BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **37**(3) (2014), 779–796

Positive Solutions for Second-order Differential Equations with Integral Boundary Conditions

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Abstract. This paper studies the existence of positive solutions for a class of second-order semipositone differential equations with a negatively perturbed term and integral boundary conditions. By using a well-known fixed-point index theorem, some new existence results are derived for the case where nonlinearity is allowed to be sign changing. Several examples are presented to demonstrate the application of our main

2010 Mathematics Subject Classification: 34B15, 34B18

Keywords and phrases: Semipositone, positive solution, boundary value problem, fixed point theorem.

1. Introduction

In this paper, we are concerned with positive solutions of the second order semipositone boundary value problem (BVP for short) with a negatively perturbed term

$$
\begin{cases} -u''(t) = h(t)F(t,u(t)) - p(t), \ t \in (0,1), \\ \alpha u(0) - \beta u'(0) = \int_0^1 u(s)d\xi(s), \\ \gamma u(1) + \delta u'(1) = \int_0^1 u(s)d\eta(s), \end{cases} \tag{1.1}
$$

where $\alpha,\beta,\gamma,\delta \geq 0$ are constants such that $\rho = \beta\gamma+\alpha\gamma+\alpha\delta > 0$, $F:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous, $\xi(s)$ and $\eta(s)$ are two nondecreasing functions of bounded variation, and the integrals in (1.1) are Riemann-Stieltjes integrals, $h,p:(0,1)\to[0,+\infty)$ are Lebesgue integrals and may have finitely many singularities in $[0,1]$.

Semipositone BVPs occur in models for steady-state diffusion with reactions [4] and interest in obtaining conditions for the existence of positive solutions of such problems has been ongoing for many years. For a small sample of such work, we refer the reader to the papers of Agarwal *et al.* [5, 6], Kosmatov [13], Lan [14–16], Liu [17], Ma *et al.* [20, 21],

Communicated by Shangjiang Guo.
Received: March 22, 2012; *Revised:* July 26, 2012.

and Xu *et al.* [25]. In [8], the second-order m-point BVP

$$
\begin{cases} -u''(t) = \lambda f(t,u(t)), \quad t \in (0,1), \\ u'(0) = \sum_{i=1}^{m-2} a_i u'(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases} \tag{1.2}
$$

is studied, where $a_i, b_i > 0$, $(i = 1,2,\ldots,m-2)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, λ is a positive parameter. By using Krasnoselskii fixed point theorem in cones, the authors established the existence results for at least one positive solution to (1.2), assuming that $0 < \sum_{i=1}^{m-2} a_i < 1$, $0 < \sum_{i=1}^{m-2} b_i < 1$, $f : [0,1] \times [0,+\infty) \to (-\infty,+\infty)$ is continuous, and there exists $A > 0$ such that $f(t,u) \geq -A$, for $(t,u) \in [0,1] \times [0,+\infty)$. Although the existence results are still true if the constant A replaced by any continuous function $A(t)$ on $[0,1]$, f also has lower bound.

Recently, Webb and Infante [24] studied arbitrary order semi-positone boundary value problems. The existence of multiple positive solutions is established via a Hammerstein integral equation of the form

$$
u(t) = \int_0^1 k(t,s) g(s) f(s,u(s)) ds,
$$

where k is the corresponding Green function, $g \in L^1[0,1]$ is non-negative and may have pointwise singularities, $f : [0,1] \times [0,+\infty) \to (-\infty,+\infty)$ satisfies Carathéodory conditions and $f(t,u) \geq -A$ for some $A > 0$. The nonlinear term is bounded below by an integral function. It is worth mentioning that the boundary conditions cover local and nonlocal types. Nonlocal boundary conditions are quite general, involving positive linear functionals on the space $C[0,1]$, given by Stieltjes integrals.

When nonlinear term takes nonnegative values, the existence of positive solutions of boundary value problems with nonlocal boundary conditions, including multi-point and integral boundary conditions, has been extensively studied by many researchers in recent years [1–3, 7, 9, 11, 12, 18, 19, 22, 23, 27]. Kong [12] studied the second order singular boundary value problem

$$
\begin{cases} u''(t) + \lambda f(t,u(t)) = 0, \ t \in (0,1), \\ u(0) = \int_0^1 u(s) d\xi(s), \\ u(1) = \int_0^1 u(s) d\eta(s), \end{cases} \tag{1.3}
$$

where λ is a positive parameter, $f : (0,1) \times (0,+\infty) \to [0,+\infty)$ is continuous, $\xi(s)$ and $\eta(s)$ are nondecreasing, and the integrals in (1.3) are Riemann-Stieltjes integrals. Sufficient conditions are obtained for the existence and uniqueness of a positive solution by using the mixed monotone operator theory.

Motivated by the previous work, in this paper, we consider the second order singular integral boundary value problem (1.1). By using Krasnoselskii's fixed point theorem, some new existence results are obtained for the case where the nonlinearity is allowed to be sign changing. We shall address here that the problem tackled has several new features. Firstly, as $p \in L^1[0,1]$, the perturbed effect of p on F may be so large that the nonlinearity may tend to negative infinity at some singular points. Secondly, BVP (1.1) can have finitely many singularities for t in $[0,1]$. Thirdly, the BVP (1.1) is more general as it includes two-point, multi-point, nonlocal problems as special cases. Finally, the nonlinearity is allowed to be sign changing. Hence, our work on (1.1) generalizes and improves some known results in the literature to some degree, for example, [8, 13, 14, 17, 20, 21].

The rest of this paper is organized as follows. In Section 2, we present some lemmas and preliminaries, and we transform the singularly perturbed problem (1.1) to an equivalent approximate problem by constructing a modified function. Sections 3 and 4 give the main results and their proofs. In Section 5, four examples are given to demonstrate the validity of our main results.

2. Preliminaries and lemmas

In this section, we first state Krasnoselskii's fixed point theorem, and then present some lemmas that are very important in the proof of the main results.

Denote

$$\phi_1(t)=\frac{1}{\rho}(\delta+\gamma(1-t)),\ \phi_2(t)=\frac{1}{\rho}(\beta+\alpha t),\ e(t)=G(t,t),\quad t\in[0,1],$$
$$k_1=1-\int_0^1\phi_1(t)d\xi(t),\quad k_2=\int_0^1\phi_2(t)d\xi(t),\quad k_3=\int_0^1\phi_1(t)d\eta(t),$$
$$k_4=1-\int_0^1\phi_2(t)d\eta(t),\quad k=k_1k_4-k_2k_3,\quad \sigma=\frac{\rho}{(\alpha+\beta)(\gamma+\delta)},$$

where

$$G(t,s)=\frac{1}{\rho}\begin{cases}(\beta+\alpha s)(\delta+\gamma(1-t)), & 0\le s\le t\le 1,\\ (\beta+\alpha t)(\delta+\gamma(1-s)), & 0\le t\le s\le 1.\end{cases}\tag{2.1}$$

Obviously,

$$e(t)=\rho\phi_1(t)\phi_2(t)=\frac{1}{\rho}(\beta+\alpha t)(\delta+\gamma(1-t)),\ t\in[0,1],\tag{2.2}$$

$$\sigma e(t)e(s)\le G(t,s)\le e(s)\ (\text{ or } e(t))\le\sigma^{-1},\ t,s\in[0,1].\tag{2.3}$$

Let $X=C[0,1]$ be a real Banach space with the norm $\|x\|=\max_{t\in[0,1]}|x(t)|$ for $x\in X$. We let $P=\{x:x\in X,x(t)\ge\sigma e(t)\|x\| \text{ for } t\in[0,1]\}$. Clearly P is a cone of X.

Lemma 2.1. [10] *Let X be a real Banach space, P is a cone in X. Assume that Ω_1 and Ω_2 are two bounded open sets of X with $\theta\in\Omega_1$ and $\overline{\Omega}_1\subset\Omega_2$. Let $T:P\cap(\overline{\Omega}_2\backslash\Omega_1)\to P$ be completely continuous operator such that either*

(i) $\|Tx\|\le\|x\|,\quad x\in P\cap\partial\Omega_1$ *and* $\|Tx\|\ge\|x\|,\quad x\in P\cap\partial\Omega_2$, *or*
(ii) $\|Tx\|\ge\|x\|,\quad x\in P\cap\partial\Omega_1$ *and* $\|Tx\|\le\|x\|,\quad x\in P\cap\partial\Omega_2$.

Then T has a fixed point in $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

Throughout this paper, we adopt the following assumptions:

($\mathbf{H}_1$) **:** $k_1>0,k_4>0,k>0$.
($\mathbf{H}_2$) **:** $h,p:(0,1)\to[0,+\infty)$ are Lebesgue integrals and there exists $\mu\in(0,\frac{1}{2})$ such that

$$\int_0^1 p(s)ds>0,\ \int_\mu^{1-\mu}e(s)h(s)ds>0.$$

($\mathbf{H}_3$)**:** $F:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous.

Remark 2.1. It follows from (2.3) and (H_2) that

$$\int_{\mu}^{1-\mu} e(s)h(s)ds \le \int_0^1 e(s)[h(s)+p(s)]ds \le \sigma^{-1}\int_0^1 [h(s)+p(s)]ds < +\infty.$$

So for convenience, in the rest of this paper, we define several notations as follows:

$$\Lambda = 1 + \frac{k_4(\gamma+\delta)+k_3(\alpha+\beta)}{\rho k}\int_0^1 d\xi(\tau) + \frac{k_2(\gamma+\delta)+k_1(\alpha+\beta)}{\rho k}\int_0^1 d\eta(\tau),$$

$$\psi(t) = e(t) + \frac{k_4\phi_1(t)+k_3\phi_2(t)}{k}\int_0^1 e(\tau)d\xi(\tau) + \frac{k_2\phi_1(t)+k_1\phi_2(t)}{k}\int_0^1 e(\tau)d\eta(\tau), \tag{2.4}$$

$$\Gamma = \min_{\mu\le 1-\mu} \sigma e(t), \quad L = \Lambda\int_0^1 e(s)h(s)ds, \quad l = \Gamma\int_{\mu}^{1-\mu} e(s)h(s)ds.$$

Remark 2.2. If $x \in C[0,1]\bigcap C^2(0,1)$ satisfies (1.1), and $x(t) > 0$ for any $t \in (0,1)$, then we say that x is a $C[0,1]\bigcap C^2(0,1)$ positive solution of BVP (1.1).

Remark 2.3. From $\alpha,\beta,\gamma,\delta \ge 0$ and $\rho = \beta\gamma+\alpha\gamma+\alpha\delta > 0$, it is easy to show that $e(t) > 0$, $t \in (0,1)$.

Lemma 2.2. *Assume that (H_1) holds. Then for any $y \in L^1[0,1]$, the problem*

$$\begin{cases} -u''(t) = y(t),\ t \in (0,1), \\ \alpha u(0) - \beta u'(0) = \int_0^1 u(s)d\xi(s), \\ \gamma u(1) + \delta u'(1) = \int_0^1 u(s)d\eta(s), \end{cases}$$

has a unique solution

$$u(t) = \int_0^1 H(t,s)y(s)ds, \tag{2.5}$$

where

$$H(t,s) = G(t,s) + \frac{k_4\phi_1(t)+k_3\phi_2(t)}{k}\int_0^1 G(\tau,s)d\xi(\tau) + \frac{k_2\phi_1(t)+k_1\phi_2(t)}{k}\int_0^1 G(\tau,s)d\eta(\tau).$$

Moreover, $u(t) \ge 0$ on $[0,1]$ provided $y \ge 0$.

Proof. The proof is similar to Lemma 2.2 of [26], so we omit it. ∎

Lemma 2.3. *Suppose that (H_1) and (H_2) hold, then for any $t,s \in [0,1]$, we have*

$$\sigma e(t)\mathscr{H}(s) \le H(t,s) \le \mathscr{H}(s), \tag{2.6}$$

where

$$\mathscr{H}(s) = e(s) + \frac{k_4(\gamma+\delta)+k_3(\alpha+\beta)}{\rho k}\int_0^1 G(\tau,s)d\xi(\tau) + \frac{k_2(\gamma+\delta)+k_1(\alpha+\beta)}{\rho k}\int_0^1 G(\tau,s)d\eta(\tau).$$

Proof. By (2.3) and the monotonicity of ϕ_1, ϕ_2, it is easy to get $H(t,s) \le \mathscr{H}(s)$.

On the other hand, by (2.2) and the monotonicity of ϕ_1, ϕ_2, we have

$$\phi_1(t) = \frac{e(t)}{\rho\phi_2(t)} = \frac{e(t)}{\alpha t+\beta} \ge \frac{e(t)}{\alpha+\beta}, \quad t \in [0,1], \tag{2.7}$$

$$\phi_2(t) = \frac{e(t)}{\rho\phi_1(t)} = \frac{e(t)}{\gamma(1-t)+\delta} \ge \frac{e(t)}{\gamma+\delta}, \quad t \in [0,1]. \tag{2.8}$$

By (2.3), (2.7) and (2.8), for any $t,s \in [0,1]$, we have

$$\begin{aligned}
H(t,s) \ge &\sigma e(t)e(s) + e(t)\left[\frac{\frac{k_4}{\alpha+\beta}+\frac{k_3}{\gamma+\delta}}{k}\int_0^1 G(\tau,s)d\xi(\tau) + \frac{\frac{k_2}{\alpha+\beta}+\frac{k_1}{\gamma+\delta}}{k}\int_0^1 G(\tau,s)d\eta(\tau)\right]\\
=&\sigma e(t)e(s) + \sigma e(t)\left[\frac{k_4(\gamma+\delta)+k_3(\alpha+\beta)}{\rho k}\int_0^1 G(\tau,s)d\xi(\tau)\right.\\
&\left. + \frac{k_2(\mu+\delta)+k_1(\alpha+\beta)}{\rho k}\int_0^1 G(\tau,s)d\eta(\tau)\right]\\
=&\sigma e(t)\mathscr{H}(s).
\end{aligned}$$

The proof of Lemma 2.3 is completed. ∎

Lemma 2.4. *Suppose that* (H_1) *and* (H_2) *hold. Then the problem*

$$\begin{cases} -w''(t) = p(t), \quad t \in (0,1),\\ \alpha w(0) - \beta w'(0) = \int_0^1 w(s)d\xi(s),\\ \gamma w(1) + \delta w'(1) = \int_0^1 w(s)d\eta(s), \end{cases}$$

has unique solution

$$w(t) = \int_0^1 H(t,s)p(s)ds, \tag{2.9}$$

which satisfies

$$w(t) \le \psi(t)\int_0^1 p(s)ds, \quad t \in [0,1]. \tag{2.10}$$

where $\psi(t)$ *is defined by* (2.4).

Proof. It follows from (2.3), (2.5), (H_1) and (H_2) that (2.9) and (2.10) hold. ∎

Remark 2.4. By (2.3), (2.4) and (2.10), it is not difficult to show that the inequality $w(t) \le \sigma^{-1}\Lambda\int_0^1 p(s)ds$ is valid.

For any $u \in X$, let us define a function $[\cdot]^+$,

$$[u(t)]^+ = \begin{cases} u(t), & u(t) \ge 0,\\ 0, & u(t) < 0. \end{cases}$$

Next we consider the following approximate problem of (1.1)

$$\begin{cases} -x''(t) = h(t)F(t,[x(t)-w(t)]^+), \ t \in (0,1),\\ \alpha x(0) - \beta x'(0) = \int_0^1 x(s)d\xi(s),\\ \gamma x(1) + \delta x'(1) = \int_0^1 x(s)d\eta(s). \end{cases} \tag{2.11}$$

Lemma 2.5. *If $x \in C[0,1]\bigcap C^2(0,1)$ is a positive solution of problem* (2.11)*, then*

$$x(t) \geq \frac{\sigma}{\Lambda \int_0^1 p(s)ds}\|x\|\omega(t). \tag{2.12}$$

Moreover, if $x(t) \geq w(t)$ for any $t \in [0,1]$, then $x - w$ is a positive solution of the singular semipositone differential equation (1.1).

Proof. (i) According to (2.5), $x \in C[0,1]\bigcap C^2(0,1)$ is a solution of equation (2.11) if and only if $x \in C[0,1]$ is a solution of the following nonlinear integral equation

$$x(t) = \int_0^1 H(t,s)h(s)F(s,[x(s)-w(s)]^+)ds.$$

Then by (2.3), we have

$$\begin{aligned}
\|x\| \leq& \int_0^1 e(s)h(s)F(s,[x(s)-w(s)]^+)ds \\
&+ \frac{k_4(\gamma+\delta)+k_3(\alpha+\beta)}{\rho k}\int_0^1 d\xi(\tau)\int_0^1 e(s)h(s)F(s,[x(s)-w(s)]^+)ds \\
&+ \frac{k_2(\gamma+\delta)+k_1(\alpha+\beta)}{\rho k}\int_0^1 d\eta(\tau)\int_0^1 e(s)h(s)F(s,[x(s)-w(s)]^+)ds \\
=&\Lambda\int_0^1 e(s)h(s)F(s,[x(s)-w(s)]^+)ds, \\
x(t) \geq& \sigma e(t)\int_0^1 e(s)h(s)F(s,[x(s)-w(s)]^+)ds \\
&+ \frac{k_4\phi_1(t)+k_3\phi_2(t)}{k}\int_0^1 \sigma e(\tau)d\xi(\tau)\int_0^1 e(s)h(s)F(s,[x(s)-w(s)]^+)ds \\
&+ \frac{k_2\phi_1(t)+k_1\phi_2(t)}{k}\int_0^1 \sigma e(\tau)d\eta(\tau)\int_0^1 e(s)h(s)F(s,[x(s)-w(s)]^+)ds \\
=&\sigma\psi(t)\int_0^1 e(s)h(s)F(s,[x(s)-w(s)]^+)ds.
\end{aligned}$$

This together with (2.10), we have

$$x(t) \geq \sigma\psi(t)\Lambda^{-1}\|x\| \geq \frac{\sigma}{\Lambda\int_0^1 p(s)ds\|x\|\omega(t).}$$

(ii) If x is a positive solution of equation (2.11) such that $x(t) \geq w(t)$ for any $t \in [0,1]$, then from (2.11) and the definition of $[u(t)]^+$, we have

$$\begin{cases} -x''(t) = h(t)F(t,x(t)-w(t)),\ t \in (0,1), \\ \alpha x(0) - \beta x'(0) = \int_0^1 x(s)d\xi(s), \\ \gamma x(1) + \delta x'(1) = \int_0^1 x(s)d\eta(s). \end{cases} \tag{2.13}$$

Let $u = x - w$, then $u'' = x'' - w''$, which implies that

$$-x'' = -u'' - w'' = -u'' + p(t).$$

Thus (2.13) becomes

$$\begin{cases} -u''(t) = h(t)F(t,u(t)) - p(t),\ t \in (0,1), \\ \alpha u(0) - \beta u'(0) = \int_0^1 u(s)d\xi(s), \\ \gamma u(1) + \delta u'(1) = \int_0^1 u(s)d\eta(s), \end{cases}$$

i.e. $x-w$ is a positive solution of (1.1). This completes the proof of Lemma 2.5. ∎

Define a nonlinear integral operator $T : X \to X$ as follows:

$$Tx(t) = \int_0^1 H(t,s)h(s)F(s,[x(s)-w(s)]^+)ds.$$

Then, problem (2.11) is equivalent to the fixed point equation $Tx = x$ in the Banach space $X = C[0,1]$.

Lemma 2.6. *Assume that (H_1)–(H_3) hold. Then $T : P \to P$ is well defined, and $T : P \to P$ is a completely continuous operator.*

Proof. For any fixed $x \in P$, there exists a constant $\widetilde{L} > 0$ such that $\|x\| \le \widetilde{L}$. And then,

$$[x(s)-w(s)]^+ \le x(s) \le \|x\| \le \widetilde{L}. \tag{2.14}$$

Thus, for any $t \in [0,1]$, it follows from (2.6) and (2.14) that

$$\begin{aligned}
|Tx(t)| \le& \int_0^1 \mathscr{H}(s)h(s)F(s,[x(s)-w(s)]^+)ds \\
\le& M\int_0^1 \mathscr{H}(s)h(s)ds \\
\le& M\left[\int_0^1 e(s)h(s)ds \right. \\
&+ \frac{k_4(\gamma+\delta)+k_3(\alpha+\beta)}{\rho k}\int_0^1 e(\tau)d\xi(\tau)\int_0^1 h(s)ds \\
&\left. + \frac{k_2(\gamma+\delta)+k_1(\alpha+\beta)}{\rho k}\int_0^1 e(\tau)d\eta(\tau)\int_0^1 h(s)ds\right] \\
<& +\infty,
\end{aligned} \tag{2.15}$$

where

$$M = \max_{(t,u)\in[0,1]\times[0,\widetilde{L}]} F(t,u). \tag{2.16}$$

Thus $T : P \to X$ is well defined.

Next for any $x \in P$, by (2.6), we have

$$\begin{aligned}
\|Tx\| =& \max_{0\le t\le 1}\int_0^1 H(t,s)h(s)F(s,[x(s)-w(s)]^+)ds \\
\le& \int_0^1 \mathscr{H}(s)h(s)F(s,[x(s)-w(s)]^+)ds.
\end{aligned}$$

On the other hand, from (2.6), we also obtain

$$\begin{aligned}
Tx(t) =& \int_0^1 H(t,s)h(s)F(s,[x(s)-w(s)]^+)ds \\
\ge& \sigma e(t)\int_0^1 \mathscr{H}(s)h(s)F(s,[x(s)-w(s)]^+)ds.
\end{aligned}$$

So

$$Tx(t) \ge \sigma e(t)\|Tx\|, \ \ t \in [0,1].$$

This yields that $T(P) \subset P$.

Let $D \subset P$ be any bounded set. Then there exists a constant $\widetilde{L} > 0$ such that $\|x\| \leq \widetilde{L}$ for any $x \in D$. Thus, for any $x \in D, s \in [0,1]$, we have $[x(s) - w(s)]^+ \leq x(s) \leq \|x\| \leq \widetilde{L}$. It follows from (2.15) and (2.16) that $T(D)$ is uniformly bounded. Now we show that $T(D)$ is equicontinuous on $[0,1]$. Since ϕ_1, ϕ_2 and $G(t,s)$ are uniformly continuous on $[0,1]$ and $[0,1] \times [0,1]$ respectively, for any $\varepsilon > 0$, there exists a constant $\delta_0 > 0$ such that for any $s \in [0,1]$, $t, t' \in [0,1]$, when $|t - t'| < \delta_0$, it holds

$$\begin{aligned}
&|G(t,s) - G(t',s)| < \left[3M \int_0^1 h(s)ds\right]^{-1} \varepsilon, \\
&|\phi_1(t) - \phi_1(t')| < \left\{3M \left[\frac{k_4}{k} \int_0^1 e(\tau)d\xi(\tau) + \frac{k_2}{k} \int_0^1 e(\tau)d\eta(\tau)\right] \int_0^1 h(s)ds\right\}^{-1} \varepsilon, \\
&|\phi_2(t) - \phi_2(t')| < \left\{3M \left[\frac{k_3}{k} \int_0^1 e(\tau)d\xi(\tau) + \frac{k_1}{k} \int_0^1 e(\tau)d\eta(\tau)\right] \int_0^1 h(s)ds\right\}^{-1} \varepsilon,
\end{aligned}$$

where M is defined by (2.16).

Consequently, for any $t, t' \in [0,1]$ and $|t - t'| < \delta_0$, we have

$$\begin{aligned}
&|Tx(t) - Tx(t')| \leq \int_0^1 |H(t,s) - H(t',s)| h(s) F(s, [x(s) - w(s)]^+) ds \\
\leq& \int_0^1 |G(t,s) - G(t',s)| h(s) F(s, [x(s) - w(s)]^+) ds + \frac{1}{k}\Big[k_4|\phi_1(t) - \phi_1(t')| \\
&+ k_3|\phi_2(t) - \phi_2(t')|\Big] \int_0^1 \int_0^1 G(\tau,s) h(s) F(s, [x(s) - w(s)]^+) d\xi(\tau) ds \\
&+ \frac{1}{k}\Big[k_2|\phi_1(t) - \phi_1(t')| + k_1|\phi_2(t) - \phi_2(t')|\Big] \\
&\times \int_0^1 \int_0^1 G(\tau,s) h(s) F(s, [x(s) - w(s)]^+) d\eta(\tau) ds \\
\leq& M \int_0^1 |G(t,s) - G(t',s)| h(s) ds \\
&+ \frac{M}{k}\Big[k_4|\phi_1(t) - \phi_1(t')| + k_3|\phi_2(t) - \phi_2(t')|\Big] \int_0^1 e(\tau)d\xi(\tau) \int_0^1 h(s)ds \\
&+ \frac{M}{k}\Big[k_2|\phi_1(t) - \phi_1(t')| + k_1|\phi_2(t) - \phi_2(t')|\Big] \int_0^1 e(\tau)d\eta(\tau) \int_0^1 h(s)ds \\
=& M \int_0^1 |G(t,s) - G(t',s)| h(s) ds \\
&+ M \left[\frac{k_4}{k} \int_0^1 e(\tau)d\xi(\tau) + \frac{k_2}{k} \int_0^1 e(\tau)d\eta(\tau)\right] \int_0^1 h(s)ds |\phi_1(t) - \phi_1(t')| \\
&+ M \left[\frac{k_3}{k} \int_0^1 e(\tau)d\xi(\tau) + \frac{k_1}{k} \int_0^1 e(\tau)d\eta(\tau)\right] \int_0^1 h(s)ds |\phi_2(t) - \phi_2(t')| \\
<& \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

This implies that $T(D)$ is equicontinuous. Thus according to the Ascoli-Arzela Theorem, $T(D)$ is a relatively compact set.

Now, from the continuity of F, it is not difficult to see that $T : P \to P$ is continuous. Thus $T : P \to P$ is a completely continuous operator. The proof is completed. ∎

3. Existence of one positive solution

Theorem 3.1. *Assume that conditions* (H_1)–(H_3) *are satisfied. Further assume that the following conditions hold:*
(S_1) *There exists a constant*

$$R_0 > \Lambda\sigma^{-1}\Gamma^{-1}\int_0^1 p(s)ds \tag{3.1}$$

such that for any $(t,u) \in [0,1]\times[0,R_0]$,

$$F(t,u) \le \frac{R_0}{L},$$

where Λ, L *and* Γ *are defined by* (2.4).
(S_2) *The function* F *satisfies*

$$\lim_{u\to+\infty}\min_{t\in[\mu,1-\mu]}\frac{F(t,u)}{u} = +\infty.$$

Then BVP (1.1) *has at least one positive solution* u *satisfying* $u(t) \ge \widetilde{l}e(t)$ *for some positive constant* $\widetilde{l}$.

Proof. Let $\Omega_1 = \{x \in P : \|x\| < R_0\}$. Then for any $x \in \partial\Omega_1, s \in [0,1]$, we have

$$[x(s)-w(s)]^+ \le x(s) \le \|x\| \le R_0.$$

It follows from (S_1) that

$$\begin{aligned}
\|Tx\| \le & \int_0^1 \mathscr{H}(s)h(s)F(s,x(s)-w(s))ds \\
< & \frac{R_0}{L}\int_0^1 \mathscr{H}(s)h(s)ds \\
\le & \frac{R_0}{L}\left[\int_0^1 e(s)h(s)ds\right. \\
& + \frac{k_4(\gamma+\delta)+k_3(\alpha+\beta)}{\rho k}\int_0^1\int_0^1 G(\tau,s)h(s)d\xi(\tau)ds \\
& \left.+ \frac{k_2(\gamma+\delta)+k_1(\alpha+\beta)}{\rho k}\int_0^1\int_0^1 G(\tau,s)h(s)d\eta(\tau)ds\right] \\
\le & \frac{R_0}{L}\left[\int_0^1 e(s)h(s)ds\right. \\
& + \frac{k_4(\gamma+\delta)+k_3(\alpha+\beta)}{\rho k}\int_0^1 d\xi(\tau)\int_0^1 e(s)h(s)ds \\
& \left.+ \frac{k_2(\gamma+\delta)+k_1(\alpha+\beta)}{\rho k}\int_0^1 d\eta(\tau)\int_0^1 e(s)h(s)ds\right] \\
= & R_0 = \|x\|,
\end{aligned} \tag{3.2}$$

which means that

$$\|Tx\| < \|x\|, \quad x \in \partial\Omega_1. \tag{3.3}$$

On the other hand, choose a real number $M > 0$ such that

$$\frac{\Gamma^2 M}{2}\int_\mu^{1-\mu} e(s)h(s)ds > 1.$$

From (S_2), there exists $N > R_0$ such that, for any $t \in [\mu, 1-\mu]$,

$$F(t,u) \geq Mu, \;\; u \geq N. \tag{3.4}$$

Take $R^* = \max\{2\Gamma^{-1}N, 2R_0\}$. Let $\Omega_2 = \{x \in P : \|x\| < R^*\}$. Then for any $x \in \partial\Omega_2, t \in [\mu, 1-\mu]$, we have

$$\begin{aligned} x(t) - w(t) &\geq x(t) - \psi(t)\int_0^1 p(s)ds \geq R^*\sigma e(t)\sigma^{-1}\Lambda\int_0^1 p(s)ds \\ &\geq R^*\Gamma - \sigma^{-1}\Lambda\int_0^1 p(s)ds \geq \frac{\Gamma}{2}R^* \geq N > 0. \end{aligned} \tag{3.5}$$

So for any $x \in \partial\Omega_2$, $t \in [\mu, 1-\mu]$, by (2.6), (3.4) and (3.5) we have

$$\begin{aligned} Tx(t) &= \int_0^1 H(t,s)h(s)F(s,[x(s)-w(s)]^+)ds \\ &\geq \sigma e(t)\int_0^1 \mathscr{H}(s)h(s)F(s,[x(s)-w(s)]^+)ds \\ &\geq \sigma e(t)\int_\mu^{1-\mu} \mathscr{H}(s)h(s)F(s,[x(s)-w(s)]^+)ds \\ &\geq \sigma e(t)\int_\mu^{1-\mu} e(s)h(s)F(s,[x(s)-w(s)]^+)ds \\ &\geq \Gamma\int_\mu^{1-\mu} e(s)h(s)ds \times \frac{\Gamma}{2}MR^* \\ &= \frac{\Gamma^2 MR^*}{2}\int_\mu^{1-\mu} e(s)h(s)ds > R^* = \|x\|. \end{aligned} \tag{3.6}$$

Therefore,

$$\|Tx\| > \|x\|, \quad x \in \partial\Omega_2. \tag{3.7}$$

By (3.3), (3.7) and Lemma 2.1, T has a fixed point x in P such that $R_0 < \|x\| < R^*$. Since $R_0 > \Lambda\sigma^{-1}\Gamma^{-1}\int_0^1 p(s)ds > \Lambda\sigma^{-1}\int_0^1 p(s)ds$, combining with (2.12) yields that

$$\begin{aligned} x(t) - w(t) \geq & x(t) - \frac{\Lambda x(t)}{\sigma\|x\|}\int_0^1 p(s)ds \geq x(t) - \frac{\Lambda x(t)}{\sigma R_0}\int_0^1 p(s)ds \\ &\geq \left(1 - \frac{\Lambda\int_0^1 p(s)ds}{\sigma R_0}\right)x(t) \geq \left(1 - \frac{\Lambda\int_0^1 p(s)ds}{\sigma R_0}\right)R_0\sigma e(t) \\ &= \widetilde{l}e(t), \;\; t \in (0,1), \end{aligned} \tag{3.8}$$

where $\widetilde{l} = \left(1 - \frac{\Lambda\int_0^1 p(s)ds}{\sigma R_0}\right)R_0\sigma > 0$.

Let $u(t) = x(t) - w(t)$, then we have $u(t) \geq \widetilde{l}e(t) > 0$. By Lemma 2.5, we know that BVP (1.1) has at least one positive solution u satisfying $u \geq \widetilde{l}e(t)$ for any $t \in [0,1]$. The proof is completed. ∎

Theorem 3.2. *Assume that conditions* (H_1)–(H_3) *are satisfied. In addition, assume that the following conditions hold:*
(S_3) *There exists a constant* $r > 2\Lambda\sigma^{-1}\Gamma^{-1}\int_0^1 p(s)ds$ *such that for any* $(t,u) \in [\mu, 1-\mu] \times \left[\frac{\Gamma}{2}r, r\right]$,

$$F(t,u) > \frac{r}{l},$$

where Λ, l *and* Γ *are defined by* (2.4).
(S_4) *The function* F *satisfies*

$$\lim_{u\to+\infty} \max_{t\in[0,1]} \frac{F(t,u)}{u} = 0.$$

Then BVP (1.1) *has at least one positive solution* u *satisfying* $u(t) \geq \widetilde{l}e(t)$ *for some positive constant* $\widetilde{l}$.

Proof. Firstly, let $\Omega_1 = \{x \in P : \|x\| < r\}$. Then for any $x \in \partial\Omega_1, t \in [\mu, 1-\mu]$, we have

$$(3.9)\quad \begin{aligned} x(t) - w(t) &\geq x(t) - \psi(t)\int_0^1 p(s)ds \geq r\sigma e(t) - \sigma^{-1}\Lambda\int_0^1 p(s)ds \\ &\geq r\Gamma - \sigma^{-1}\Lambda\int_0^1 p(s)ds \geq \frac{\Gamma}{2}r > 0. \end{aligned}$$

So for any $x \in \partial\Omega_1, t \in [\mu, 1-\mu]$, by (3.9) we have

$$(3.10)\quad \frac{\Gamma}{2}r \leq x(t) - w(t) \leq r.$$

It follows from (S_3), (3.10) and (2.6) that, for any $x \in \partial\Omega_1$, $t \in [\mu, 1-\mu]$, we have

$$(3.11)\quad \begin{aligned} Tx(t) &= \int_0^1 H(t,s)h(s)F(s,[x(s)-w(s)]^+)ds \\ &\geq \sigma e(t)\int_0^1 \mathscr{H}(s)h(s)F(s,[x(s)-w(s)]^+)ds \\ &\geq \sigma e(t)\int_\mu^{1-\mu} \mathscr{H}(s)h(s)F(s,[x(s)-w(s)]^+)ds \\ &\geq \sigma e(t)\int_\mu^{1-\mu} e(s)h(s)F(s,[x(s)-w(s)]^+)ds \\ &> \Gamma\int_\mu^{1-\mu} e(s)h(s)ds \times \frac{r}{l} \\ &= r = \|x\|. \end{aligned}$$

Therefore,

$$(3.12)\quad \|Tx\| > \|x\|, \quad x \in \partial\Omega_1.$$

Next, let us choose $\varepsilon > 0$ such that $\varepsilon\int_0^1 \mathscr{H}(s)h(s)ds < 1$. Then for the above ε, by (S_4), there exists $N > r > 0$ such that, for any $t \in [0,1]$ and for any $u > N$, $F(t,u) \leq \varepsilon u$.

Take

$$R^* = \frac{\tau\Big\{\int_0^1 \mathscr{H}(s)[h(s)+p(s)]ds + \int_0^1 \mathscr{H}(s)p(s)ds\Big\}}{1-\varepsilon\int_0^1 \mathscr{H}(s)h(s)ds} + N,$$

where $\tau = \max_{t\in[0,1],u\le N} F(t,u)$. Then $R^* > N > r$.

Now let $\Omega_2 = \{x \in P : \|x\| < R^*\}$. Then for any $x \in \partial\Omega_2$, we have

$$\begin{aligned}
\|Tx\| \le & \int_0^1 \mathscr{H}(s)h(s)F(s,[x(s)-w(s)]^+)ds \\
& < \max_{t\in[0,1],u\le N} F(t,u)\int_0^1 \mathscr{H}(s)h(s)ds \\
& \quad + \int_0^1 \mathscr{H}(s)h(s)\varepsilon[x(s)-w(s)]^+ds \\
& \le \tau\int_0^1 \mathscr{H}(s)h(s)ds + \int_0^1 \mathscr{H}(s)h(s)\varepsilon\|x\|ds \\
& \le \tau\int_0^1 \mathscr{H}(s)h(s)ds + \varepsilon R^*\int_0^1 \mathscr{H}(s)h(s)ds \\
& < \left(1-\varepsilon\int_0^1 \mathscr{H}(s)h(s)ds\right)R^* + \varepsilon R^*\int_0^1 \mathscr{H}(s)h(s)ds \\
& = R^* = \|x\|.
\end{aligned} \tag{3.13}$$

Thus we get

$$\|Tx\| < \|x\|, \quad x \in \partial\Omega_2. \tag{3.14}$$

By (3.12), (3.14) and Lemma 2.1, T has a fixed point x in P such that $r < \|x\| < R^*$. Since $r > 2\Lambda\sigma^{-1}\Gamma^{-1}\int_0^1 p(s)ds > 2\Lambda\sigma^{-1}\int_0^1 p(s)ds$, combining with (2.12) yields

$$\begin{aligned}
x(t)-w(t) &\ge x(t) - \frac{\Lambda x(t)}{\sigma\|x\|}\int_0^1 p(s)ds \ge x(t) - \Lambda\frac{x(t)}{\sigma r}\int_0^1 p(s)ds \\
&\ge \frac{1}{2}x(t) \ge \frac{1}{2}r\sigma e(t) = \widetilde{l}e(t), \quad t\in(0,1),
\end{aligned} \tag{3.15}$$

where $\widetilde{l} = \frac{1}{2}r\sigma > 0$.

Let $u(t) = x(t)-w(t)$, then we have $u(t) \ge \widetilde{l}e(t) > 0$. By Lemma 2.5, we know that BVP (1.1) has at least one positive solution u satisfying $u \ge \widetilde{l}e(t)$ for any $t\in[0,1]$. The proof is completed. ∎

4. Existence of two positive solutions

Theorem 4.1. *Assume that conditions (H_1)–(H_3) and $(S_1),(S_4)$ are satisfied. Further assume that the following condition holds:*
(C_1) There exists a constant $R > 2R_0$ such that for any $(t,u)\in[\mu,1-\mu]\times\left[\frac{\Gamma}{2}R,R\right]$,

$$F(t,u) > \frac{R}{l},$$

where l,Γ and R_0 are defined by (2.4) and (3.1), respectively.

Then BVP (1.1) *has at least two positive solutions* u_1,u_2 *satisfying* $u_1(t) \geq l_1e(t)$, $u_2(t) \geq l_2e(t)$ *for* $l_1 = \left(1 - \frac{\Lambda\int_0^1 p(s)ds}{\sigma R_0}\right) R_0\sigma$, $l_2 = \frac{1}{2}r\sigma$.

Proof. Set $\Omega_1 = \{x \in P : \|x\| < R_0\}$. From (S_1) and proceeding as in (3.2), we have

$$\|Tx\| < \|x\|, \ \ x \in \partial\Omega_1. \tag{4.1}$$

On the other hand, Let $\Omega_2 = \{x \in P : \|x\| < R\}$. Similar to the proof of Theorem 3.2, by (C_1) and (2.6), we have

$$\|Tx\| > \|x\|, \ \ x \in \partial\Omega_2. \tag{4.2}$$

Next, let us choose $\varepsilon > 0$ such that $\varepsilon\int_0^1 \mathscr{H}(s)h(s)ds < 1$. Similar to the proof of Theorem 3.2, combining with (S_4) we can take R^* as Theorem 3.2. Then $R^* > N > R$. Let $\Omega_3 = \{x \in P : \|x\| < R^*\}$. Similar to (3.13), we have

$$\|Tx\| < \|x\|, \ \ x \in \partial\Omega_3. \tag{4.3}$$

By (4.1), (4.2), (4.3) and Lemma 2.1, T has two fixed points x_1,x_2 in P such that $R_0 < \|x_1\| < R < \|x_2\|$. By arguments similar to (3.8), there exists $l_1 = \left(1 - \frac{\Lambda\int_0^1 p(s)ds}{\sigma R_0}\right) R_0\sigma$, such that $x_1(t) - w(t) \geq l_1e(t),\ \ t \in (0,1)$.

Let $u_1(t) = x_1(t) - w(t)$, then we have $u_1(t) \geq l_1e(t) > 0$. On the other hand, let $u_2(t) = x_2(t) - w(t)$, as (3.15), it is easy to find a constant $l_2 = \frac{1}{2}r\sigma$ such that $u_2(t) \geq l_2e(t) > 0$. By Lemma 2.5, we know BVP (1.1) has at least two positive solutions u_1,u_2 satisfying $u_1(t) \geq l_1e(t), u_2(t) \geq l_2e(t)$ for any $t \in [0,1]$. The proof is completed. ∎

Theorem 4.2. *Assume that conditions* (H_1)–(H_3) *and* $(S_2),(S_3)$ *are satisfied. In addition, assume that the following condition holds:*
(C_2) *There exists a constant* $R > \max\left\{r, \frac{Lr}{l}\right\}$ *such that for any* $(t,u) \in [0,1] \times [0,R]$,

$$F(t,u) \leq \frac{R}{L},$$

where L,l *are defined by* (2.4) *and* r *is defined in* (S_3).
Then BVP (1.1) *has at least two positive solutions* u_1,u_2 *satisfying* $u_1(t) \geq m_1e(t)$, $u_2(t) \geq m_2e(t)$ *for* $m_1 = \frac{1}{2}r\sigma$, $m_2 = \left(1 - \frac{\Lambda\int_0^1 p(s)ds}{\sigma R}\right) R\sigma$.

Proof. Firstly, let $\Omega_1 = \{x \in P : \|x\| < r\}$. From (S_3) and proceeding as in (3.11), we obtain

$$\|Tx\| > \|x\|, \ \ x \in \partial\Omega_1. \tag{4.4}$$

Next, by (C_2), we have $R > r$ and $\frac{R}{L} > \frac{r}{l} > 0$. Let $\Omega_2 = \{x \in P : \|x\| < R\}$. Then for any $x \in \partial\Omega_2, s \in [0,1]$, we have

$$[x(s) - w(s)]^+ \leq x(s) \leq \|x\| \leq R.$$

It follows from (C_2), (2.6) and proceeding as in (3.2), we have

$$\|Tx\| < \|x\|, \ \ x \in \partial\Omega_2. \tag{4.5}$$

On the other hand, choose a real number $M > 0$ such that

$$\frac{\Gamma^2 M}{2}\int_\mu^{1-\mu} e(s)h(s)ds > 1.$$

From (S_2), there exists $N > R$ such that, for any $t \in [\mu, 1-\mu]$, $F(t,u) \geq Mu$. Take $R^* = \max\{2\Gamma^{-1}N, R\}$, then $R^* > R > r$. Let $\Omega_3 = \{x \in P : \|x\| < R^*\}$. Similar to (3.6), we have

$$\|Tx\| > \|x\|, \ \ x \in \partial\Omega_3. \tag{4.6}$$

By (4.4), (4.5), (4.6) and Lemma 2.1, T has two fixed points x_1, x_2 in P such that $r < \|x_1\| < R < \|x_2\|$. By similar arguments to (3.15), there exist $m_1 = \frac{1}{2}r\sigma$, $m_2 = \left(1 - \frac{\Lambda\int_0^1 p(s)ds}{\sigma R}\right)R\sigma$ such that $x_i(t) - w(t) \geq m_i e(t)$, $t \in (0,1)(i = 1,2)$.

Let $u_i(t) = x_i(t) - w(t)(i = 1,2)$, then we have $u_i(t) \geq m_i e(t) > 0(i = 1,2)$. By Lemma 2.5, we know that BVP (1.1) has at least two positive solutions u_1, u_2 satisfying $u_1(t) \geq m_1 e(t), u_2(t) \geq m_2 e(t)$ for any $t \in [0,1]$. The proof is completed. ∎

5. Applications

In this section, we construct several examples to demonstrate the application of our main results.

Example 5.1. Consider the following four-point boundary value problem

$$\begin{cases} -u'' = \dfrac{1}{\sqrt{t(1-t)}}F(t,u) - \dfrac{\sqrt{2}}{\sqrt[4]{t^3(1-t)}}, \quad 0 < t < 1, \\ u(0) = \dfrac{1}{2}u\Big(\dfrac{1}{8}\Big), \quad u(1) = \dfrac{1}{2}u\Big(\dfrac{7}{8}\Big), \end{cases} \tag{5.1}$$

where

$$F(t,u) = \begin{cases} \dfrac{1}{675}u^2, & 0 \leq u \leq 225, \\ \dfrac{14269}{15}u - 213960, & 225 \leq u \leq 300, \\ u + 71120, & 300 \leq u \leq 3200, \\ 929\sqrt{2u}, & u \geq 3200, \end{cases}$$

Then the BVP (5.1) has at least two positive solutions.

Proof. Let

$$h(t) = \frac{1}{\sqrt{t(1-t)}}, \qquad p(t) = \frac{\sqrt{2}}{\sqrt[4]{t^3(1-t)}},$$

then h, p are singular at $t = 0$ and $t = 1$. BVP (5.1) can be regard as a boundary value problem of the form of (1.1). In this situation, $\alpha = \gamma = 1, \beta = \delta = 0$ and

$$\xi(s) = \begin{cases} 0, & s \in \Big[0, \dfrac{1}{8}\Big), \\ \dfrac{1}{2}, & s \in \Big[\dfrac{1}{8}, 1\Big], \end{cases} \qquad \eta(s) = \begin{cases} 0, & s \in \Big[0, \dfrac{7}{8}\Big), \\ \dfrac{1}{2}, & s \in \Big[\dfrac{7}{8}, 1\Big]. \end{cases}$$

Take $[\frac{1}{4}, \frac{3}{4}] \subset [0,1]$, by direct calculation, we have

$$\int_0^1 h(t)dt = \int_0^1 \frac{1}{\sqrt{t(1-t)}}dt = \pi, \quad \int_0^1 p(t)dt = \int_0^1 \frac{\sqrt{2}}{\sqrt[4]{t^3(1-t)}}dt = 2\pi,$$

$$\rho = \sigma = 1, \quad \phi_1(t) = 1-t, \quad \phi_2(t) = t, \quad e(t) = t(1-t), \quad t \in [0,1],$$

$$\Gamma=\frac{3}{16},\quad k_1=\frac{9}{16},\quad k_2=\frac{1}{16},\quad k_3=\frac{1}{16},\quad k_4=\frac{9}{16},\quad k=\frac{5}{16},\quad \Lambda=3,$$

$$l=\Gamma\int_{\frac{1}{4}}^{\frac{3}{4}}e(s)h(s)ds=\frac{2\pi+3\sqrt{3}}{256},\quad L=\Lambda\int_0^1 e(s)h(s)ds=\frac{5\pi}{16}.$$

Clearly, the conditions (H_1)–(H_3) hold. Choose $R_0=225$, then

$$R_0=225>\max\left\{L,\Lambda\sigma^{-1}\Gamma^{-1}\int_0^1 p(s)ds\right\}=64\pi,$$

and for any $(t,u)\in[0,1]\times[0,225]$,

$$F(t,u)=\frac{u^2}{675}\leq 75\leq\frac{R_0}{L}.$$

So the condition (S_1) is satisfied.

On the other hand, we take $R=3200$, then $R>2R_0$. For any $(t,u)\in[\frac{1}{4},\frac{3}{4}]\times\left[\frac{\Gamma}{2}R,R\right]=[\frac{1}{4},\frac{3}{4}]\times[300,3200]$, we have

$$F(t,u)=u+71120\geq 71420>\frac{R}{l},$$

so the condition (C_1) holds. Next, we have

$$\lim_{u\to+\infty}\max_{t\in[0,1]}\frac{F(t,u)}{u}=0.$$

Thus (S_4) also holds. Consequently, by Theorem 4.1, we infer that singular BVP (5.1) has at least two positive solutions. ∎

Example 5.2. Let $F_1(t,u)=\frac{1}{675}u^2,(t,u)\in[0,1]\times[0,+\infty)$. It is easy to verify that the conditions of Theorem 3.1 are satisfied when the function F replaced by F_1 in Example 5.1. By Theorem 3.1 we know that BVP (5.1) with F_1 has at least one positive solution $u(t)$ such that $u(t)\geq(225-12\pi)t(1-t)$, $t\in[0,1]$.

Example 5.3. Consider the following 4-point boundary value problem

$$(5.2)\quad\begin{cases}-u''=\dfrac{F(t,u)}{\sqrt{t(1-t)}}-\dfrac{105}{512+384\sqrt[3]{4}}\left[\dfrac{1}{\sqrt{t}}+\dfrac{1}{\sqrt{1-t}}+\dfrac{1}{\sqrt[3]{(t-\frac{1}{2})^2}}\right], & 0<t<1,\\ u(0)-u'(0)=\dfrac{1}{4}u\Big(\dfrac{1}{3}\Big)+\dfrac{1}{9}u\Big(\dfrac{2}{3}\Big),\\ u(1)+u'(1)=\dfrac{3}{8}u\Big(\dfrac{1}{3}\Big)+u\Big(\dfrac{2}{3}\Big),\end{cases}$$

where

$$F(t,u)=\begin{cases}40\sqrt{2u}, & 0\leq u\leq 128,\\ 640, & 128\leq u\leq 21153,\\ 10(u-21145)^2, & u\geq 21153,\end{cases}$$

Then the BVP (5.2) has at least two positive solutions.

Proof. Let

$$h(t)=\frac{1}{\sqrt{t(1-t)}},\qquad p(t)=\frac{105}{512+384\sqrt[3]{4}}\left[\frac{1}{\sqrt{t}}+\frac{1}{\sqrt{1-t}}+\frac{1}{\sqrt[3]{(t-\frac{1}{2})^2}}\right].$$

BVP (5.2) can be regard as a boundary value problem of the form of (1.1). In this situation, $\alpha=\beta=\gamma=\delta=1$ and

$$\xi(s)=\begin{cases}0, & s\in\left[0,\frac{1}{3}\right),\\ \frac{1}{4}, & s\in\left[\frac{1}{3},\frac{2}{3}\right),\\ \frac{13}{36}, & s\in\left[\frac{2}{3},1\right],\end{cases}\qquad \eta(s)=\begin{cases}0, & s\in\left[0,\frac{1}{3}\right),\\ \frac{3}{8}, & s\in\left[\frac{1}{3},\frac{2}{3}\right),\\ \frac{11}{8}, & s\in\left[\frac{2}{3},1\right].\end{cases}$$

Take $[\frac{1}{4},\frac{3}{4}]\subset[0,1]$, by direct calculation, we have

$$\int_0^1 h(t)dt=\pi,\quad \int_0^1 p(t)dt=\frac{105}{512+384\sqrt[3]{4}}\int_0^1\left[\frac{1}{\sqrt{t}}+\frac{1}{\sqrt{1-t}}+\frac{1}{\sqrt[3]{(t-\frac{1}{2})^2}}\right]dt=\frac{105}{128},$$

$$\rho=3,\quad \sigma=\frac{3}{4},\quad \phi_1(t)=\frac{2-t}{3},\quad \phi_2(t)=\frac{1+t}{3},\quad e(t)=\frac{1}{3}(2-t)(1+t),\quad t\in[0,1],$$

$$\Gamma=\frac{35}{64},\quad k_1=\frac{263}{324},\quad k_2=\frac{14}{81},\quad k_3=\frac{47}{72},\quad k_4=\frac{5}{18},\quad k=\frac{73}{648},\quad \Lambda=11,$$

$$l=\Gamma\int_{\frac{1}{4}}^{\frac{3}{4}}e(s)h(s)ds=\frac{595\pi}{4608}+\frac{35\sqrt{3}}{3072},\quad L=\Lambda\int_0^1 e(s)h(s)ds<33.$$

Clearly, the conditions (H_1)–(H_3) hold. Set $r=128$, then

$$r=128>2\Lambda\sigma^{-1}\Gamma^{-1}\int_0^1 p(s)ds=44,$$

and for any $(t,u)\in[\frac{1}{4},\frac{3}{4}]\times\left[\frac{\Gamma}{2}r,r\right]=[\frac{1}{4},\frac{3}{4}]\times[35,128]$,

$$F(t,u)=40\sqrt{2u}\geq 40\sqrt{70}>\frac{r}{l}.$$

So the condition (S_3) is satisfied.

On the other hand, we take $R=21153$, then $R>\max\{r,\frac{Lr}{l}\}$. For any $(t,u)\in[0,1]\times[0,21153]$, we have

$$F(t,u)\leq 640\leq\frac{R}{L},$$

so the condition (C_2) holds. Finally, we have

$$\lim_{u\to+\infty}\min_{t\in[\frac{1}{4},\frac{3}{4}]}\frac{F(t,u)}{u}=+\infty.$$

Thus (S_2) also holds. Consequently, by Theorem 4.2, we infer that singular BVP (5.2) has at least two positive solutions. ∎

Example 5.4. Let $F_2(t,u) = 40\sqrt{2u}, (t,u) \in [0,1] \times [0,+\infty)$. It is easy to verify that the conditions of Theorem 3.2 are satisfied if the function F replaced by F_2 in Example 5.3. By Theorem 3.2 we know that BVP (5.2) with F_2 has at least one positive solution $u(t)$ such that $u(t) \geq 16(2-t)(1+t)$, $t \in [0,1]$.

Acknowledgement. The first and second authors were supported financially by the National Natural Science Foundation of China (11371221, 11126231) and the Project of Shandong Province Higher Educational Science and Technology Program (J14LI08), the Specialized Research Foundation for the Doctoral Program of Higher Education of China (20123705110001) and the Program for Scientific Research Innovation Team in Colleges and Universities of Shandong Province. The third author was supported financially by the Australia Research Council through an ARC Discovery Project Grant.

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