(0)-\frac{f(0)}{K_1(0,0)}\frac{\partial K_1(t,0)}{\partial t}\bigg|_{t=0}, \tag{4.7}$$

where $B(0)=0, B^{(1)}(0)\neq 0$. Hence the coefficient $x_0(z)$ is linear wrt z and depends on the arbitrary constant. So, it the 2nd case

$$x_0(z)=\left(f^{(1)}(0)-\frac{f(0)}{K_1(0,0)}\frac{\partial K_1(0,0)}{\partial t}\right)\frac{1}{B^{(1)}(0)}z+c,$$

where c is arbitrary constant.
3rd case. Let $j=0$ be root of the equation $B(j)=0$ with order of multiplicity of $k+1$, i.e. $B(0)=B'(0)=\dots B^{(k)}(0)=0, B^{(k+1)}(0)\neq 0$, $k\geq 1$. Solution $x_0(z)$ of the difference equation (4.3) we search in the form of a polynomial

$$x_0(z)=x_{01}z^{k+1}+x_{02}z^k+\cdots+x_{0k+1}z+x_{0k+2}. \tag{4.8}$$

Let us substitute polynomial (4.8) into equation (4.4) and take into account the identity

$$\frac{d^k}{dj^k}B(j)=\sum_{i=1}^{n-1}(\alpha_i'(0))^{1+j}a_i^k(K_i(0,0)-K_{i+1}(0,0)),$$

where $a_i=\ln\alpha_i'(0)$. Next lets equate the coefficients of powers

$$z^{k+1},z^k,\dots,z,z^0$$

to zero. Finally we get recurrent sequence of linear algebraic equations wrt $x_{01}, x_{02}, \dots, x_{0k+2}$:

$$
(4.9) \qquad \begin{cases}
B(0)x_{01} = 0, \\
B(0)x_{02} + B^{(1)}(0)\binom{k+1}{k}x_{01} = 0, \\
B(0)x_{0l+1} + B^{(l)}(0)\binom{k+1}{k+1-l}x_{01} + B^{(l-1)}(0)\binom{k}{k+1-l}x_{02} + \dots \\
\dots + B^{(1)}(0)\binom{k+1-l+1}{k+1-l}x_{ol} = 0, \, l = 1, \dots, k,
\end{cases}
$$

(4.10)

$$
B(0)x_{0k+2} + B^{(k+1)}(0)x_{01} + B^{(k)}(0)x_{02} + \dots B^{(1)}(0)x_{0k+1} = f'(0) - \frac{f(0)}{K_1(0,0)} \frac{\partial K_1(t,0)}{\partial t}\bigg|_{t=0}.
$$

In our case $B(0) = B'(0) = \dots = B^{(k)}(0) = 0$, $B^{(k+1)}(0) \neq 0$. Hence in polynomial (4.8) we let

$$
x_{01} = \frac{1}{B^{(k+1)}(0)}\left(f'(0) - \frac{f(0)}{K_1(0,0)}\frac{\partial K_1(0,0)}{\partial t}\right).
$$

Equations of system (4.9) become identities $B(0)x_{0j} = 0$, $j = \overline{1,k+1}$, $B(0) = 0$. Hence coefficients $x_{02}, \dots, x_{0k+2}$ of polynomial (4.8) remain arbitrary constants. Next, let's employ the method of undetermined coefficients and take into account the identity

$$
\int t^j \ln^k t \, dt = t^{j+1} \sum_{s=0}^{k} (-1)^s \frac{k(k-1)\dots(k-(s-1))}{(j+1)^{s+1}} \ln^{k-s} t.
$$

By this means we construct the difference equations for determination of the coefficient $x_1(z)$ $(z = \ln t)$ and next coefficients of the asymptotic expansion (4.1). Indeed,

$$
(4.11) \qquad L(x)\bigg|_{x = x_0(z) + x_1(z)t} \stackrel{\text{def}}{=} \Bigg[K_n(0,0)x_1(z) + \sum_{i=1}^{n-1} (\alpha_i'(0))^2 (K_i(0,0)
$$
$$
- K_{i+1}(0,0))x_1(z + a_i) + P_1(x_0(z))\Bigg] t + r(t), \quad r(t) = o(t).
$$

Here $P_1(x_0(z))$ is the polynomial of z. It's degree is equal to the multiplicity of solution $j = 0$ of equation $B(j) = 0$ as have been proved. From the relation (4.11) due to $r(t) = o(t)$ for $t \to 0$ it follows that coefficient $x_1(z)$ have to satisfy the difference equation

$$
(4.12) \qquad K_n(0,0)x_1(z) + \sum_{i=1}^{n-1} (\alpha'(0))^2 \big(K_i(0,0) - K_{i+1}(0,0)\big) x_1(z + a_i) + P_1(x_0(z)) = 0.
$$

If $B(1) \neq 0$, then the equation (4.12) has solution $x_1(z)$ as the same degree polynomial as multiplicity order of solution $j = 0$ of equation $B(j) = 0$. If $j = 1$ is also the solution of equation $B(j) = 0$ the solution $x_1(z)$ can be constructed as polynomial of the power $k_0 + k_1$, where k_0 and k_1 are corresponding multiplicities of solutions $j = 0$ and $j = 1$ of equation $B(j) = 0$. Coefficient $x_1(z)$ depends on $k_0 + k_1$ arbitrary constants.

Let us introduce the following condition

(C) Let equation $B(j) = 0$ in array $(0, 1, \dots, N)$ has solutions $j_1, \dots, j_\nu$ of multiplicities k_i, $i = \overline{1,\nu}$.

Then, in a similar way we can calculate the remaining coefficients $x_2(z),\dots,x_N(z)$ of asymptotic approximation $\hat{x}(t)$ of solution of equation (2.4) from the following sequence of difference equation

$$K_n(0,0)x_j(z)+\sum_{i=1}^{n-1}(\alpha'(0))^{1+j}\big(K_i(0,0)-K_{i+1}(0,0)\big)x_j(z+a_i)+\mathscr{P}_j(x_0(z),\dots,x_{j-1}(z)))=0,$$

$j=\overline{2,N}$. Thus we have the following lemma.

Lemma 4.1. *Let conditions (B) and (C) be fulfilled. Then exists the function* $\hat{x}(t)=\sum_{i=0}^{N}x_i(\ln t)t^i$, *such as for* $t\to+0$ *the residual solution of equation* (2.4) *satisfies the estimate*

$$\left|F(\hat{x}(t))-f^{(1)}(t)+K^{(1)}(t,0)\frac{f(0)}{K_1(0,0)}\right|=o(t^N).$$

The coefficients $x_i(\ln t)$ *are polynomials of* $\ln t$. *The degrees of these polynomials are increasing and do not exceed the sum of the multiplicities of* $\sum_j k_j$ *of solutions of equation* $B(j)=0$ *from the array* $(0,1,\dots,i)$. *The coefficients* $x_i(\ln t)$ *depend on* $\sum_{j=0}^{i}k_j$ *arbitrary constants.*

Remark 4.1. If $B(j)\neq 0$, then in the sum $\sum_{j=0}^{i}k_j$ we zero the corresponding k_j.

5. An existence theorem for continuous parametric solutions families

Since $0<\alpha_i'(0)<1$, $\alpha_i(0)=0$, $i=\overline{1,n-1}$, then for any $0<\varepsilon<1$ exists $T'\in(0,T]$ such as the following estimates are fulfilled

$$\max_{i=\overline{1,n-1},t\in[0,T']}|\alpha_i'(t)|\le\varepsilon,$$

$$\sup_{i=\overline{1,n-1},t\in(0,T']}\frac{\alpha_i(t)}{t}\le\varepsilon.$$

Let us introduce the condition

(D) Let $K_n(t,t)\neq 0$ for $t\in[0,T']$ and N^* is chosen so large that the following equality

$$\sup_{t\in(0,T')}\varepsilon^{N^*}|A(t)|\le q<1$$

is fulfilled, where function $A(t)$ is defined the Section 3 with formula $(*)$.

Lemma 5.1. *Let condition (D) be fulfilled. Let in* $\mathbb{C}_{(0,T')}$ *class of continuous functions for* $t\in(0,T']$ *which have the limit (which could be infinite) for* $t\to+0$ *exists an element* $\hat{x}(t)$ *such as for* $t\to+0$ *error of the solution of equation* (2.4) *satisfy the estimate*

$$\left|F(\hat{x}(t))-f'(t)+K_1'(t,0)\frac{f(0)}{K_1(0,0)}\right|=o(t^N),$$

where $N\ge N^*$. *Then equation* (2.4) *in* $\mathbb{C}_{(0,T')}$ *has the solution*

$$x(t)=\hat{x}(t)+t^Nv(t), \tag{5.1}$$

where $v(t)$ *is uniquely determined by successive approximations.*

Proof. Substitution of (5.1) in equation (2.4) gives us the following integral-functional equation for determination of the function $v(t)$

$$\begin{aligned} v(t)+K_n(t,t)\Bigg\{\sum_{i=1}^{n-1}\alpha_i'(t)\left(\frac{\alpha_i(t)}{t}\right)^{N^*}\Big(K_i(t,\alpha_i(t))-K_{i+1}(t,\alpha_i(t))\Big)v(\alpha_i(t)) \\ +\sum_{i=1}^{n}\int\limits_{\alpha_{i-1}(t)}^{\alpha_i(t)}K_i^{(1)}(t,s)\left(\frac{s}{t}\right)^{N^*}v(s)\,ds\Bigg\}=\Bigg\{f'(t)-\frac{\partial K_1(t,0)}{\partial t}\frac{f(0)}{K_1(0,0)}-F(\hat{x}(t))\Bigg\}(t^{N^*}K_n(t,t))^{-1}. \end{aligned} \tag{5.2}$$

Let us introduce the linear operators

$$Mu \overset{\text{def}}{=} K_n^{-1}(t,t)\sum_{i=1}^{n-1}\alpha_i'(t)\left(\frac{\alpha_i(t)}{t}\right)^{N^*}\Big\{K_i(t,\alpha_i(t))-K_{i+1}(t,\alpha_i(t))\Big\}v(\alpha_i(t)),$$

$$Kv \overset{\text{def}}{=} \sum_{i=1}^{n}\int\limits_{\alpha_{i-1}(t)}^{\alpha_i(t)}K_n^{-1}(t,t)K_i^{(1)}(t,s)(s/t)^{N^*}v(s)\,ds.$$

Then equation (5.2) can be presented as following operator equation

$$u+(M+K)u=\gamma(t),$$

where $\gamma(t)$ is the right hand side of the equation (5.2). This function is continuous due to condition of the Lemma 5.1. Let us introduce the Banach space X of continuous functions $v(t)$ with norm

$$||v||_l=\max_{0\le t\le T'}e^{-lt}|v(t)|,\, l>0.$$

Then due to the inequalities $\sup\limits_{t\in(0,T']}\frac{\alpha_i(t)}{t}\le\varepsilon<1$ and due to the condition (D) for $\forall l\ge 0$ norm of a linear function of the operator M satisfies

$$||M||_{\mathscr{L}(X\to X)}\le q<1.$$

In addition, for the integral operator K for sufficiently large l the following estimate is correct

$$||K||_{\mathscr{L}(X\to X)}\le q_1<1-q.$$

For sufficiently large $l>0$ this implies that

$$||M+K||_{\mathscr{L}(X\to X)}<1,$$

i.e. the linear operator $M+K$ is a contraction operator in the space X. Hence the sequence $\{v_n\}$ converge where $v_n=-(M+K)v_{n-1}+\gamma(t)$, $v_0=\gamma(t)$. This completes the proof of the theorem. ∎

Theorem 5.1. *Let the following conditions be fulfilled (B), (C), (D),* $f(0)\neq 0$, $K_1(0,0)\neq 0$. *Then equation* (1.3) *for* $0<t\le T'\le T$ *has the solution*

$$x(t)=\frac{f(0)}{K_1(0,0)}\delta(t)+\hat{x}(t)+t^{N^*}v(t),$$

which depends on $\sum_{i=1}^{\nu} k_i$ *arbitrary constants, where* k_i *are determined in condition (C). Function* $\hat{x}$ *is constructed in the form of* (4.1)*, then* $v(t)$ *is uniquely determined with successive approximations. And we have the following asymptotic estimate*

$$\left| x(t) - \frac{f(0)}{K_1(0,0)}\delta(t) - \hat{x}(t) \right| = \mathscr{O}(t^{N^*})$$

for $t \to +0$.

Proof. Based on the Lemma 4.1 because of the conditions of the theorem is possible to construct an asymptotic approximation of the regular part $\hat{x}(t)$ of the solution in the form of the following log-power polynomial:

$$\sum_{i=0}^{N} x_i(\ln t)t^i.$$

In this case, by construction, the coefficients $x_i(\ln t)$ depend on the certain number of arbitrary constants. Due to Lemma 5.1 the substitution $x(t) = \hat{x}(t) + t^{N^*}u(t)$ enable the construction of the continuous function $u(t)$ using the successive approximations method. ▮

The solution constructed on $[0, T']$ can be extended on the whole interval $[0, T]$, based on known method of steps [3, c. 199].

In simple cases one can use the solution of the equivalent equation (2.1) in order to construct the solution of integral equation (1.3) in closed form.

Example 5.1.

$$\int_0^{t/2} x(s)ds + 2\int_{t/2}^{t} x(s)ds = 2 + t, t > 0.$$

An equivalent equation (2.1) in this example has the following form $-\frac{t}{2}x(\frac{t}{2}) + 2x(t) = 2\delta(t) + 1$. The desired solution is as follows $x(t) = 2\delta(t) + 2/3$.

Example 5.2.

$$\int_0^{t/2} x(s)ds - \int_{t/2}^{t} x(s)ds = 1 + t, t > 0.$$

The equivalent equation here is as follows $x(\frac{t}{2}) - x(t) = \delta(t) + 1$. It has c–parametric family of generalized solutions $x(t) = \delta(t) + c - \frac{\ln t}{\ln 2}$, c is constant.

Conclusion. The method proposed in this article does not covering all the feasible generalized solutions of such new class of linear Volterra integral equations with piece-wise continuous kernels. The future work may involve development of new theory with a view to relaxing the smoothness conditions on $K_i(t,s)$ and $f(t)$. Some related work can be found in the monograph [17]. We may also address this equation in the form $\int_0^t K(t,s)\,dg(s) = f(t)$ where $g(s)$ is unknown bounded variation which can be presented as Lebesgue's decomposition. In this case it make sense to seek solution in the form $g = a\mu + \nu$, where a is arbitrary constant value, μ and ν are measures, e.g. the Borel charges of bounded variation on certain intervals.

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